Japan. J. Math.
Vol. 8, No. 2, 1982

On the Rochlin invariants of Z_2-homology 3-spheres with cyclic actions

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(Received August 17, 1981)

The purpose of this paper is to study how a cyclic action on a Z_2-homology 3-sphere contributes to the Rochlin invariant of the Z_2-homology 3-sphere. Let S be a Z_2-homology 3-sphere with Z_n-action. The induced projection $pr: S \to \underset{\sim}{S} = S/Z_n$ is clearly the composite of a sequence of the induced projections of cyclic actions on Z_2-homology 3-spheres, arising in the following four cases (1)–(4): (1) Free cyclic action of an order which is a power of 2, (2) Non-free involution, (3) Free cyclic action of odd-prime order, (4) Non-free cyclic action of odd-prime order. Therefore, the problem is reduced to the actions on Z_2-homology 3-spheres of the cases (1)–(4). In each case, we shall establish a congruence in Q/Z containing the Rochlin invariant and the Atiyah-Singer invariant of the action. Then one could derive from these congruences a general congruence for any cyclic action on any Z_2-homology 3-sphere, although we do not state it in this paper.

In Section 1 we introduce a notion of the slope with value in $Q/Z \cup \{\infty\}$ of a knot in an oriented 3-manifold. A geometric meaning of the slope discussed there will be used often in this paper. In Section 2 we shall discuss an invariant of a knot in a Z_2-homology 3-sphere, generalizing the Robertello invariant of a classical knot by Robertello [30] or a knot in a Z-homology 3-sphere by Gordon [12]. This enables us to calculate the Rochlin invariant from a novel viewpoint. In Section 3 several elementary calculations of the Rochlin invariants will be made. In Section 4 we shall discuss the Atiyah-Singer invariant of a cyclic action on a closed oriented 3-manifold. It is well-known for a free cyclic action. We shall also define it for a certain semi-free cyclic action on a closed oriented 3-manifold, e.g., for any semi-free cyclic action on any rational homology 3-sphere. Section 5 is devoted to a remark concerning a cyclic action of odd order on a spin 4-manifold. In Sections 6, 8, 9, 10, 11 and 12, our desired congruences will be established. Section 6 deals with the case of a free involution. In Section 7 is given the application of the result of Section 6. Section 8 is concerned with the case of a non-free involution, and Section 9 with the case of a free cyclic action of order four. Section 10 takes care of the case of a free cyclic action of an order which is a power of 2 greater than four, Section 11 the case of a free cyclic action of odd-prime order, and Section 12 the case of a non-free cyclic action of odd-prime order.

Throughout this paper the following conventions will be adopted unless otherwise specified: Spaces and maps are in the piecewise-linear category. Manifolds are orientable and oriented suitably. Cyclic groups acting on manifolds are written as multiplicative groups. Actions on manifolds are orientation-preserving actions. Actions are faithful (i.e., each element of the group of action except for the identity acts non-trivially). Let X be an oriented manifold. In case $\partial X \neq \emptyset$, the boundary ∂X is oriented by the orientation induced from X. $-X$ is the same manifold as X but with the opposite orientation. Let $X \times [-1, 1]$ have an orientation whose boundary has an orientation such that the natural injections $X \times 1 \rightarrow X$ and $X \times (-1) \rightarrow -X$ are orientation-preserving.

§ 1. The slope of a knot in a 3-manifold

Let k be a knot (i.e., an imbedded oriented 1-sphere) in an oriented 3-manifold M with a tubular neighborhood $T \subset \text{Int}\, M$. A *meridian*, m, of T (or k) is an oriented, simple closed curve on ∂T, bounding an oriented disk D in T with intersection number $D.k = +1$. A *longitude*, ℓ, of T (or k) is any oriented, simple closed curve on ∂T, homotopic to k in T. Then $[m]$, $[\ell] \in H_1(\partial T; Z)$ form a basis with intersection number $[m] \cdot [\ell] = +1$ with respect to the orientation of ∂T induced from $T \subset M$. An *m. ℓ. pair* of T (or k) is a pair (m, ℓ) of a meridian m and a longitude ℓ of T (or k) such that the intersection $m \cap \ell$ is one point. A link (i.e., an imbedded, closed, oriented, possibly disconnected 1-manifold), $P \subset M$, is *parallel* on T (or k) if $P \subset \partial T$ and any two (oriented) components of P are isotopic on ∂T. The number of the components of a link L is denoted by $\sharp L$. For a knot k in M, $o(k)$ denotes the order of the element $[k] \in H_1(M; Z)$, called the *order* of the knot k in M. Note that $o(k) = 1$ if $[k] = 0$.

LEMMA 1.1. *Given a knot $k \subset M$ of finite order with tubular neighborhood T, there exists exactly one (up to isotopy) parallel link P on T such that*

(1) *$[P] = o(k)[k]$ in $H_1(T; Z)$,*

(2) *P bounds a compact oriented surface in $E = M - \text{Int}\, T$.*

The link P, any component K of P and any compact oriented surface in E, bounded by P are called the *characteristic parallel link*, the *characteristic knot* and a *characteristic surface* for the knot k in M, respectively. In case $o(k) = 1$, P is a longitude of T and we see that k bounds a surface in M, obtained by extending any characteristic surface for the knot k, called a

Seifert surface in the classical knot theory.

Proof of Lemma 1.1. We may assume $\partial M=\emptyset$. Then consider the following commutative diagram with natural homomorphisms:

$$\begin{array}{ccccc} H_2(E, \partial E; Z) & \xrightarrow{\partial} & H_1(\partial E; Z) & & \\ \cong \downarrow e & & \downarrow & & \\ H_2(M, T; Z) & \xrightarrow{\partial'} & H_1(T; Z) & \xrightarrow{i_*} & H_1(M; Z). \end{array}$$

Since $i_*o(k)[k]=0$ in $H_1(M; Z)$, we find $x \in H_2(M, T; Z)$ such that $\partial' x= o(k)[k]$. Let $x' \in H_2(E, \partial E; Z)$ be the preimage of x under the excision isomorphism e. Choose a compact 3-submanifold X of E so that ∂E is a component of ∂X and x' is the image of some $x'' \in H_2(X, \partial E; Z)$ under the natural homomorphism $H_2(X, \partial E; Z) \to H_2(E, \partial E; Z)$. Let $A=\partial X-\partial E$. By Poincaré duality,

$$[X/A, S^1]=H^1(X/A; Z)=H^1(X, A; Z) \xrightarrow{\cong} H_2(X, \partial E; Z).$$

We have a piecewise-linear map $f: (X, A) \to (S^1, \{pt\})$ corresponding to x''. By t-regularity, there is a point $q(\neq pt) \in S^1$ such that $f^{-1}(q)=F$ is a compact oriented surface in X with $\partial F \subset \partial E$. We have $[F]=x''$ in $H_1(X, \partial E; Z)$, so that $[F]=x'$ in $H_2(E, \partial E; Z)$. By construction, $[\partial F]=o(k)[k]$ in $H_1(T; Z)$. If ∂F has a component null-homologous in ∂T, then ∂F has necessarily a component c bounding a disk D in ∂T such that $\text{Int}\, D \cap \partial F=\emptyset$. By pushing D into Int E, we obtain a new surface F_1 with $\partial F_1=\partial F-c$. Clearly, $[\partial F_1]=[\partial F]$ in $H_1(T; Z)$. By induction, we can assume that ∂F has no component null-homologous in ∂T. Then if we neglect the orientations of the components of ∂F, any two components would be isotopic. So, if ∂F has two components, not isotopic with respect to the orientation induced from F, then ∂F has necessarily two (oriented) components whose union bounds an oriented annulus $A(\cong S^1 \times [0, 1])$ in ∂T such that $\text{Int}\, A \cap \partial F=\emptyset$. By pushing A into Int E, we obtain a new surface F_2 with $\partial F_2=\partial F-\partial A$. Clearly, $[\partial F_2]=[\partial F]$ in $H_1(T; Z)$. By induction, we obtain a compact oriented proper surface F_* in E such that the link $\partial F_*=P$ is a parallel link on T and $[P]=o(k)[k]$ in $H_1(T; Z)$. P has (1) and (2). To prove the uniqueness of P, we use the following lemma:

Lemma 1.2. *Any 1-cycle c_1 in ∂T with intersection number $c_1 \cdot P \neq 0$ in ∂T represents an element of infinite order in $H_1(E; Z)$.*

By Lemma 1.2, $o(m)=\infty$ in $H_1(E; Z)$, where m is the meridian of T. This implies that the boundary homomorphism $\bar{\partial}: H_2(M, E; Z) \to H_1(E; Z)$ coming from the pair (M, E) is injective, so that the natural homomorphism $H_2(E; Z)$

$\to H_2(M; Z)$ is onto. To prove the uniqueness of P, let P' be another parallel link on T which bounds a surface F' in E such that $[P']=o(k)[k]$ in $H_1(T; Z)$. Since $H_2(E; Z)\to H_2(M: Z)$ is onto, we see that $[F]-[F']\in H_2(M, T; Z)$ is represented by a 2-cycle c_2 in E. Hence $[F]=[F']+[c_2]$ in $H_2(E, \partial E; Z)$. Thus, we have

$$[P]=[\partial F]=\partial[F]=\partial[F']+\partial[c_2]=[\partial F']=[P']$$

in $H_1(\partial E; Z)$. Put $r=\sharp P$ and $r'=\sharp P'$. Write $[P]=ra[m]+rb[\ell]$ and $[P']=r'a'[m]+r'b'[\ell]$ for an m. ℓ. pair (m, ℓ) of T. Since $[P]=[P']$, we have that $ra=r'a'$ and $rb=r'b'$. But, $(a, b)=(a', b')=1$, so that $r=(ra, rb)=(r'a', r'b')=r'$, that is, $a=a'$ and $b=b'$. This implies that P and P' are isotopic on ∂T. This completes the proof except for the proof of Lemma 1.2.

PROOF OF LEMMA 1.2. Let c_1' be a 1-cycle in E obtained by pushing c_1 into Int E. Let F be a compact oriented surface in E bounded by P. The intersection number, $F\cdot c_1'$ (in E)$=\partial F\cdot c_1$ (in ∂E)$=c_1\cdot P$ (in ∂T)$\neq 0$. This implies that $[c_1]=[c_1']$ is an element of infinite order in $H_1(E; Z)$, proving Lemma 1.2.

COROLLARY 1.3. *The characteristic parallel link P up to orientation of P is determined uniquely by the space $E=M-$*Int *T.*

PROOF. It suffices to show that assuming $\partial M=\emptyset$, $[P]$ is a generator of $G=\mathrm{Im}[\partial: H_2(E, \partial E; Z)\to H_1(\partial E; Z)]\cong Z$ ($G\cong Z$ follows from Lemma 1.2). Clearly, $0\neq[P]\in G$. Suppose $[P]$ is not a generator. Write $[P]=ng$ for an integer $n>1$ and a generator g of G. Since $[P]=o(k)[k]$ in $H_1(T; Z)$, the image g' of g under the composite $G\subset H_1(\partial E; Z)\to H_1(T; Z)$ has $g'=n'[k]$ for some n' with $0<n'<o(k)$. The natural homomorphism $H_1(T; Z)\to H_1(M; Z)$ maps g' to 0. So, the order of k must be at most $n'<o(k)$, a contradiction. Therefore, $[P]$ is a generator of G, completing the proof.

Let K be the characteristic knot of a knot $k\subset M$ of finite order. Write $[K]=a[m]+b[\ell]=a'[m']+b'[\ell']$ in $H_1(\partial T; Z)$ for any m. ℓ. pairs (m, ℓ), (m', ℓ') of T. Clearly, $b=b'$. By definition, $[m]=[m']$ and $[\ell']=[\ell]+c[m]$ for an integer c, so that $a=a'+bc$. Thus, $a/b=(a'/b)+c=a'/b'$ in Q/Z. We showed that $a/b\in Q/Z$ is a knot type invariant of k in M.

DEFINITION 1.4. $a/b\in Q/Z$ is called the *slope* of the knot $k\subset M$ of finite order and denoted by $s(k)=s(k\subset M)$. If $s(k)=0$, then we say the knot k is *flat*. When k is a knot of infinite order, we say the slope of k is infinite and denote $s(k)=\infty$.

A flat knot has properties analogous to those of a classical knot. For example, any flat knot has a unique m. ℓ. pair with the longitude, being the characteristic knot. A *normal presentation* of the slope $s(k)$ of a knot k of finite order is a rational number a/b with coprime integers $a \geq 0$ and $b > 0$ such that $s(k) = a/b$ (mod 1).

The following shows that the complement $M - k$ never contributes to the slope $s(k)$.

PROPOSITION 1.5. *Let E be an oriented 3-manifold with ∂E, a torus. Suppose the natural homomorphism $H_1(\partial E; Q) \to H_1(E; Q)$ is not injective. Then for each $s \in Q/Z \cup \{\infty\}$ there exists a knot $k \subset M$ with $s(k) = s$ such that $M - \mathrm{Int}\, T$ is homeomorphic to E.*

REMARK 1.6. When E is compact, the homomorphism $H_1(\partial E; Q) \to H_1(E; Q)$ is not injective, since by Poincaré duality it is Q-dual to the boundary homomorphism $\partial: H_2(E, \partial E; Q) \to H_1(\partial E; Q)$.

PROOF OF PROPOSITION 1.5. Since $H_1(\partial E; Q) \to H_1(E; Q)$ is not injective, there is a knot $k_* \subset M_*$ of finite order such that $M_* - \mathrm{Int}\, T(k_*) = E$. Let K be the characteristic knot ($\subset \partial T(k_*)$) of k_*. For $s \neq \infty$ choose coprime integers a, b so that $s = a/b$ (mod 1), $b > 0$. For $s = \infty$, let $a = 1$, $b = 0$. Take simple closed curves m, ℓ on ∂E so that $m \cap \ell$ is one point with $[\ell] \cdot [m] = +1$ on ∂E and $[K] = a[m] + b[\ell]$. Construct an oriented 3-manifold $M = E \cup S^1 \times D^2$ identifying ∂E with $S^1 \times \partial D^2$ so that $m = p \times \partial D^2$ ($p \in S^1$) and $\ell = S^1 \times q$ ($q \in \partial D^2$). Let $k = S^1 \times 0 \subset M$. When $s \neq \infty$, we see easily that k has the slope s. For $s = \infty$, by Lemma 1.2 $o(\ell) = \infty$ in $H_1(E; Z)$. From the Mayer-Vietoris sequence we see that $o(k) = o(\ell) = \infty$ in $H_1(M; Z)$ and $s(k) = \infty = s$. This completes the proof.

COROLLARY 1.7. *The space M of a knot k with $s(k) = \infty$, constructed from E of Proposition 1.5 is unique in the following sense: For another space M' of a knot k' with $s(k') = \infty$, constructed from E and a homeomorphism $h: E \to E$, there exists a homeomorphism $\bar{h}: M' \to M$ extending h.*

PROOF. To be $s(k \subset M) = \infty$, it is necessary that the attaching homeomorphism $\partial(S^1 \times D^2) \to \partial E$ sends the meridian of $S^1 \times D^2$ to a curve isotopic to the characteristic knot K (in the proof of Proposition 1.5), which is unique up to orientation by Corollary 1.3. The construction must be as in the proof of Proposition 1.5. It is easy to see that the homeomorphism type of M does not depend on a choice of the longitude ℓ. This completes the proof.

Let M be an oriented 3-manifold. Let $\tau H_1(M)$ be the torsion part of $H_1(M; Z)$. The linking pairing $\phi: \tau H_1(M) \times \tau H_1(M) \to Q/Z$ is defined as usual (cf. Seifert-Threlfall [33]).

LEMMA 1.8. *For any knot $k\subset M$ of finite order, we have*

$$s(k)=-\phi([k],[k]).$$

PROOF. Let P be the characteristic parallel link of $k\subset M$. Let $r=\sharp P$. Write $[P]=ra[m]+rb[\ell]$ for an m. ℓ. *pair* (m,ℓ) of $T=T(k)$. An integral 2-chain $\hat{F}$ in M with $\partial\hat{F}=rbk$ can easily be constructed by using a characteristic surface F bounded by P in $E=M-\text{Int}\,T$. By definition, $\phi([k],[k])=\hat{F}\cdot\ell/rb$ (mod 1), where $\hat{F}\cdot\ell$ is the intersection number in M. But, $\hat{F}\cdot\ell=F\cdot\ell$ (in E)$=P\cdot\ell$ (in ∂E)$=-ra$ (Note that $m\cdot\ell=-1$ in ∂E). So, $\phi([k],[k])=-ra/rb=-a/b=-s(k)$. This completes the proof.

The following corollaries for knots of finite order are easily obtained from Lemma 1.8 and the well-known properties of the linking pairing.

COROLLARY 1.9. *The slope $s(k\subset M)$ depends only on the homology class $[k]\in H_1(M;Z)$. More generally, for two knots $k\subset M$, $k'\subset M'$ and a homeomorphism $f\colon M\to M'$ of degree $\varepsilon(\varepsilon=\pm1)$ such that $f_*[k]=\pm[k']$, we have $s(k\subset M)=\varepsilon s(k'\subset M')$.*

COROLLARY 1.10. $s(-k\subset M)=s(k\subset M)$, $s(k\subset -M)=-s(k\subset M)$.

COROLLARY 1.11. *The knot $k\subset M$ is flat if $o(k)$ is odd and $k\subset M$ is amphicheiral (i.e., $\exists$ an orientation-reversing homeomorphism of M onto itself sending k to $\pm k$).*

COROLLARY 1.12. *For the knot sum $k\sharp k'\subset M\sharp M'$ of any two knots $k\subset M$ and $k'\subset M'$, we have $s(k\sharp k'\subset M\sharp M')=s(k\subset M)+s(k'\subset M')$.*

COROLLARY 1.13. *Let n be the order (≥ 1) of $s(k\subset M)$ in Q/Z. The knot sum of n copies of the knot $k\subset M$ is flat.*

COROLLARY 1.11 was used implicitly in the proof of Theorem 5.1 in [18], which provides a key to the argument of [19].

§2. A generalization of the Robertello invariant of a classical knot

Let M be a closed oriented 3-manifold with $H_1(M;Z_2)=0$. Each component of M is a Z_2-homology 3-sphere. The *Rochlin invariant* (or *μ-invariant*), $\mu(M)$, of M is defined by

$$\mu(M)=-\text{sign}\,W/16\in Q/Z$$

for any compact oriented spin $(w_2=0)$ 4-manifold W with $\partial W=M$ (e.g. [13]). Throughout this section, S denotes a Z_2-homology 3-sphere. Let k be a knot in S. A pair $D\subset W$ of a proper disk D in a compact oriented 4-manifold W is *admissible* for a knot $k\subset S$, when S is a component of ∂W, $\partial D=k$, $H_1(\partial W; Z_2)=0$ and the Z_2-intersection number $[D_2^+]\cdot x=x\cdot x$ for all $x\in H_2(W; Z_2)$, where D_2^+ is a (mod 2) 2-cycle obtained from D by attaching a (mod 2) 2-chain c_2 in S with $\partial c_2=-k$. For each knot $k\subset S$, there is an admissible pair $D\subset W$ with D being locally flat. [PROOF. Let a/b be a normal presentation of $s(k)$ with a odd. Note that b is odd. Let (m, ℓ) be an $m.\ \ell.$ *pair* of $T=T(k)$ such that $[K]=a[m]+b[\ell]$ in $H_1(\partial T; Z)$ for the characteristic knot K of k in S. Construct $W=S\times[-1, 1]\cup D^2\times D^2$ identifying $T\times 1$ with $\partial D^2\times D^2$ so that $m\times 1=p\times\partial D^2$, $\ell\times 1=\partial D^2\times q$ $(p, q\in\partial D^2)$. Let $E=S-\text{Int}\,T$. Since $[K]=0$ in $H_1(E; Z_2)$, we have $[\ell]=[m]\neq 0$ in $H_1(E; Z_2)=Z_2$. It follows that $H_1(\partial W; Z_2)=0$ and the pair $\bar{D}=k\times[-1, 1]\cup D\times 0\subset W$ is admissible for $k\times(-1)\subset S\times(-1)$.] Let D_Q^+ be a rational 2-cycle in W obtained from D by attaching a rational 2-chain c_Q in S with $\partial c_Q=-k$.

DEFINITION 2.1. $\delta(k)=\delta(k\subset S)=([D_Q^+]^2\text{-sign}\,W)/16-\mu(\partial W)\in Q/Z$ for any admissible pair $D\subset W$ for $k\subset S$ with a locally flat disk D. ($[D_Q^+]^2$ is the Q-intersection number of $[D_Q^+]\in H_2(W; Q)$.)

This invariant was defined by Robertello [30] for a classical knot and by Gordon [12] for a knot in a Z-homology 3-sphere. In their cases, it takes the value 0 or 1/2, but in our general case, it takes more values depending on the slope of the knot.

To check the well-definedness, let $D'\subset W'$ be another admissible pair with a locally flat disk D'. The union $\Sigma=D\cup -D'$ is a locally flat 2-sphere in a compact oriented 4-manifold $W''=W\cup -W'$, identifying two copies of S. Note that $[\Sigma]\cdot x=x\cdot x$ for all $x\in H_2(W''; Z_2)$, since $H_2(S; Z_2)=0$. Then the Rochlin theorem ([10], [13], [22], [23], [31]) asserts that

$$\mu(\partial W'')=([\Sigma]^2\text{-sign}\,W'')/16$$

in Q/Z. (We understand that $\mu(\partial W'')=0$ if $\partial W''=\phi$.) But, $[\Sigma]^2=[D_Q^+]^2-[-D_Q'^+]^2=[D_Q^+]^2-[D_Q'^+]^2$ (since $H_2(S; Q)=0$), $\text{sign}\,W''=\text{sign}\,W-\text{sign}\,W'$ and $\mu(\partial W'')=\mu(\partial W-S)-\mu(\partial W'-S)$, where we count $[-D_Q'^+]^2$, $[D_Q'^+]^2$ in W'. It follows that $([D_Q^+]^2\text{-sign}\,W)/16-\mu(\partial W)=([D_Q'^+]^2\text{-sign}\,W')/16-\mu(\partial W')$ in Q/Z, showing the well-definedness of $\delta(k\subset S)$.

LEMMA 2.2. *For the knot sum $k_1\# k_2\subset S_1\# S_2$ of two knots $k_i\subset S_i$, $i=1, 2$, $\delta(k_1\# k_2\subset S_1\# S_2)=\delta(k_1\subset S_1)+\delta(k_2\subset S_2)$.*

This follows easily from the definition of δ-invariant.

LEMMA 2.3. *Let $D\subset W$ be an admissible pair for a knot $k\subset S$. Suppose D has just one non-locally flat point represented by a classical knot $k_0\subset S^3$.*

Then we have

$$\delta(k\subset S)+\delta(k_0\subset S^3)=([D_Q^+]^2\text{-sign } W)/16-\mu(\partial W).$$

PROOF. The inclusion $(D, \partial D)\subset(W, S)$ is homotopic to a locally flat imbedding $i: (D, \partial D)\rightarrow(W, S)$ such that $\partial(iD\subset W)=(k\sharp k_0\subset S\sharp S^3=S)$ (cf. Fox-Milnor [9]). Since $[D]=[iD]$ in $H_2(W, S; Z)$ and $H_1(S; Z_2)=H_2(S; Z_2)=0$, we have $[D_2^+]=[iD_2^+]$ in $H_2(W; Z_2)$, so that $iD\subset W$ is admissible for the knot $k\sharp k_0$. Using $H_1(S; Q)=H_2(S; Q)=0$, we have also $[D_Q^+]=[iD_Q^+]$ in $H_2(W; Q)$. Now the result follows from the definition of δ-invariant and Lemma 2.2.

By Remark 1.6 and Corollary 1.7 we can obtain from the knot complement $E=S-\text{Int } T(k)$ a unique (up to homeomorphism) closed connected oriented 3-manifold M such that $H_1(M; Z)$ is infinite. Then, since $H_1(E; Z_2)=Z_2$, we have that $H_1(M; Z)/(\text{odd torsion})\cong Z$. Such a manifold is called a Z_2-homology handle [18]. Let $Z\langle t\rangle$ be the integral group ring of the infinite cyclic group $\langle t\rangle$ generated by t. Let $A(t)$ be the Alexander polynomial of M in $Z\langle t\rangle$ associated with an epimorphism $\gamma: \pi_1(M)\rightarrow\langle t\rangle$ (See [20]), and $A(t)_2$ be its modulo 2 reduction in $Z_2\langle t\rangle$. $A(t)$ has the properties that $A(t)=A(t^{-1})$ up to a multiple of t and $A(1)$ is odd, so that we can deduce that $A(t)_2=A(t^{-1})_2$ up to a multiple of t and $A(1)_2=1\in Z_2$, by a method analogous to [21]. In particular, $A(t)$ (up to a multiple of t) does not depend on any choice of γ and is an invariant of M. In [18], we defined an invariant $\varepsilon(M)$, being 0 or 1, of M such that

(I) $\varepsilon(M)=0$ if and only if M is the boundary of a compact, connected, oriented spin 4-manifold W with an isomorphism

$$H_1(M; Z)/(\text{odd torsion}) \longrightarrow H_1(W; Z)/(\text{odd torsion})\ (\cong Z),$$

induced by inclusion, and

(II) $A(t)_2=\varepsilon(M)(t^2+1)+t$ in $Z_2\langle t\rangle/(t^4+1)$ up to a multiple of t.

We have also the following (cf. [28], [12]):

LEMMA 2.4. $\varepsilon(M)/2=(A(1)^2A(-1)^2-1)/16$ in Q/Z.

PROOF. Write $A(t)=a_0+a_1(t+t^{-1})+\cdots+a_m(t^m+t^{-m})$, $a_i\in Z$. Since $A(1)$ is odd, a_0 is odd. Further,

$$A(1)-A(-1)=4(a_1+a_3+\cdots+a_{m'}),$$

where m' is the greatest odd number such that $m'\leq m$. Using that a_0 is odd, $A(t)_2$ is written in $Z_2\langle t\rangle/(t^4+1)$ as follows:

$$1+(a_1+a_3+\cdots+a_{m'})(t+t^{-1}).$$

So, we have $\varepsilon(M)\equiv a_1+a_3+\cdots+a_{m'}$ (mod 2). Using $(A(1)-A(-1))/4=a_1+a_3+\cdots+a_{m'}$, we see that

$$(A(1)-A(-1))/8\equiv\varepsilon(M)/2 \pmod 1.$$

Since $A(1)\equiv A(-1)$ (mod 4), let $A(-1)=4a+A(1)$ for some $a\in Z$. Then $A(1)A(-1)=4aA(1)+A(1)^2$ and hence

$$\begin{aligned}(A(1)^2A(-1)^2-1)/16 &\equiv(8aA(1)^3+A(1)^4-1)/16 \pmod 1\\ &\equiv a/2 \pmod 1, \text{ for } A(1) \text{ is odd}\\ &\equiv(A(1)-A(-1))/8 \pmod 1\\ &\equiv\varepsilon(M)/2 \pmod 1,\end{aligned}$$

as desired.

The following theorem characterizes the δ-invariant of a flat knot.

THEOREM 2.5. *For any flat knot $k\subset S$ with Z_2-homology handle M, we have $\delta(k\subset S)=\varepsilon(M)/2\in\{0, 1/2\}\subset Q/Z$.*

PROOF. Let (m, ℓ) be an $m.\ \ell.$ pair of $T(k)$ with $\ell=K(k)$ the characteristic knot of k. Suppose $\varepsilon(M)=0$. Then M bounds a compact connected oriented spin 4-manifold W such that $H_1(M; Z)/(\text{odd torsion})\overset{\cong}{\to}H_1(W; Z)/(\text{odd torsion})$. Construct $W^*=S\times[-1, 1]\cup D^2\times D^2$, identifying $T(k)\times 1$ with $\partial D^2\times D^2$ so that $m\times 1=p\times\partial D^2$, $\ell\times 1=\partial D^2\times q$, $p, q\in\partial D^2$. Clearly, $\partial W^*-S\times(-1)=M$. Then $\overline{W}=W^*\cup -W$, with the two copies of M identified, is a spin 4-manifold with boundary $S\times(-1)\cong -S$. Note that $\overline{D}=k\times[-1, 1]\cup D^2\times 0\subset\overline{W}$ is admissible for $k\times(-1)\subset S\times(-1)$. Since $[\overline{D}{}_Q^+]^2=0$ and $\mu(S\times(-1))=-\operatorname{sign}\overline{W}/16$, it follows that $\delta(k\subset S)=-\delta(k\times(-1)\subset S\times(-1))=0$. Next, let $\varepsilon(M)=1$. Let $k_0\subset S^3$ be a classical knot with knot polynomial, congruent to t^2+t+1 in $Z_2\langle t\rangle/(t^4+1)$. By Robertello [30], we have $\delta(k_0\subset S^3)=1/2$. Let M' be the Z_2-homology handle of the knot sum $k\# k_0\subset S\# S^3=S$ which is also a flat knot. By an argument analogous to the proof of Theorem 5.1 of [18], we have $\varepsilon(M')=0$. Hence $\delta(k\# k_0\subset S)=\delta(k\subset S)+\delta(k_0\subset S^3)=0$, so that $\delta(k\subset S)=\delta(k_0\subset S^3)=1/2$. This completes the proof.

LEMMA 2.6. Let *k be a knot of finite order in the interior of an oriented 3-manifold M. Let a/b be a normal presentation of the slope $s(k\subset M)$. Let (m, ℓ) be an $m.\ \ell.$ pair of $T=T(k)$ such that $[K]=a[m]+b[\ell]$. Construct $W=M\times[-1, 1]\cup D^2\times D^2$, identifying $T\times 1$ with $\partial D^2\times D^2$ so that $m\times 1=p\times\partial D^2$, $\ell\times 1=\partial D^2\times q$, $p, q\in\partial D^2$. Let $\overline{D}=k\times[-1, 1]\cup D^2\times 0$ be a disk. We have $[\overline{D}{}_Q^+]^2=-a/b$, and $[\overline{D}{}_Q^+]^2 \pmod 1=s(k\times(-1)\subset M\times(-1))=-s(k\subset M)$.*

PROOF. $\overline{D}'=\ell\times[-1, 1]\cup D^2\times q$ is a disk with $\partial\overline{D}'=\ell\times(-1)$. Let c be a rational 2-chain in $S\times(-1)$ with $\partial c=-k\times(-1)$. Then,

$$[\bar{D}_Q^+]^2=\bar{D}'\cdot(c+\bar{D})(\text{in } W)=\bar{D}'\cdot c \text{ (in } W)=\partial\bar{D}'\cdot c$$

(in $M\times(-1))=\ell\times(-1)\cdot c$ (in $M\times(-1))=c\cdot\ell\times(-1)$ (in $M\times(-1)$).

Now let c be the rational 2-chain $(-1/rb)\hat{F}\times(-1)$ with $\hat{F}$ being as used in the proof of Lemma 1.8. Then,

$$\begin{aligned} c\cdot\ell\times(-1) \text{ (in } M\times(-1))&=(-1/rb)P\times(-1)\cdot\ell\times(-1) \text{ (in } \partial E\times(-1))\\ &=(1/rb)P\times(-1)\cdot\ell\times(-1) \text{ (in } \partial T\times(-1))\\ &=-ra/rb=-a/b. \end{aligned}$$

This completes the proof.

Since the characteristic knot of any knot of finite order is flat, the following theorem gives a reduction of the δ-invariant of any knot in S to that of a flat knot in S.

THEOREM 2.7. *For any normal presentation a/b of the slope $s(k\subset S)$ with a odd, we have*

$$\delta(k\subset S)=\delta(K\subset S)+(a/b-ab)/16$$

in Q/Z, where K is the characteristic knot of k in S.

PROOF. Let (m, ℓ) be an m. ℓ. pair of $T=T(k)$ so that $[K]=a[m]+b[\ell]$. Construct $W=S\times[-1, 1]\cup D^2\times D^2$, identifying $T\times 1$ with $\partial D^2\times D^2$ so that $m\times 1=p\times\partial D^2$, $\ell\times 1=\partial D^2\times q$, $p, q\in\partial D^2$. Since a, b are odd, we have $H_1(\partial W; Z_2)=0$. Note that $\bar{D}=k\times[-1, 1]\cup D^2\times 0\subset W$ is admissible for $(k\times(-1)\subset S\times(-1))=(-k\subset -S)$. So,

$$-\delta(k\subset S)=\delta(-k\subset -S)=([\bar{D}_Q^+]^2-\text{sign } W)/16-\mu(\partial W)$$

Let D' be a disk obtained from the knot $K\times 1\subset T(k)\times 1\subset\partial(D^2\times D^2)$ by taking a cone with vertex in $\text{Int}(D^2\times D^2)$. $\bar{D}'=K\times[-1, 1]\cup D'$ is a disk in W with just one non-locally flat point represented by the torus knot $k_{a,b}\subset S^3$ of type (a, b). From the composite isomorphism

$$H_2(W, S\times(-1); Z)\xleftarrow{\cong} H_2(D^2\times D^2, T\times 1; Z)\xrightarrow{\cong} H_1(T; Z).$$

and $[K]=b[k]$ in $H_1(T; Z)$, we see that $[\bar{D}']=b[\bar{D}]$ in $H_2(W, S; Z)$. Since b is odd, $\bar{D}'\subset W$ is admissible for $(K\times(-1)\subset S\times(-1))=(-K\subset -S)$. By Lemma 2.3,

$$\delta(-K\subset -S)+\delta(k_{a,b}\subset S^3)=([\bar{D}_Q']^2-\text{sign } W)/16-\mu(\partial W).$$

The knot polynomial $A(t)$ of $k_{a,b}$ with a, b odd has $|A(\pm 1)|=1$ (cf. [9]), so that $\delta(k_{a,b}\subset S^3)=0$ (cf. Lemma 2.4 and Theorem 2.5). Thus, we have

$$\delta(k\subset S)=\delta(K\subset S)+([\bar{D}_Q'^+]^2-[\bar{D}_Q^+]^2)/16.$$

But, $[\bar{D}_Q^+]^2=-a/b$ by Lemma 2.6. Since $[\bar{D}']=b[\bar{D}]$ in $H_2(W, S\times(-1); Z)$, we have $[\bar{D}_Q'^+]=b[\bar{D}_Q^+]$ in $H_2(W; Q)$, so that $[\bar{D}_Q'^+]^2=b^2[\bar{D}_Q^+]^2=-ab$. The result follows.

§ 3. Several elementary calculations of the Rochlin invariants

Let T_i be oriented solid tori with m. ℓ. pairs (m_i, ℓ_i), $i=1, 2$. Let $h: \partial T_1\to \partial T_2$ be an orientation-reversing homeomorphism such that $h_*[m_1]=a[m_2]+b[\ell]$ $(b\neq 0)$. The adjunction space $T_1\cup_h T_2$ is the lens space $-L(b, a)=L(b, -a)=L(-b, a)$. Our orientation convention coincides with that of [13] (or [12]). For example, the boundary of a disk bundle over a 2-sphere with Euler number $b(\neq 0)$ is $-L(b, 1)$.

The following reciprocity law is obtainable by a long, elementary number-theoretic calculation, arising from the Atiyah-Singer index theorem (cf. [14], [29]), but we shall prove it here by a different simple method using the δ-invariant.

LEMMA 3.1 (Reciprocity Law). *For coprime odd $a, b>0$,*

$$\mu(L(a, b))+\mu(L(b, a))=(1-ab)/16 \qquad \text{in } Q/Z.$$

PROOF. Write $-L(b, a)=T_1\cup_h T_2$ by h in the above remark. Construct $W=-L(b, a)\times[-1, 1]\cup D^2\times D^2$, identifying $T_2\times 1$ with $\partial D^2\times D^2$ so that $m_2\times 1=p\times\partial D^2$, $\ell_2\times 1=\partial D^2\times q$, $p, q\in\partial D^2$. Note that $\partial W=-L(b, a)\times(-1)+(-L(a, b))=L(b, a)+L(a, b)$. Let k be a core of T_2. $\bar{D}=-k\times[-1, 1]\cup D^2\times 0\subset W$ is admissible for $k\subset L(b, a)$. By Lemma 2.6, $[\bar{D}_Q^+]^2=-a/b$ and hence sign $W=-1$. It follows that

$$\delta(k\subset L(b, a))=(-a/b+1)/16-\mu(\partial W).$$

By Theorem 2.7, $\delta(k\subset L(b, a))=\delta(K\subset L(b, a))+(ab-a/b)/16$, since $s(k\subset L(b, a))=-a/b$, where K is the characteristic knot of $k\subset L(b, a)$. K bounds a disk in $L(b, a)$, so $\delta(K\subset L(b, a))=0$. We have

$$\mu(L(a, b))+\mu(L(b, a))=\mu(\partial W)=(-a/b+1)/16-(ab-a/b)/16$$
$$=(1-ab)/16.$$

This completes the proof.

Using the fact that $L(b, a)\cong L(b, a')$ if and only if $a^{\pm 1}\cdot a'\equiv 1 \pmod b$, one can compute the μ-invariant of any lens space $L(b, a)$ with b odd from this reciprocity law.

COROLLARY 3.2 [13]. *For coprime integers n, a with $a \neq 0$ and odd $n \geq 1$,*

$$2\mu(L(n, a)) = -(n+1)/8 + \frac{a}{n}(1/4),$$

where $\frac{a}{n}$ is the Legendre-Jacobi symbol (cf. [14]).

PROOF. We use the induction on n. It is true for $n=1$. Let $n>1$. Write $a = n_1 n + a_1$, $0 < |a_1| < n$, a_1, odd. For $a_1 > 0$, by Lemma 3.1 and the induction hypothesis,

$$\begin{aligned} 2\mu(L(n, a_1)) &= (1 - na_1)/8 - 2\mu(L(a_1, n)) \\ &= (1 - na_1)/8 - \left[-(a_1+1)/8 + \frac{n}{a_1}(1/4)\right] \\ &= -(n+1)/8 + \frac{a_1}{n}(1/4) + 1/2 - (n-1)(a_1-1)/8 - \left[\frac{n}{a_1} + \frac{a_1}{n}\right](1/4). \end{aligned}$$

By the formula $\frac{n}{a_1} = (-1)^{[(n-1)/2]\cdot[(a_1-1)/2]}\frac{a_1}{n}$, we have that

$$1/2 - (n-1)(a_1-1)/8 - \left[\frac{n}{a_1} + \frac{a_1}{n}\right](1/4) = 0$$

in Q/Z. Therefore,

$$2\mu(L(n, a)) = 2\mu(L(n, a_1)) = -(n+1)/8 + \frac{a_1}{n}(1/4) = -(n+1)/8 + \frac{a}{n}(1/4).$$

For $a_1 < 0$, by the above calculation, we obtain

$$2\mu(L(n, -a_1)) = -(n+1)/8 + \frac{-a_1}{n}(1/4),$$

so that

$$\begin{aligned} 2\mu(L(n, a_1)) &= (n+1)/8 - \frac{-a_1}{n}(1/4) \\ &= -(n+1)/8 + \frac{a_1}{n}(1/4) + (n+1)/4 - \left[\frac{a_1}{n} + \frac{-a_1}{n}\right](1/4). \end{aligned}$$

Since $\frac{-a_1}{n} = \frac{-1}{n}\frac{a_1}{n} = (-1)^{(n-1)/2}\frac{a_1}{n}$, we see that

$$(n+1)/4 - \left[\frac{a_1}{n} + \frac{-a_1}{n}\right](1/4) = 0$$

in Q/Z. Therefore,

$$2\mu(L(n,a))=2\mu(L(n,a_1))=-(n+1)/8+\frac{a_1}{n}(1/4)=-(n+1)/8+\frac{a}{n}(1/4).$$

This completes the proof.

THEOREM 3.3. *Suppose a flat knot k' in a Z_2-homology 3-sphere S' is obtained from a knot k in a Z_2-homology 3-sphere S so that*

$$S'-\mathrm{Int}\,T(k')=S-\mathrm{Int}\,T(k)$$

and

$$\begin{aligned}&[m']=c[m]+d[\ell],\\ &[\ell']=[K(k)]=a[m]+b[\ell],\\ &ad-bc=-1,\end{aligned}$$

in $H_1(\partial T(k);Z)$ for m. ℓ. pairs (m,ℓ), (m',ℓ') of $T(k)$, $T(k')$, respectively. Then b is odd and

$$\mu(S')=\mu(S)+\mu(L(b,a))+d\delta(k'\subset S').$$

PROOF. Note that $[m']\neq 0$, $[\ell']=0$ in $H_1(S-\mathrm{Int}\,T(k);Z_2)=Z_2$. By Mayer-Vietoris sequence, b is odd if and only if $H_1(S;Z_2)=0$. So, b is odd. Let $-L(b,a)=T_0\cup_h T_1$ by an orientation-reversing homeomorphism $h\colon\partial T_0\to\partial T_1$ such that

$$\begin{aligned}&h_*[m_0]=a[m_1]+b[\ell_1],\\ &h_*[\ell_0]=c[m_1]+d[\ell_1].\end{aligned}$$

Let k_i be a core of T_i. Let T^* be a small regular neighborhood of m' in $S-\mathrm{Int}\,T(k)$. Regard T^* as a framed solid torus with framing determined by the annulus $T^*\cap\partial T(k)$. Next, push T^* into $S-T(k)$. Then consider the manifold $\overline{W}=S\times[-1,1]\cup -W\cup D^2\times D^2$, identifying $T(k)\times 1$ with T_1 so that $m\times 1=m_1$, $\ell\times 1=\ell_1$, and identifying the framed $T^*\times 1$ with $\partial D^2\times D^2$ having the product framing, where W is a simply connected, compact spin 4-manifold with $\partial W=-L(b,a)$ (cf. [16], [24]). The identifications $m\times 1=m_1$, $\ell\times 1=\ell_1$ imply the identifications $m'\times 1=h\ell_0$, $K(k)\times 1=\ell'\times 1=hm_0$ (up to isotopy of $\partial T(k)$). Then note that $\partial W=S\times(-1)+S'=-S+S'$ and $H_2(\overline{W};Q)=H_2(-W;Q)\oplus Q\oplus Q$. One factor Q is represented by a closed surface $\overline{F}$, extending a characteristic surface F for $k\times 1\subset S\times 1$ and contained in the 3-manifold $M=\partial(S\times[-1,1]\cup -W)-S\times(-1)$. M is a Z_2-homology handle obtained from $S-\mathrm{Int}\,T(k)$. The other factor Q is represented by a rational cycle suspending a core of $T^*\times 1$. An intersection matrix on $H_2(\overline{W};Q)$ is non-singular and is a block sum of an intersection matrix on $H_2(-W;Q)$ and $\begin{pmatrix}0&1\\1&x\end{pmatrix}$, since $\overline{F}\subset M$ implies $[\overline{F}]^2=0$. So, sign $\overline{W}=-$sign W. Now suppose d is even. We show that $\overline{W}$ is spin. Note that $H_2(\overline{W}; Z_2)=H_2(-W; Z_2)\oplus Z_2\oplus Z_2$ and the first factor of $Z_2\oplus Z_2$ is represented by $\overline{F}$ and the second, a Z_2-cycle c_2 suspending a core of $T^*\times 1$. Since $\overline{F}\subset M$, $[\overline{F}]^2=0$ in $H_2(\overline{W}; Z_2)$. Using the fact that d is even, we see that the Z_2-linking number in S of a core of T^* and a curve on ∂T^* giving the framing of T^* is 0 (mod 2) (Recall the construction of T^*). So, $[c_2]^2=0$ in $H_2(\overline{W}; Z_2)$. Since $-W$ is spin, it follows that $\overline{W}$ is spin. Therefore,

$$-\mu(S)+\mu(S')=\mu(\partial W)=-\text{sign}\,\overline{W}/16=\text{sign}\,W/16=\mu(L(b, a)),$$

and

$$\mu(S')=\mu(S)+\mu(L(b, a)).$$

Next, let d be odd. Let $k''\subset S''$ be a flat knot obtained from the flat knot $k'\subset S'$ such that $S''-\text{Int}\,T(k'')=S'-\text{Int}\,T(k')$, $K(k'')=K(k')=\ell''=\ell'$, and $[m'']=[m']+[\ell']$ in $H_1(\partial T(k'); Z)$, where (m'', ℓ'') is an $m.\ \ell.$ pair of $T(k'')$. We proceed to the proof by assuming the following lemma:

LEMMA 3.4. $\mu(S'')=\mu(S')+\delta(k'\subset S')$.

Since $[m'']=(c+a)[m]+(d+b)[\ell]$, $[\ell'']=a[m]+b[\ell]$ and $d+b$ is even, we see from the above argument that $\mu(S'')=\mu(S)+\mu(L(b, a))$. By Lemma 3.4,

$$\mu(S')=\mu(S)+\mu(L(b, a))-\delta(k'\subset S').$$

Using that $2\delta(k'\subset S')=0$ by Theorem 2.5, we have

$$\mu(S')=\mu(S)+\mu(L(b, a))+d\mu(k'\subset S')$$

for any given d. This completes the proof except for the proof of Lemma 3.4.

PROOF OF LEMMA 3.4. Let T'' be a small regular neighborhood of m'' in $S'-\text{Int}\,T(k')$ with framing determined by the annulus $T''\cap\partial T(k')$. Push T'' into $S'-T(k')$. Construct $W=S'\times[-1, 1]\cup D^2\times D_1^2\cup D^2\times D_2^2$, identifying $T(k')\times 1$ with $\partial D^2\times D_1^2$ so that $m'\times 1=p\times\partial D_1^2$, $\ell'\times 1=\partial D^2\times q_1$ ($p, q\in\partial D^2$) and identifying the framed $T''\times 1$ with $\partial D^2\times D_2^2$ having the product framing. Let $\overline{D}=k'\times[-1, 1]\cup D^2\times 0_1$ be a disk. Note that $\partial W=S'\times(-1)+S''=-S'+S''$. By Lemma 2.6, $[\overline{D}_Q^+]^2=0$. Hence sign $W=0$, since $H_2(W; Q)=Q\oplus Q$ and $[\overline{D}_Q^+]$ generates one factor Q and the intersection pairing on $H_2(W; Q)$ is non-singular. From construction, $\overline{D}\subset W$ is admissible for $(k'\times(-1)\subset S'\times(-1))=(-k'\subset -S')$. So,

$$-\delta(k'\subset S')=-\mu(\partial W)=\mu(S')-\mu(S'')$$

and

$$\mu(S'')=\mu(S')+\delta(k'\subset S').$$

This proves the lemma.

The following is a generalization of a result of Gordon [12].

COROLLARY 3.5. *Let k_i be a flat knot in a Z_2-homology 3-sphere S_i with an m. ℓ. pair (m_i, ℓ_i) on $T(k_i)$ such that $\ell_i = K(k_i)$, $i=1, 2$. Let $S = S_1 - \mathrm{Int}\, T(k_1) \cup_h S_2 - \mathrm{Int}\, T(k_2)$ be the adjunction space obtained by an orientation-reversing homeomorphism $h: \partial T(k_1) \to \partial T(k_2)$ such that*

$$h_*[\ell_1] = a[\ell_2] + b[m_2],$$
$$h_*[m_1] = c[\ell_2] + d[m_2],$$
$$ad - bc = -1.$$

Suppose S is a Z_2-homology 3-sphere. Then b is odd and

$$\mu(S) = \mu(S_1) + \mu(S_2) - \mu(L(b, a)) + d\delta(k_1 \subset S_1) + a\delta(k_2 \subset S_2).$$

PROOF. Note that $[m_i] \neq 0$, $[\ell_i] = 0$ in $H_1(S_i - \mathrm{Int}\, T(k_i); Z_2) = Z_2$. By Mayer-Vietoris sequence, we obtain that b is odd if and only in $H_1(S; Z_2) = 0$. So, b is odd. Let $S_1^+ = S_1 - \mathrm{Int}\, T(k_1) \cup_{h'} S^1 \times D^2$ by a homeomorphism $h': \partial T(k_1) \to \partial(S^1 \times D^2)$ such that

$$h'_*[\ell_1] = a[p \times \partial D^2] + b[S^1 \times q],$$
$$h'_*[m_1] = c[p \times \partial D^2] + d[S^1 \times q],$$

$p \in S^1$, $q \in \partial D^2$. By Theorem 3.3,

$$\mu(S_1) = \mu(S_1^+) + \mu(L(b, a)) + d\delta(k_1 \subset S_1).$$

Similarly, let $S_2^+ = S^1 \times D^2 \cup_{h''} S_2 - \mathrm{Int}\, T(k_2) = S_2 - \mathrm{Int}\, T(k_2) \cup_{h''^{-1}} S^1 \times D^2$ by a homeomorphism $h'': \partial(S^1 \times D^2) \to \partial T(k_2)$ or $h''^{-1}: \partial T(k_2) \to \partial(S^1 \times D^2)$ such that

$$h''_*[p \times \partial D^2] = a[\ell_2] + b[m_2],$$
$$h''_*[S^1 \times q] = c[\ell_2] + d[m_2],$$

or

$$(h''^{-1})_*[\ell_2] = -d[p \times \partial D^2] + b[S^1 \times q],$$
$$(h''^{-1})_*[m_2] = c[p \times \partial D^2] - a[S^1 \times q].$$

By Theorem 3.3,

$$\mu(S_2) = \mu(S_2^+) + \mu(L(b, -d)) + a\delta(k_2 \subset S_2).$$

Construct $W = S \times [-1, 1] \cup S_1^+ \times [-1, 1] \cup S_2^+ \times [-1, 1]$, identifying, for each i, $i=1, 2$, two copies of $S_i - \mathrm{Int}\, T(k_i)$ in $S \times 1$ and $S_i^+ \times (-1)$. Note that W is spin with signature 0 (cf. Siebenmann [34]). Since $\partial W = S \times (-1) + S_1^+ + S_2^+$

$+L(b, a)$ (Note the orientation convention of $L(b, a)$), it follows that

$$\mu(S)=\mu(S_1^+)+\mu(S_2^+)+\mu(L(b, a)).$$

Noting that $L(b, -d)=L(b, a)$ and $2\delta(k_i\subset S_i)=0$, we have that

$$\mu(S)=\mu(S_1)+\mu(S_2)-\mu(L(b, a))+d\delta(k_1\subset S_1)+a\delta(k_2\subset S_2).$$

This completes the proof.

The following is a generalization of Theorem 2.5.

COROLLARY 3.6. *For any knot k in a Z_2-homology 3-sphere S with Z_2-homology handle M, we have $\delta(K(k)\subset S)=\varepsilon(M)/2$.*

PROOF. Let $k'\subset S'$ be a flat knot stated in Theorem 3.3 with an extra condition that d is odd. Let $\overline{W}$ be the 4-manifold constructed in the proof of Theorem 3.3. $K(k\times(-1))=K(k)\times(-1)$ bounds a locally flat disk D in $\overline{W}$, coming from a meridian disk of T_0. Because d is odd, we can see that $D\subset \overline{W}$ is admissible for $K(k\times(-1))\subset S\times(-1)$. Hence

$$\begin{aligned}-\delta(K(k)\subset S)&=\delta(K(k)\times(-1)\subset S\times(-1))\\&=([D_Q^+]^2-\operatorname{sign}\overline{W})/16-\mu(\partial\overline{W})\\&=\mu(L(b, a))+\mu(S)-\mu(S'),\ \text{for } [D_Q^+]^2=0\\&=-\delta(k'\subset S') \text{ by Theorem 3.3.}\end{aligned}$$

Thus, $\delta(K(k)\subset S)=\delta(k'\subset S')=\varepsilon(M)/2$ by Theorem 2.5, since $k'\subset S'$ has M as its Z_2-homology handle by Corollary 1.7. This completes the proof.

This corollary and Theorem 2.7 show that the δ-invariant of any knot in a Z_2-homology 3-sphere is determined completely by the Z_2-homology handle and the slope of the knot.

§ 4. The Atiyah-Singer invariant of a cyclic action on a 3-manifold

Let W be a compact oriented piecewise-linear w-manifold with piecewise-linear Z_n-action, $n>1$. For $t\in Z_n$, we denote $F(t, W)=\{x\in W\,|\,tx=x\}$ and $F(Z_n, W)=\cup_{t(\neq 1)\in Z_n}F(t, W)$. The Z_n-action on W is semi-free if $F(Z_n, W)=F(t, W)$ for any $t(\neq 1)\in Z_n$, and free if $F(Z_n, W)=\phi$. When n is prime, the Z_n-action is necessarily semi-free. By Smith theory ([2], [3]), $F=F(Z_n, W)$ is the disjoint union of compact, proper, simplicial Z_n-homology submanifolds of even codimensions, since each $t\in Z_n$ is orientation-preserving by assumption. In particular, when $w\leq 5$, F consists of compact proper submanifolds of even codimensions. Thus, if $F\neq\phi$, then $w\geq 2$, and according as $w=2$, 3, 4, $\cdots$, each component of F is a point in Int W, a compact proper 1-manifold, (a point in Int W or a compact proper surface), $\cdots$. If the Z_n-action on W is semi-free and F is a locally flat $(w-2)$-submanifold of W, then the orbit space $\underset{\sim}{W}=W/Z_n$ is also a compact oriented piecewise-linear w-manifold with orientation induced from W and $\underset{\sim}{F}=F/Z_n$ is a locally flat $(w-2)$-submanifold of $\underset{\sim}{W}$ and the natural projection $pr\colon W\to\underset{\sim}{W}$ is a branched Z_n-covering, branched over $\underset{\sim}{F}$, which is induced by an epimorphism $\theta\colon H_1(\underset{\sim}{W}-\underset{\sim}{F};Z)\to Z_n$ sending each meridian of $\underset{\sim}{F}$ in $\underset{\sim}{W}$ to a unit of Z_n. The following is a well-known fact:

4.1. *The collection of locally flat compact proper $(w-2)$-submanifolds $\underset{\sim}{F}$ in a compact oriented w-manifold $\underset{\sim}{W}$ with an epimorphism $\theta\colon H_1(\underset{\sim}{W}-\underset{\sim}{F};Z)\to Z_n$, sending each meridian of $\underset{\sim}{F}$ in $\underset{\sim}{W}$ to a unit of Z_n, corresponds, by the branched covering, to the collection of compact oriented w-manifolds W with semi-free Z_n-action such that $W/Z_n=\underset{\sim}{W}$ and $F(Z_n, W)=F$ is a locally flat compact proper $(w-2)$-submanifold.*

[Note that for each $x\in\underset{\sim}{F}$ $(D_x^{w-2}\subset D_x^w)=(st(x,\underset{\sim}{F})\subset st(x,\underset{\sim}{W}))$ is an unknotted disk pair and the epimorphism $\theta\colon H_1(\underset{\sim}{W}-\underset{\sim}{F};Z)\to Z_n$ induces an epimorphism $\theta'\colon H_1(D_x^w-D_x^{w-2};Z)=Z\to Z_n$, giving a unique branched Z_n-covering space of D_x^w, branched over D_x^{w-2}, that is a w-ball.]

First we consider a closed connected oriented 3-manifold M with free Z_n-action. It is known by Casson-Gordon [5] that M is the equivariant boundary of a compact connected oriented 4-manifold W with semi-free Z_n-action such that $F=F(Z_n, W)$ is ϕ or a locally flat closed orientable surface.

Definition 4.2. $\alpha(Z_n, M)=-\operatorname{sign} W+n\operatorname{sign}\underset{\sim}{W}-[F]^2(n^2-1)/3$, where $[F]^2=0$ when $F=\phi$.

The well-definedness of the above definition follows easily from the Novikov addition theorem of signatures [1] and the following well-known fact (derived from the Atiyah-Singer G-signature theorem [1]) (See [14], p. 181):

4.3. *For a closed connected oriented 4-manifold W with semi-free Z_n-action such that $F=F(Z_n, W)$ is ϕ or a locally flat closed orientable surface F, we have*

$$-\operatorname{sign} W+n\operatorname{sign}\underset{\sim}{W}-[F]^2(n^2-1)/3=0.$$

Clearly, $\alpha(Z_n, M)$ is an invariant of the equivariant, orientation-preserving homeomorphism type of (Z_n, M). Two kind of finer invariants but depending on each $t\in Z_n$ are widely known. One is the Atiyah-Singer α-invariant, $\alpha(t, M)$ (cf. [14], p. 72) and the other, the Casson-Gordon invariant, $\sigma_r(\underset{\sim}{M},\theta)$, $0<r<n$, associated with the homomorphism $\theta\colon H_1(\underset{\sim}{M};Z)\to Z_n$

corresponding to the covering $pr: M \to \underset{\sim}{M}$ (cf. [5], p. 42). From the definitions, we see the following:

4.4. $\alpha(Z_n, M) = \Sigma_{t(\neq 1) \in Z_n} \alpha(t, M) = \Sigma_{r=1}^{n-1} \sigma_r(\underset{\sim}{M}, \theta)$.

When $n=2$, $\alpha(Z_2, M) = \alpha(t, M) = \sigma_1(\underset{\sim}{M}, \theta)$, $t \neq 1$, is an integer and called the *Browder-Livesay invariant* (cf. [14]).

Next we consider a closed connected oriented 3-manifold M with semi-free Z_n-action such that $F(Z_n, M) = L \neq \phi$. Note that L is a link in M. We shall define an analogous invariant $\alpha(Z_n, M)$ of the equivariant, orientation-preserving homeomorphism type of (Z_n, M) only when each component of L is a knot of finite order in M. A difficult point is that $\alpha(Z_n, M)$ does not depend on any particular choice of the orientations of the components of L.

LEMMA 4.5. *Let M be a closed oriented 3-manifold with a semi-free or free Z_n-action. Let k be a Z_n-invariant knot in M such that k is a component of $L = F(Z_n, M)$ or $k \cap L = \phi$. Then k is of finite order in M if and only if $\underset{\sim}{k} = k/Z_n$ is so in $\underset{\sim}{M} = M/Z_n$. Further, in this case, we have*

$$ns(k \subset M) = s(\underset{\sim}{k} \subset \underset{\sim}{M}) \ (\text{if } k \subset L), \text{ or}$$
$$s(k \subset M) = ns(\underset{\sim}{k} \subset \underset{\sim}{M}) \ (\text{if } k \cap L = \phi).$$

Thus, for example, if $H_1(\underset{\sim}{M}; Q) = 0$ and $L \neq \phi$, then each component of L is a knot of finite order in M.

PROOF OF LEMMA 4.5. The projection $pr: M \to \underset{\sim}{M}$ induces a homomorphism $pr_*: H_1(M; Z) \to H_1(\underset{\sim}{M}; Z)$ sending $[k]$ to $[\underset{\sim}{k}]$ (if $k \subset L$) or $n[\underset{\sim}{k}]$ (if $k \cap L = \phi$). Hence if k is of finite order in M, then so is $\underset{\sim}{k}$ in $\underset{\sim}{M}$. Conversely, assume $\underset{\sim}{k}$ is of finite order in $\underset{\sim}{M}$. By Lemma 1.1, $\underset{\sim}{k}$ has a characteristic surface $\underset{\sim}{F}$ in $\underset{\sim}{M} - \text{Int } T(\underset{\sim}{k})$ with characteristic parallel link $P(\underset{\sim}{k}) \subset \partial T(\underset{\sim}{k})$. We may assume that $T(\underset{\sim}{k}) \cap (\underset{\sim}{L} - \underset{\sim}{k}) = \phi$ and $\underset{\sim}{L} - \underset{\sim}{k}$ intersects $\underset{\sim}{F}$ transversally (when $(\underset{\sim}{L} - \underset{\sim}{k}) \cap \underset{\sim}{F} \neq \phi$). Then we see that the lift F of $\underset{\sim}{F}$ is a compact orientable surface in $M - \text{Int } T(k)$, $T(k) = pr^{-1} T(\underset{\sim}{k})$. In fact, F is an unbranched Z_n-covering space of $\underset{\sim}{F}$ or a branched Z_n-covering space of $\underset{\sim}{F}$ branched over $(\underset{\sim}{L} - \underset{\sim}{k}) \cap \underset{\sim}{F}$, according as whether $(\underset{\sim}{L} - \underset{\sim}{k}) \cap \underset{\sim}{F}$ is ϕ or not. Let $r = \sharp P(\underset{\sim}{k})$ and write $[P(\underset{\sim}{k})] = ra[\underset{\sim}{m}] + rb[\underset{\sim}{\ell}]$ in $H_1(\partial T(\underset{\sim}{k}); Z)$ for an $m.\ \ell.$ pair $(\underset{\sim}{m}, \underset{\sim}{\ell})$ of $T(\underset{\sim}{k})$ such that the lift m of $\underset{\sim}{m}$ is connected and the lift of $\underset{\sim}{\ell}$ has n components (when $k \subset L$), or the lift of $\underset{\sim}{m}$ has n components and the lift, ℓ of $\underset{\sim}{\ell}$ is connected (when $k \cap L = \phi$). (We find such a pair easily when $k \cap L = \phi$ or by noting 4.1 when $k \subset L$.) First, consider the case that $k \subset L$. For a component ℓ of $pr^{-1}(\underset{\sim}{\ell})$, (m, ℓ) is an $m.\ \ell.$ pair of $T(k)$ and we have $[\partial F] = ra[m] + rbn[\ell]$ in $H_1(\partial T(k); Z)$. So, $rbn[k] = rbn[\ell] = 0$ in $H_1(M; Z)$, showing that k is of finite order in M. Note that k bounds a rational 2-chain $(1/rbn)\hat{F}$ with $\hat{F}$, an integral 2-chain extending F. By Lemma 1.8, $s(k \subset M) = -\phi([k], [k]) = -\hat{F} \cdot \ell / rbn \pmod 1 = ra/rbn \pmod 1 = a/bn \pmod 1$, so that $ns(k\subset M)=a/b \pmod 1 = s(\underset{\sim}{k}\subset \underset{\sim}{M})$. Next, consider the case that $k\cap L=\phi$. For a component m of $pr^{-1}(\underset{\sim}{m})$, (m, ℓ) is an $m.\ \ell.$ pair of $T(k)$ and we have $[\partial F]=ran[m]+rb[\ell]$ in $H_1(\partial T(k); Z)$. So, $rb[k]=rb[\ell]=0$ in $H_1(M; Z)$. By Lemma 1.8, $s(k\subset M)=-\phi([k],[k])=an/b \pmod 1 = ns(\underset{\sim}{k}\subset \underset{\sim}{M})$. This completes the proof.

We assume $L=F(Z_n, M)$ is a link in M with components $k_1, \cdots, k_r$ being knots of finite order. Let W be a compact connected oriented 4-manifold with semi-free Z_n-action such that $\partial(Z_n, W)=(Z_n, M)$ and $F=F(Z_n, W)$ is a locally flat, compact proper orientable surface. Such a 4-manifold always exists. For example, construct $W_1=-M\times[-1,1]\cup D^2\times D_1^2\cup\cdots\cup D^2\times D_r^2$ identifying a Z_n-invariant $T(k_i)\times 1$ with $\partial D^2\times D_i^2$ so as to admit a Z_n-action such that $F(Z_n, W_1)=D_1^*\cup\cdots D_r^*$ with $D_i^*=k_i\times[-1,1]\cup D^2\times 0_i$, a disk. $M_1=\partial W_1-(-M)\times(-1)$ is a closed connected oriented 3-manifold with free Z_n-action. By [5], $-M_1$ bounds a compact connected oriented 4-manifold W_2 with a semi-free Z_n-action such that $F(Z_n, W_2)=F_2^*$ is ϕ or a locally flat, closed orientable surface. The manifold $W^*=W_1\cup W_2$ is the desired one. Orient F and then $k_1, \cdots, k_r$ so that $\partial F=L=k_1\cup\cdots\cup k_r$. Let F_Q^+ be a rational 2-cycle in W obtained from F by attaching rational 2-chains $c_1, \cdots, c_r$ in M with $\partial c_i=-k_i$, $i=1, 2, \cdots, r$. Define

$$\lambda(L)=\lambda(L\subset M)=\sum_{i>j}\mathrm{Link}_M(k_i, k_j)\in Q,$$

where $\mathrm{Link}_M(k_i, k_j)$ is the Q-linking number of k_i and k_j in M. This is called the *total Q-linking number* of the oriented link $L\subset M$. $\lambda(L)=0$ if L is connected.

DEFINITION 4.6. $\alpha(Z_n, M)=-\mathrm{sign}\, W+n\,\mathrm{sign}\, \underset{\sim}{W}-([F_Q^+]^2+2\lambda(L))(n^2-1)/3.$

To check the well-definedness, let W^* be the 4-manifold constructed above, so that $\partial(Z_n, W^*)=(Z_n, M)$ and $F^*=F=(Z_n, W^*)=D_1^*\cup\cdots\cup D_r^*\cup F_2^*$. Orient F^* so that $\partial F^*=-L=\bigcup_{i=1}^r -k_i$. Let $\overline{W}=W\cup -W^*$. $\overline{W}$ is a closed connected oriented 4-manifold with semi-free Z_n-action such that $\overline{F}=F(Z_n, \overline{W})=F\cup F^*$ is a locally flat, closed oriented surface. By 4.3,

$$-\mathrm{sign}\,\overline{W}+n\,\mathrm{sign}\,\underset{\sim}{\overline{W}}-[\overline{F}]^2(n^2-1)/3=0.$$

By the Novikov addition theorem, $\mathrm{sign}\,\overline{W}=\mathrm{sign}\, W-\mathrm{sign}\, W^*$ and $\mathrm{sign}\,\underset{\sim}{\overline{W}}=\mathrm{sign}\,\underset{\sim}{W}-\mathrm{sign}\,\underset{\sim}{W}^*$. We show that $[\overline{F}]^2=[F_Q^+]^2-[F_Q^{*+}]^2$, where we count $[F_Q^{*+}]^2$ in W^*. Note that $\overline{F}$ is Q-homologous to a sum

$$F_Q^+ + F_Q^{*+} + z_1 + \cdots + z_r,$$

where z_i are rational 2-cycles in $M\subset\overline{W}$. Using a collar of M in W, we see that the Q-intersection numbers, $[F_Q^+]\cdot[F_Q^{*+}]$, $[F_Q^+]\cdot[z_i]$, $[F_Q^{*+}]\cdot[z_j]$ and $[z_i]\cdot[z_j]$ are all 0 for all i, j. Thus, $[\bar{F}]^2=[F_Q^+]^2-[F_Q^{*+}]^2$ and

$$\begin{aligned}&-\operatorname{sign} W+n \operatorname{sign} \underset{\sim}{W}-([F_Q^+]^2+2\lambda(L))(n^2-1)/3\\ &\quad=-\operatorname{sign} W^*+n \operatorname{sign} \underset{\sim}{W}^*-([F_Q^{*+}]^2+2\lambda(L))(n^2-1)/3.\end{aligned}$$

Since $[F_Q^{*+}]^2+2\lambda(L)=\sum_{i=1}^r[D_{iQ}^{*+}]^2+[F_2^*]^2$ is not altered by any change of the orientations of $D_1^*, \cdots, D_r^*$ (that is, $k_1, \cdots, k_r$), we have proven the well-definedness of $\alpha(Z_n, M)$. [Note that for $i\neq j$, $[D_{iQ}^{*+}]\cdot[D_{jQ}^{*+}]$ (in W^*)$=D_i^*\cdot D_{jQ}^{*+}$ (in W^*)$=k_i\cdot c_j$ (in M)$=\operatorname{Link}_M(k_i, -k_j)=-\operatorname{Link}_M(k_i, k_j)$, where c_j is a rational 2-chain in M with $\partial c_j=-k_j$, and $D_{jQ}^{*+}=D_j^*+c_j$.]

Consider an oriented link $L=\bigcup_{i=1}^r k_i$ in a closed connected oriented 3-manifold M such that each k_i is a knot of finite order and L bounds a compact oriented surface F in M. Push $\operatorname{Int}(-F)\times(-1)$ into the interior of $-M\times[-1, 1]$. Let F' be the result. Let W_n be the natural branched Z_n-covering space of $-M\times[-1, 1]$, branched over F', so that $\partial W_n=M_n+n(-M)$, where M_n is the natural branched Z_n-covering space of M, branched over the link L, constructed by splitting M along F.

PROPOSITION 4.7. $\alpha(Z_n, M_n)=-\operatorname{sign} W_n-2\lambda(L\subset M)(n^2-1)/3n.$

REMARK 4.8. In the case of the double branched covering space $S(L)_2$ of a Z-homology 3-sphere S, branched over a link L, we have sign $W_2=\sigma(L)$ and $\alpha(Z_2, S(L)_2)=-\sigma(L)-\lambda(L\subset S)$, where $\sigma(L)$ is the Murasugi signature of L, that is, $\sigma(L)=\operatorname{sign}(A+A')$ for a link matrix A associated with a Seifert surface for L (cf. [17], [4]). It follows that $\sigma(L)+\lambda(L\subset S)$ is an invariant of the unoriented link type of $L\subset S$, since $\alpha(Z_2, S(L)_2)$ is such (cf. [27], [17]). See also [11] for an analogous invariant in case L is a knot.

PROOF OF PROPOSITION 4.7. Since the Z_n-action on $\partial W_n-M_n=n(-M)$ is the cyclic translation of order n on n copies of $-M$, we obtain that

$$\alpha(Z_n, M_n)=-\operatorname{sign} W_n-([\tilde{F}_Q'^+]^2+2\lambda(\tilde{L}\subset M_n))(n^2-1)/3,$$

where $\tilde{F}'$, $\tilde{L}$ are the lifts of F', L, respectively. Let ℓ_i be the longitude determined by $F\cap\partial T(k_i)$ of k_i. Write $[K(k_i)]=a_i[m_i]+b_i[\ell_i]$, where m_i is a meridian of $T(k_i)$. Let $\tilde{k}_i$, $\tilde{m}_i$ be the lifts of k_i, m_i, and $\tilde{\ell}_i$ be a component of the lift of ℓ_i. We see that $(\tilde{m}_i, \tilde{\ell}_i)$ is an $m.\ \ell.$ pair of $\tilde{k}_i$ and $[K(\tilde{k}_i)]=\tilde{a}_i[\tilde{m}_i]+\tilde{b}_i[\tilde{\ell}_i]$ with $\tilde{a}_i$, $\tilde{b}_i$ such that $\tilde{a}_i/\tilde{b}_i=a_i/nb_i$ by the proof of Lemma 4.5, since by the proof of Corollary 1.3 $[\tilde{P}(k_i)]=u[P(\tilde{k}_i)]$ for an integer $u>0$ in $H_1(\partial T(\tilde{k}_i); Z)$, where $\tilde{P}(k_i)$ is the lift of the characteristic parallel link $P(k_i)$ on $T(k_i)$. Clearly, $[\tilde{F}_Q'^+]^2=[\tilde{F}']\cdot[\tilde{F}_Q'^+]$ (where $[\tilde{F}']\in H_2(W_n, M_n; Z)$)$=-2\lambda(\tilde{L}\subset M_n)-\sum_{i=1}^r\operatorname{Link}_{M_n}(\tilde{\ell}_i, \tilde{k}_i)$. By the proof of Lemma 1.8, $\operatorname{Link}_{M_n}(\tilde{\ell}_i, \tilde{k}_i)=-\tilde{a}_i/\tilde{b}_i=-$

a_i/nb_i. But, $-a_i/b_i=\mathrm{Link}_M(\ell_i, k_i)=-\sum_{\substack{1\le j\le r\\ j\ne i}}\mathrm{Link}_M(k_j, k_i)$, because ℓ_i is homologous to $\bigcup_{j\ne i}-k_j$ in $M-k_i$. Therefore,

$$[\tilde{F}_Q'^+]^2+2\lambda(\tilde{L}\subset M_n)=-\sum_{i=1}^{r}\mathrm{Link}_{M_n}(\tilde{\ell}_i, \tilde{k}_i)=2\lambda(L\subset M)/n.$$

This completes the proof.

LEMMA 4.9. *Let M be a closed connected oriented 3-manifold with semi-free $Z_{nn'}$-action. If $L=F(Z_{nn'}, M)\neq\phi$, then assume that each component of L is a knot of finite order in M. Then*

$$\alpha(Z_{nn'}, M)=\alpha(Z_n, M)+n\alpha(Z_{n'}, M/Z_n),$$

where Z_n is a subgroup of $Z_{nn'}$ of order n and $Z_{n'}=Z_{nn'}/Z_n$.

PROOF. Let $k_1, \cdots, k_k$ be the components of L. Let W be a compact connected oriented 4-manifold with semi-free $Z_{nn'}$-action such that $\partial(Z_{nn'}, W)=(Z_{nn'}, M)$ and $F=F(Z_{nn'}, W)=D_1\cup\cdots\cup D_r\cup F_2$, where D_i is a locally flat proper disk with $\partial D_i=k_i$ and F_2 is a locally flat closed orientable surface or ϕ. Let $W'=W/Z_n$, $D_i'=D_i/Z_n$, $F_2'=F_2/Z_n$, $k'=k/Z_n$ and $\underset{\sim}{W}=W/Z_{nn'}$. By definition,

$$\alpha(Z_{nn'}, M)=-\mathrm{sign}\, W+nn'\,\mathrm{sign}\,\underset{\sim}{W}-\Big(\sum_{i=1}^{r}[D_{iQ}^+]^2+[F_2]^2\Big)(n^2n'^2-1)/3,$$

$$\alpha(Z_n, M)=-\mathrm{sign}\, W+n\,\mathrm{sign}\, W'-\Big(\sum_{i=1}^{r}[D_{iQ}^+]^2+[F_2]^2\Big)(n^2-1)/3,$$

$$\alpha(Z_{n'}, M/Z_n)=-\mathrm{sign}\, W'+n'\,\mathrm{sign}\,\underset{\sim}{W}-\Big(\sum_{i=1}^{r}[D_{iQ}'^+]^2+[F_2']^2\Big)(n'^2-1)/3.$$

Clearly, $[F_2']^2/n=[F_2]^2$. We show that $[D_{iQ}'^+]/n=[D_{iQ}^+]^2$. To see this, let $D_1'^*$ be a slight translation of D_1' so that $D_i'^*\cap D_i'=\phi$. $\partial D_i'^*$ gives a longitude ℓ_i' of k_i'. Write $[K(k_i')]=a_i'[m_i']+b_i'[\ell_i']$, where m_i' is a meridian of k_i'. Let (m_i, ℓ_i) be an $m\cdot\ell$-pair of k_i obtained from the lift of (m_i', ℓ_i'). [Note that the lift of $D_i'^*$ has n components.] We have $[K(k_i)]=a_i[m_i]+b_i[\ell_i]$ with a_i, b_i such that $a_i/b_i=a_i'/b_i'n$ (cf. the proof of Proposition 4.7). By Lemma 2.6, $[D_{iQ}^+]^2=-a_i/b_i=-a_i'/b_i'n=[D_{iQ}'^+]^2/n$. Therefore,

$$\begin{aligned}&\alpha(Z_{nn'}, M)-\alpha(Z_n, M)-n\alpha(Z_{n'}, M/Z_n)\\ &\quad=\{-(n^2n'^2-1)/3+(n^2-1)/3+n^2(n'^2-1)/3\}\Big(\sum_{i=1}^{r}[D_{iQ}^+]^2+[F_2]^2\Big)=0.\end{aligned}$$

The proof of the case $L=\phi$ is easier. This completes the proof.

Consider two closed connected oriented 3-manifolds M_i, $i=1, 2$, with semi-free Z_n-actions such that $L_i=F(Z_n, M_i)\neq\phi$ and each component of L_i is of finite order. Let k_i be any component of L_i. Construct a knot sum $k_1 \# k_2 \subset M_1 \# M_2$ which admits a semi-free Z_n-action induced by the Z_n-actions on M_1 and M_2, so that $F(Z_n, M_1 \# M_2) = (L_1 - k_1) \cup k_1 \# k_2 \cup (L_2 - k_2)$. By definition, we see the following:

LEMMA 4.10. $\alpha(Z_n, M_1 \# M_2) = \alpha(Z_n, M_1) + \alpha(Z_n, M_2)$.

Actual calculations of the α-invariants will be made at another opportunity.

§ 5. A cyclic action of odd order on a spin 4-manifold

Let W be a compact oriented 4-manifold with semi-free Z_n-action such that n is odd and $F = F(Z_n, W)$ is ϕ or a locally flat surface. Let $\underset{\sim}{W} = W/Z_n$ and $\underset{\sim}{F} = F/Z_n$ ($\cong F$). In case $F \neq \phi$, $\underset{\sim}{F}$ and F are orientable by 4.1, because a meridian of a non-orientable component of $\underset{\sim}{F}$ represents an element x with $2x = 0$ in $H_1(\underset{\sim}{W} - \underset{\sim}{F}; Z)$.

LEMMA 5.1. *W is spin if and only if $\underset{\sim}{W}$ is so.*

PROOF. A compact 4-manifold X is spin if and only if its double $D(X) = \partial(X \times [-1, 1])$ is so. Hence we may assume that $\partial W = \partial F = \phi$. First, suppose W is spin. By the Wu formula, $w_2(W)$ is equal to the second Wu class $v_2(W)$, so that $v_2(W) = 0$. Let pr: $W \to \underset{\sim}{W}$ be the natural projection. A transfer argument shows that $pr^*: H^*(\underset{\sim}{W}; Z_2) \to H^*(W; Z_2)$ is injective, for n is odd. For any $x \in H^2(\underset{\sim}{W}; Z_2)$, $pr^*(v_2(\underset{\sim}{W}) \cup x) = pr^*(x \cup x) = pr^*(x) \cup pr^*(x) = v_2(W) \cup pr^*(x) = 0$, so that $v_2(\underset{\sim}{W}) \cup x = 0$. By Poincaré duality, $v_2(\underset{\sim}{W}) = 0$. Thus, $\underset{\sim}{W}$ is spin. Next, suppose $\underset{\sim}{W}$ is spin. If $F = \phi$, then W is spin, because the unbranched covering projection preserves the characteristic classes. Let $F \neq \phi$. Let $\underset{\sim}{N}$ be a regular neighborhood of $\underset{\sim}{F}$, and $\underset{\sim}{E} = \underset{\sim}{W} - \mathrm{Int}\, \underset{\sim}{N}$. Since $\underset{\sim}{W}$ is spin, so are $\underset{\sim}{N}$ and $\underset{\sim}{E}$. Let $pr^{-1}(\underset{\sim}{E}) = E$. Since $pr|E$ is an unbranched covering, $v_2(W)|E = 0$. Let $\underset{\sim}{N}_j$ be a component of $\underset{\sim}{N}$ and $pr^{-1}(\underset{\sim}{N}_j) = N_j$. Note that N_j is a disk bundle over a component F_j of F with Euler number, say e_j. $\underset{\sim}{N}_j$ is a disk bundle over $\underset{\sim}{F}_j$ with Euler number ne_j. Note that $\underset{\sim}{N}_j$ is spin if and only if ne_j is even. Using that n is odd, we see that N is spin, i.e., $v_2(W)|N = 0$, and $pr^*: H^1(\partial \underset{\sim}{N}; Z_2) \to H^1(\partial N; Z_2)$ is an isomorphism. By the Mayer-Vietoris sequence of $(W; E, N)$, we find $u \in H^1(\partial N; Z_2)$ such that $\delta(u) = v_2(W)$, where $\delta: H^1(\partial N; Z_2) \to H^2(W; Z_2)$ is the coboundary. We have $\underset{\sim}{u} \in H^1(\partial \underset{\sim}{N}; Z_2)$ such that $pr^*(\underset{\sim}{u}) = u$. Consider the following commutative square:

$$\begin{array}{ccc} H^1(\partial N; Z_2) & \xrightarrow{\delta} & H^2(W; Z_2) \\ \uparrow{\scriptstyle pr^*} & & \uparrow{\scriptstyle pr^*} \\ H^1(\partial \underset{\sim}{N}; Z_2) & \xrightarrow{\delta} & H^2(\underset{\sim}{W}; Z_2). \end{array}$$

Then we see that $pr^*\delta(\underset{\sim}{u})=v_2(W)$. Then $pr^*(\delta(\underset{\sim}{u})\cup x)=pr^*(\delta(\underset{\sim}{u}))\cup pr^*(x)=v_2(W)\cup pr^*(x)=pr^*(x)\cup pr^*(x)=pr^*(x\cup x)$. Since $pr^*\colon H^*(\underset{\sim}{W}; Z_2)\to H^*(W; Z_2)$ is injective, $\delta(\underset{\sim}{u})\cup x=x\cup x$ for all $x\in H^2(\underset{\sim}{W}; Z_2)$. By definition, $\delta(\underset{\sim}{u})=v_2(\underset{\sim}{W})$. Since $\underset{\sim}{W}$ is spin, $\delta(\underset{\sim}{u})=v_2(\underset{\sim}{W})=0$. So, $v_2(W)=pr^*(\delta(\underset{\sim}{u}))=0$. That is, W is spin. This completes the proof.

REMARK 5.2. When n is even, Lemma 5.1 is not true in general. Such examples are obtained easily in the case of a non-free Z_n-action. To give an example of the case of a free Z_n-action, let T be a solid torus with $m\cdot\ell\cdot$pairs (m, ℓ), (m, ℓ') such that $[\ell']=[\ell]+[m]$. Let $h\colon T\to T$ be a homeomorphism sending (m, ℓ) to (m, ℓ'). Let $\underset{\sim}{W}$ and W be the mapping tori of h and h^n, respectively. $\underset{\sim}{W}$ and W are compact orientable 4-manifolds with natural Z_n-covering $W\to\underset{\sim}{W}$. $\underset{\sim}{W}$ is not spin, but for even n, W is spin.

§ 6. The case of a free involution

Let S be a Z_2-homology 3-sphere with a free involution, i.e., a free Z_2-action. Let $\underset{\sim}{S}=S/Z_2$. Let k be a knot in $\underset{\sim}{S}$ with $[k]\neq 0$ in $H_1(\underset{\sim}{S}; Z_2)=Z_2$. The slope $s(k)$ has a normal presentation of the form $a/2b$ (a, b are odd). [In fact, the linking pairing ϕ, restricted to the 2-primary component $H_1(\underset{\sim}{S})_2$ of $H_1(\underset{\sim}{S}; Z)$ is non-singular. For an odd b', $0\neq b'[k]\in H_1(\underset{\sim}{S})_2=Z_2$. So, $b'^2\phi([k], [k])=\phi(b'[k], b'[k])=1/2$. By Lemma 1.8, the assertion follows.] Construct $W=\underset{\sim}{S}\times[-1, 1]\cup D^2\times D^2$, identifying $T(k)\times 1$ with $\partial D^2\times D^2$ such that $m\times 1=p\times\partial D^2$, $\ell\times 1=\partial D^2\times q$ $(p, q\in\partial D^2)$ for an $m\cdot\ell\cdot$pair (m, ℓ) of $T(k)$ with $[K(k)]=a[m]+2b[\ell]$. $\partial W-\underset{\sim}{S}\times(-1)$ is a Z_2-homology 3-sphere. Denote it by $\underset{\sim}{S}(k; -2b/a)$. Note that $\pm\bar{k}=0\times\partial D^2$ $(\subset D^2\times\partial D^2)\subset\underset{\sim}{S}(k; -2b/a)$ has the slope $-2b/a$. We say the knot $\bar{k}$ with the orientation, specified by $K(\bar{k})=K(k)$ is the *dual knot* of $k\subset\underset{\sim}{S}$ with respect to the normal presentation $a/2b$ of $s(k)$. Let $k'\subset\underset{\sim}{S}$ be another knot with $[k']\neq 0$ in $H_1(\underset{\sim}{S}; Z_2)=Z_2$, and $a'/2b'$ be a normal presentation of $s(k')$.

LEMMA 6.1. *$\mu(\underset{\sim}{S}(k'; -2b'/a'))=\mu(\underset{\sim}{S}(k; -2b/a))$ if $a'b'\equiv ab$ (mod 4).*

PROOF. Let $W'=\underset{\sim}{S}\times[-1, 1]\cup D^2\times D^2$, identifying $T(k')\times 1$ with $\partial D^2\times D^2$ so that $m'\times 1=p\times\partial D^2$, $\ell'\times 1=\partial D^2\times q$ $(p, q\in\partial D^2)$ for an $m\cdot\ell\cdot$pair (m', ℓ') of $T(k')$ with $[K(k')]=a'[m']+2b'[\ell']$. Construct $\overline{W}=W'\cup -W$, identifying two copies of $\underset{\sim}{S}\times(-1)$ in W' and W. Since $ab>0$, $a'b'>0$, by Lemma 2.6 we see that sign $W=$ sign $W'=-1$. Hence sign $\overline{W}=0$. We show that $\overline{W}$ is spin. Let $\#P(k)=r$, $\#P(k')=r'$. We have $o(k)=2br$ and $o(k')=2b'r'$. Note that brk is Z-homologous to $b'r'k'$ in $\underset{\sim}{S}$. Let c be a 2-chain in $\underset{\sim}{S}\times(-1)$ with $\partial c=brkx(-1)-b'r'k'\times(-1)$. Let $\bar{c}=b'r'\overline{D}'+c-br\overline{D}$ be an integral 2-cycle in $\overline{W}$, where $\overline{D}=k\times[-1, 1]\cup D^2\times 0\subset W$, $\overline{D}'=k'\times[-1, 1]\cup D^2\times 0\subset W'$. Let c_2 be a

Z_2-cycle in $\underset{\sim}{S}\times(-1)$ representing a generator of $H_2(\underset{\sim}{S}\times(-1); Z_2)=Z_2$. The Z_2-cycles c_2 and $\bar{c}$ (mod 2) form a basis of $H_2(\overline{W}; Z_2)=Z_2\oplus Z_2$. Clearly, $[c_2]^2=0$. We show that $[\bar{c}]^2\equiv 0$ (mod 2). Since $H_2(\underset{\sim}{S}; Q)=0$, $\bar{c}$ is Q-homologous to $b'r'\overline{D}_Q'^{+}-br\overline{D}_Q^{+}$, hence by Lemma 2.6,

$$\begin{aligned}[\bar{c}]^2&=b'^2r'^2(-a'/2b')-b^2r^2(-a/2b)\\&=(abr^2-a'b'r'^2)/2\\&\equiv(ab-a'b')/2 \text{ (mod 2), for } r^2\equiv 1\equiv r'^2 \text{ (mod 4)}\\&\equiv 0 \text{ (mod 2), for } ab-a'b'\equiv 0 \text{ (mod 4)}.\end{aligned}$$

Thus, $\overline{W}$ is spin and

$$\mu(\underset{\sim}{S}(k'; -2b'/a'))-\mu(\underset{\sim}{S}(k; 2b/a))=-\operatorname{sign}\overline{W}/16=0.$$

This completes the proof.

DEFINITION 6.2. $\mu(Z_2, S)=\mu(\underset{\sim}{S}(k; -2b/a))\in Q/Z$ for any knot k in $\underset{\sim}{S}$ with $[k]\neq 0$ in $H_1(\underset{\sim}{S}; Z_2)$ and any normal presentation $a/2b$ of $s(k)$ with $ab\equiv 1$ (mod 4).

Note that the slope $s(k)$ of a knot k in $\underset{\sim}{S}$ with $[k]\neq 0$ in $H_1(\underset{\sim}{S}; Z_2)$ has necessarily a normal presentation $a/2b$ with $ab\equiv 1$ (mod 4). [In fact, if $ab\equiv -1$ (mod 4), then take $(a+2b)/2b$ as the normal presentation of $s(k)$.] By Lemma 6.1, $\mu(Z_2, S)$ is well-defined and is an invariant of the equivariant, orientation-preserving homeomorphism type of (Z_2, S).

LEMMA 6.3. *For any knot k in $\underset{\sim}{S}$ with $s(k)=1/2$, we have*

$$\mu(\underset{\sim}{S}(k; -2/1))=\mu(\underset{\sim}{S}(k; -2/3))-1/8+\delta(\tilde{k}\subset S),$$

where $\tilde{k}\subset S$ is a knot which is the lift of k.

PROOF. Let $\bar{k}\subset\underset{\sim}{S}(k; -2/1)$, $\bar{\bar{k}}\subset\underset{\sim}{S}(k; -2/3)$ be the dual knots of k in $\underset{\sim}{S}$ with respect to the normal presentations 1/2, 3/2 of $s(k)$, respectively. $s(\bar{k})=-2/1=0$, $s(\bar{\bar{k}})=-2/3$. We may consider that $E=\underset{\sim}{S}-\operatorname{Int} T(k)=\underset{\sim}{S}(k; -2/1)-\operatorname{Int} T(\bar{k})=\underset{\sim}{S}(k; -2/3)-\operatorname{Int} T(\bar{\bar{k}})$ and that k, $\bar{k}$, $\bar{\bar{k}}$ have the same characteristic surface in E (cf. Corollary 1.3). We find $m.\ \ell.$ pairs $(\bar{m}, \bar{\ell})$, $(\bar{\bar{m}}, \bar{\bar{\ell}})$ of $T(\bar{k})$, $T(\bar{\bar{k}})$ such that

$$\begin{aligned}&[\bar{m}]=[\bar{\bar{m}}]-[\bar{\bar{\ell}}],\\&[\bar{\ell}]=[K(\bar{k})]=-2[\bar{\bar{m}}]+3[\bar{\bar{\ell}}]\end{aligned}$$

in $H_1(\partial E; Z)$. By Theorem 3.3,

$$\mu(\underset{\sim}{S}(k; -2/1))=\mu(\underset{\sim}{S}(k; -2/3))+\mu(L(3, -2))-\delta(\bar{k}\subset\underset{\sim}{S}(k; -2/1)).$$

Here, $\mu(L(3, -2))=\mu(L(3, 1))=(1-3)/16=-1/8$ by Lemma 3.1. By Corollary 1.7, the knots $k\subset \underset{\sim}{S}$ and $\bar{k}\subset \underset{\sim}{S}(k; -2/1)$ have the same Z_2-homology handle M. By Lemma 4.5, $\tilde{k}\subset S$ is flat. We see easily that the Z_2-homology homology handle of $\tilde{k}\subset S$ is a double covering space $\tilde{M}$ of M. By Lemma 4.2 of [18], $\varepsilon(\tilde{M})=\varepsilon(M)$. Since $\bar{k}$, $\tilde{k}$ are flat, it follows from Theorem 2.5 that

$$\delta(\tilde{k}\subset S)=\varepsilon(\tilde{M})/2=\varepsilon(M)/2=\delta(\bar{k}\subset \underset{\sim}{S}(k; -2/1)).$$

This completes the proof.

Corollary 6.4. *$\delta(k\subset S)$ is invariant for any Z_2-invariant flat knot $k\subset S$.*

Proof. By the natural projection $S\rightarrow \underset{\sim}{S}$, any Z_2-invariant flat knot k in S corresponds to a knot $\underset{\sim}{k}$ in $\underset{\sim}{S}$ with $s(\underset{\sim}{k})=1/2$ by Lemma 4.5. This corollary follows from Lemma 6.3, since both $\mu(\underset{\sim}{S}(\underset{\sim}{k}; -2/1))$ and $\mu(\underset{\sim}{S}(\underset{\sim}{k}; -2/3))$ are invariat for any knot $\underset{\sim}{k}$ in $\underset{\sim}{S}$ with $s(\underset{\sim}{k})=1/2$ by Lemma 6.1.

This corollary is generalized as follows:

Lemma 6.5. *$\delta(K(k)\subset S)+(b^2-1)/16 \in Q/Z$ is invariant for any Z_2-invariant knot k in S, where b is the order (≥ 1) of the slope $s(k)\in Q/Z$.*

Proof. Let $a/2b$ be a normal presentation of the slope $s(\underset{\sim}{k}\subset \underset{\sim}{S})$. Let $(\underset{\sim}{m}, \underset{\sim}{\ell})$ be an $m.\,\ell.$ pair of $T(\underset{\sim}{k})$ such that $[K(\underset{\sim}{k})]=a[\underset{\sim}{m}]+2b[\underset{\sim}{\ell}]$. Let (m, ℓ) be an $m.\,\ell.$ pair of the lift $T(k)$ of $T(\underset{\sim}{k})$, obtained by the lift of $(\underset{\sim}{m}, \underset{\sim}{\ell})$, so that ℓ is Z_2-invariant. We have $[K(k)]=a[m]+b[\ell]$. Let $k'\subset\partial T(k)$ be a Z_2-invariant knot in S such that $[k']=2[m]+b[\ell]$ (i.e., $[\underset{\sim}{k}']=[\underset{\sim}{m}]+b[\underset{\sim}{\ell}]$). The knot k' is flat. In fact, by Lemma 1.8, $s(k')=-\phi([k'], [k'])=-b^2\phi([k], [k])=-b^2(-a/b)=0$ in Q/Z. We show that

$$\delta(k'\subset S)=\delta(K(k)\subset S)+(b^2-1)/16.$$

Then the assertion follows from Corollary 6.4. Construct $W=S\times[-1, 1]\cup D^2\times D^2$, identifying $T(k)\times 1$ with $\partial D^2\times D^2$ so that $m\times 1=p\times\partial D^2$, $\ell\times 1=\partial D^2\times q$ ($p, q\in\partial D^2$). $\bar{D}=k\times[-1, 1]\subset D^2\times 0\subset W$ is admissible for $-k\subset -S$. So, $\delta(-k\subset -S)=([\bar{D}_Q^+]^2-\text{sign}\, W)/16-\mu(\partial W)$. The knot $-k'\subset -S$ bounds a disk $\bar{D}'$ in W with just one non-locally flat point represented by a torus knot $k_{2,b}\subset S^3$ of type $(2, b)$. $\bar{D}'\subset W$ is also admissible. Hence by Lemma 2.3, $\delta(-k'\subset -S)+\delta(k_{2,b}\subset S^3)=([\bar{D}_Q'^+]^2-\text{sign}\, W)/16-\mu(\partial W)$. $\delta(k_{2,b}\subset S^3)=(b^2-1)/16$, for example, by Lemma 2.4 and Theorem 2.5, since the knot polynomial $A(t)$ of $k_{2,b}$ has $|A(1)|=1$, $|A(-1)|=b$ (cf. [7]). Note that $[\bar{D}_Q^+]^2=-a/b$, $[\bar{D}_Q'^+]^2=-ab$ and $\delta(k\subset S)=\delta(K(k)\subset S)+(a/b-ab)/16$ (cf. Lemma 2.6, Theorem 2.7). We have

$$\begin{aligned}\delta(k'\subset S)&=[\bar{D}_Q^+]^2/16-[\bar{D}_Q'^+]^2/16+\delta(k\subset S)+\delta(k_{2,b}\subset S^3)\\&=\delta(K(k)\subset S)+(b^2-1)/16.\end{aligned}$$

This completes the proof.

DEFINITION 6.6. $\delta(Z_2, S)=\delta(K(k)\subset S)+(b^2-1)/16 \in \{0, 1/2\}\subset Q/Z$ for any Z_2-invariant knot $k\subset S$, where b is the order (≥ 1) of $s(k) \in Q/Z$.

By Lemma 6.5, $\delta(Z_2, S)$ is an invariant of the equivariant homeomorphism type of (Z_2, S). Clearly, $\delta(Z_2, -S)=-\delta(Z_2, S)=\delta(Z_2, S)$.

PROPOSITION 6.7. *Let k be a knot in $\underset{\sim}{S}$ with $[k]\neq 0$ in $H_1(\underset{\sim}{S}; Z_2)$.*

(1) $\mu(\underset{\sim}{S}(k; -2b/a))=\mu(Z_2, S)+1/8+\delta(Z_2, S)$ *for any normal presentation $a/2b$ of $s(k\subset \underset{\sim}{S})$ with* $ab\equiv -1 \pmod 4$,

(2) $\mu((-\underset{\sim}{S})(k; -2b/a))=-\mu(Z_2, S)+1/8$ *for any normal presentation $a/2b$ of $s(k\subset -\underset{\sim}{S})$ with* $ab\equiv -1 \pmod 4$,

(3) $\mu(Z_2, -S)=-\mu(Z_2, S)+\delta(Z_2, S)$.

PROOF. (1) follows from Lemma 6.3 since by Lemma 6.1 $\mu(\underset{\sim}{S}(k; -2b/a)) =\mu(\underset{\sim}{S}(k'; -2/3))$ for any knot $k'\subset\underset{\sim}{S}$ with $s(k')=1/2$. To see (2), it suffices to show that $\mu((-\underset{\sim}{S})(k; -2/3))=-\mu(\underset{\sim}{S}(k; -2/1))+1/8$ for any knot k in $\underset{\sim}{S}$ with $s(k)=1/2$. Write $[K(k\subset\underset{\sim}{S})]=[m]+2[\ell]$ and $[K(k\subset -\underset{\sim}{S})]=3[m']+2[\ell']$ for m. ℓ. pairs (m, ℓ) and (m', ℓ') of $k\subset\underset{\sim}{S}$ and $k\subset -\underset{\sim}{S}$. Construct $W=\underset{\sim}{S}\times[-1, 1]\cup D^2\times D_1^2$ and $W'=(-\underset{\sim}{S})\times[-1, 1]\cup D^2\times D_2^2$, identifying $T(k\subset\underset{\sim}{S})\times 1$ with $\partial D^2\times D_1^2$ and $T(k\subset -\underset{\sim}{S})\times 1$ with $\partial D^2\times D_2^2$ so that $(m\times 1, \ell\times 1)=(p\times\partial D_1^2, \partial D^2\times q_1)$ and $(m'\times 1, \ell'\times 1)=(p'\times\partial D_2^2, \partial D^2\times q_2')$ $(p, p', q, q' \in \partial D^2)$. Construct $\overline{W}= W'\cup W$ identifying $\underset{\sim}{S}\times(-1)$ with $(-\underset{\sim}{S})\times(-1)$. By Lemma 2.6, $\operatorname{sign} W'=\operatorname{sign} W=-1$, so that $\operatorname{sign}\overline{W}=-2$. Let $\bar{D}=k\times[-1, 1]\cup D^2\times 0_1\subset W$, $\bar{D}'=k\times[-1, 1]\cup D^2\times 0_2\subset W'$. The 2-sphere $\bar{D}'\cup -\bar{D}$ has the self-intersection number -2 in $\overline{W}$, so that $\overline{W}$ is spin (cf. the proof of Lemma 6.1). Since $\partial\overline{W}=\underset{\sim}{S}(k; -2/1)\cup(-\underset{\sim}{S})(k; -2/3)$, we have

$$\mu(\underset{\sim}{S}(k; -2/1))+\mu((-\underset{\sim}{S})(k; -2/3))=-\operatorname{sign}\overline{W}/16=1/8,$$

showing (2). To see (3), note that $\mu((-\underset{\sim}{S})(k; -2/3))=\mu(Z_2, -S)+1/8+\delta(Z_2, -S)$ by (1) for any knot $k\subset -\underset{\sim}{S}$ with $s(k)=1/2$. But, $\mu((-\underset{\sim}{S})(k, -2/3))= -\mu(Z_2, S)+1/8$ by (2). Hence $\mu(Z_2, -S)=-\mu(Z_2, S)+\delta(Z_2, S)$. This completes the proof.

LEMMA 6.8. *Suppose $H_1(S; Z_p)=0$ for any prime $p\equiv\pm 3 \pmod 8$. Then* $\delta(Z_2, S)=0$.

PROOF. Let $k\subset S$ be a Z_2-invariant flat knot with $\underset{\sim}{k}=k/Z_2\subset\underset{\sim}{S}=S/Z_2$. Let $\underset{\sim}{M}$ be the Z_2-homology handle of $\underset{\sim}{k}\subset\underset{\sim}{S}$ (cf. the proof of Lemma 6.3). The Z_2-homology handle of $k\subset S$ is a double covering space, M of $\underset{\sim}{M}$. Let $A(t)$, $\underset{\sim}{A}(t) \in Z\langle t\rangle$ be the Alexander polynomials of M, $\underset{\sim}{M}$, respectively. Take the infinite cyclic covering $\overline{\underset{\sim}{M}}\to\underset{\sim}{M}$ associated with an isomorphism $H_1(\underset{\sim}{M}; Z)/($odd torsion)$\to\langle t\rangle$. Clearly, $M=\underset{\sim}{\bar{M}}/\langle t^2\rangle$. The Z_p-reductions $A(t)_p$, $\underset{\sim}{A}(t)_p$ of $A(t)$, $\underset{\sim}{A}(t)$ are given as follows:

$$\underset{\sim}{A}(t)_p=\text{the characteristic polynomial of } t_*: H_1(\underset{\sim}{\bar{M}}; Z_p)\to H_1(\underset{\sim}{\bar{M}}; Z_p),$$
$$A(t)_p=\text{the characteristic polynomial of } t_*^2: H_1(\underset{\sim}{\bar{M}}; Z_p)\to H_1(\underset{\sim}{\bar{M}}; Z_p)$$

up to units of $Z_p\langle t\rangle$. Hence $A(t^2)_p=\underset{\sim}{A}(t)_p\underset{\sim}{A}(-t)_p$ up to units of $Z_p\langle t\rangle$. $H_1(S; Z_p)=0$ implies $H_1(M; Z_p)=Z_p$, so that by the Wang exact sequence $t_*^2-1: H_1(\underset{\sim}{\bar{M}}; Z_p)\to H_1(\underset{\sim}{\bar{M}}; Z_p)$ is an isomorphism. So, $A(1)_p\neq 0$ in Z_p. Noting that $p\equiv\pm 3 \pmod 8$, we see that $|\underset{\sim}{A}(1)\underset{\sim}{A}(-1)|=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ for prime numbers p_i with $p_i\equiv\pm 1 \pmod 8$ $(a_i\geq 0)$. Therefore, Lemma 4.2 of [18], Lemma 2.4 and Theorem 2.5 show that

$$\delta(Z_2, S)=\delta(k\subset S)=\varepsilon(M)/2=\varepsilon(\underset{\sim}{M})/2=(\underset{\sim}{A}(1)^2\underset{\sim}{A}(-1)^2-1)/16=0$$

in Q/Z. This proves Lemma 6.8.

The following is our main assertion of this section, but the proof will be given in Section 8.

THEOREM 6.9. *For a Z_2-homology 3-sphere S with free Z_2-action,*

$$\mu(S)=\alpha(Z_2, S)/16+2\mu(Z_2, S)+\delta(Z_2, S)$$

in Q/Z.

Here are examples of irreducible Z_2-homology 3-spheres with free Z_2-action, which show the independence of the invariants $\alpha(Z_2, S)/16(\in Q/Z)$, $2\mu(Z_2, S)$ and $\delta(Z_2, S)$ appearing in Theorem 6.9.

EXAMPLES 6.10. Let $k\subset S^3$ be a classical knot with m. ℓ. pair (m, ℓ) on $T(k)$ such that $\ell=K(k)$. Let n be an odd integer >0. Let (m, ℓ') be an m. ℓ. pair on $T(k)$ such that $[\ell]=2n[m]+[\ell']$. Construct $\underset{\sim}{W}=S^3\times[-1, 1]\cup D^2\times D^2$ identifying $T(k)\times 1$ with $\partial D^2\times D^2$ so that $m\times 1=p\times\partial D^2$, $\ell'\times 1=\partial D^2\times q$ $(p, q \in\partial D^2)$. Let $S(k)_2$ be the double branched covering space of S^3 branched over k, and $W=S(k)_2\times[-1, 1]\cup D^2\times D^2$ be the double branched covering space of $\underset{\sim}{W}$ branched over the disk $\bar{D}=k\times[-1, 1]\cup D^2\times 0$. Note that $S(k)_2$ is a Z_2-homology 3-sphere, so that $S=\partial W-S(k)_2\times(-1)$ is a Z_2-homology 3-sphere which is the double covering space of $\underset{\sim}{S}=\partial\underset{\sim}{W}-S^3\times(-1)$ with $H_1(\underset{\sim}{S}; Z)=Z_{2n}$. Also, note that $\alpha(Z_2, S(k)_2)=-\sigma(k)$ by Remark 4.8, where $\sigma(k)$ is the knot signature of k, and $k'=0\times\partial D^2\subset S$ has the slope $-1/n$. We have the following:

(1) $\alpha(Z_2, S)=-\sigma(k)+n-1,$

$$(2)\quad \mu(Z_2, S)=\begin{cases}\delta(Z_2, S) & (n\equiv 1 \pmod 4)\\ 1/8 & (n\equiv -1 \pmod 4)\end{cases}$$

$$(3)\quad \delta(Z_2, S)=\delta(k\subset S^3)+(n^2-1)/16.$$

(1) and (3) follow easily. To check (2), use Proposition 6.7.

(6.10.1) Let k be a trefoil with $\sigma(k)=\pm 2$ and $n=8b-3$ for any odd $b\geq 1$. $\underset{\sim}{S}$ is a Seifert 3-manifold (cf. [25]), and therefore so is S. In particular, S is irreducible. We have $\alpha(Z_2, S)/16=1/8$ and $\mu(Z_2, S)=\delta(Z_2, S)=0$ (since $\delta(k\subset S^3)=1/2$).

(6.10.2) Let k be a torus knot of type $(a, 2)$ $(a>0)$ with $\sigma(k)=a-1$ and $a\equiv -1 \pmod 4$. Let $n=16b+a>0$ for an non-zero integer b. $\underset{\sim}{S}$ is a Seifert 3-manifold (cf. [25]), and so is S and hence S is irreducible. We have $\sigma(k)\equiv n-1 \pmod{16}$ and $\delta(k\subset S^3)=(a^2-1)/16\equiv(n^2-1)/16 \pmod 1$. So, $\alpha(Z_2, S)/16=\delta(Z_2, S)=0$ in Q/Z and $2\mu(Z_2, S)=1/4$.

(6.10.3) Let k be a figure eight knot with $\sigma(k)=0$ and $\delta(k\subset S^3)=1/2$. For $n=16b+1$ with any non-negative integer b, $\underset{\sim}{S}$ is a Seifert 3-manifold $(b=0)$ or a hyperbolic 3-manifold $(b>0)$ by Thurston [35], and therefore so is S and hence S is irreducible. We have $\alpha(Z_2, S)/16=2\mu(Z_2, S)=0$ in Q/Z and $\delta(Z_2, S)=1/2$.

LEMMA 6.11. *For any two Z_2-homology 3-spheres S_i with free Z_2-action, $i=1, 2$, there exists a Z_2-homology 3-sphere S with free Z_2-action such that*

$$\alpha(Z_2, S)=\alpha(Z_2, S_1)+\alpha(Z_2, S_2),$$
$$\mu(Z_2, S)=\mu(Z_2, S_1)+\mu(Z_2, S_2),$$
$$\delta(Z_2, S)=\delta(Z_2, S_1)+\delta(Z_2, S_2).$$

PROOF. Let k_i be a knot in $\underset{\sim}{S}_i$ with $s(k_i)=1/2$. Let $\bar{k}_i\subset\underset{\sim}{S}_i(k_i, -2/1)$ be the dual knot of k_i with respect to the normal presentation $1/2$. Consider the knot sum $\bar{k}_1\#\bar{k}_2\subset\underset{\sim}{S}_1(k_1, -2/1)\#\underset{\sim}{S}_2(k_2, -2/1)$ which is a flat knot. Let $k\subset\underset{\sim}{S}$ be a knot with $H_1(\underset{\sim}{S}; Z)/(\text{odd torsion})=Z_2$ and $s(k)=1/2$ whose dual knot is the above knot sum. The double covering space S of $\underset{\sim}{S}$ is a Z_2-homology 3-sphere with desired properties (See Lemma 4.10, Lemma 2.2 and the proof of Theorem 6.9 in Section 8). This completes the proof.

The following is obtained from Examples 6.10 and Lemma 6.11.

COROLLARY 6.12. *For any integers m_1, m_2 and m_3, there exists a Z_2-homology 3-sphere S with free Z_2-action such that $\alpha(Z_2, S)/16=m_1/8$, $2\mu(Z_2, S)=m_2/4$ and $\delta(Z_2, S)=m_3/2$ in Q/Z.* [Note that $8\mu(Z_2\text{-homology 3-sphere})=0$ *in Q/Z, so that by Theorem 6.9 $\alpha(Z_2, S)$ is an even integer.*]

§ 7. Some applications of the case of a free involution

In this section, we shall obtain some consequences of Theorem 6.9 (which will be proved in Section 8).

COROLLARY 7.1 [37]. *For a Z-homology 3-sphere S with free Z_2-action,*

$$\mu(S)=\alpha(Z_2, S)/16.$$

PROOF. Since 2μ(Z-homology 3-sphere)$=0$, we have $2\mu(Z_2, S)=0$. By Lemma 6.8, $\delta(Z_2, S)=0$. By Theorem 6.9, the proof is completed.

COROLLARY 7.2 [29]. *Let Z_2 act freely on the lens space $L(b, a)$, so that $L(b, a)/Z_2=L(2h, a)$. For any odd b, we have*

$$\mu(L(b, a))=-\alpha(Z_2, L(b, a))/16.$$

PROOF. We may assume that a, b are odd integers >0 and $ab\equiv 1 \pmod 4$. Then

$$2\mu(Z_2, L(b, a))=-2\mu(L(a, 2b))=(a+1)/8-\frac{2b}{a}(1/4),$$
$$\delta(Z_2, L(b, a))=(b^2-1)/16, \text{ and}$$
$$2\mu(L(b, a))=-(b+1)/8+\frac{a}{b}(1/4).$$

We show that

$$I=2\mu(Z_2, L(b, a))+\delta(Z_2, L(b, a))-2\mu(L(b, a))$$

is 0 in Q/Z. In fact,

$$I=(a+b)/8+1/4+(b^2-1)/16-\{1+(-1)^{(a^2-1)/8+(a-1)(b-1)/4}\}\frac{a}{b}(1/4),$$

because $\frac{2b}{a}=\frac{2}{a}\frac{b}{a}=(-1)^{(a^2-1)/8}\frac{b}{a}$ and $\frac{b}{a}=(-1)^{(a-1)(b-1)/4}\frac{a}{b}$. Since $ab\equiv 1 \pmod 4$, we can write $a=4n+\varepsilon$, $b=4n'+\varepsilon$ for the same $\varepsilon=\pm 1$. So,

$$I=n/2+(\varepsilon+1)/4-\{1+(-1)^{n+(1-\varepsilon)/2}\}(1/4),$$

since $\varepsilon-1$ and $1+(-1)^q$ $(q\in Z)$ are even. Thus,

$$I=n/2+\varepsilon/4-(-1)^{n+(1-\varepsilon)/2}(1/4).$$

Now we can check directly that $I=0$, according as n is odd or even and ε is 1 or -1. Therefore, by Theorem 6.9

$$\begin{aligned}\mu(L(b, a))&=\alpha(Z_2, L(b, a))/16+2\mu(Z_2, L(b, a))+\delta(Z_2, L(b, a))\\&=\alpha(Z_2, L(b, a))/16+2\mu(L(b, a)), \text{ and}\end{aligned}$$

$$\mu(L(b, a)) = -\alpha(Z_2, L(b, a))/16.$$

This completes the proof.

THEOREM 7.3. *For any two Z_2-homology 3-spheres S, S' with free Z_2-action, $\delta(Z_2, S)=\delta(Z_2, S')$ if and only if there exists a compact connected oriented 4-manifold W with free Z_2-action such that $\partial(Z_2, W)=(Z_2, S'\cup -S)$ and $\underset{\sim}{W}=W/Z_2$ is spin. In this case, we have*

$$\mu(Z_2, S')-\mu(Z_2, S)+(\alpha(Z_2, S')-\alpha(Z_2, S))/32 = -\operatorname{sign} W/32$$

in Q/Z for any such 4-manifold W. Further, we can take W so that $H_1(W; Z_2)=0$.

REMARK 7.4. We can use Theorem 7.3 to see the existence of a fake P^4 and the existence of the exotic free involution of S^4 by Cappell-Shaneson [6] and Fintushel-Stern [8].

PROOF OF THEOREM 7.3. Assume the existence of W in the statement of Theorem 7.3. Let $k\subset \underset{\sim}{S}$, $k'\subset \underset{\sim}{S}'$ be knots of the slope 1/2. Attaching two 2-handles to $\underset{\sim}{W}$ along $T(k)$ and $T(k')$, we can obtain a spin 4-manifold $\underset{\sim}{W}'$ such that $\operatorname{sign}\underset{\sim}{W}'=\operatorname{sign}\underset{\sim}{W}$ and $\partial\underset{\sim}{W}'=\bar{\underset{\sim}{S}}'\cup -\bar{\underset{\sim}{S}}$ with $\bar{\underset{\sim}{S}}'=\underset{\sim}{S}'(k'; -2/1)$ and $\bar{\underset{\sim}{S}}=\underset{\sim}{S}(k, -2/1)$ or $\underset{\sim}{S}(k, -2/3)$. Then we show that only the case $\bar{\underset{\sim}{S}}=\underset{\sim}{S}(k; -2/1)$ can occur. Note that

$$\mu(\bar{\underset{\sim}{S}}')-\mu(\bar{\underset{\sim}{S}}) = -\operatorname{sign}\underset{\sim}{W}/16,$$
$$\mu(S')-\mu(S) = -\operatorname{sign} W/16,$$
$$\alpha(Z_2, S')-\alpha(Z_2, S) = -\operatorname{sign} W+2\operatorname{sign}\underset{\sim}{W}.$$

Hence,

$$\mu(S')-\alpha(Z_2, S')/16-2\mu(\bar{\underset{\sim}{S}}')=\mu(S)-\alpha(Z_2, S)/16-2\mu(\bar{\underset{\sim}{S}})$$

in Q/Z. By Theorem 6.9,

$$\mu(S)=\alpha(Z_2, S)/16+2\mu(Z_2, S)+\delta(Z_2, S),$$
$$\mu(S')=\alpha(Z_2, S')/16+2\mu(Z_2, S')+\delta(Z_2, S').$$

Since $\mu(\bar{\underset{\sim}{S}}')=\mu(Z_2, S')$, it follows that

$$\delta(Z_2, S')=\delta(Z_2, S)+2\mu(Z_2, S)-2\mu(\bar{\underset{\sim}{S}}).$$

If $\bar{\underset{\sim}{S}}=\underset{\sim}{S}=(k; -2/3)$, then by Proposition 6.7, $2\mu(\bar{\underset{\sim}{S}})=2\mu(Z_2, S)+1/4$. But, $2\delta(Z_2, S)=2\delta(Z_2, S')=0$. So, this is impossible. Thus, we have $\bar{\underset{\sim}{S}}=\underset{\sim}{S}(k, -2/1)$, so that $\mu(\bar{\underset{\sim}{S}})=\mu(Z_2, S)$ and hence $\delta(Z_2, S')=\delta(Z_2, S)$ and $\mu(Z_2, S')-\mu(Z_2, S)=-\operatorname{sign}\underset{\sim}{W}/16$. Then we also have

$$\mu(Z_2, S') - \mu(Z_2, S) + (\alpha(Z_2, S') - \alpha(Z_2, S))/32 = -\text{sign}\, W/32.$$

Conversely, assume that $\delta(Z_2, S) = \delta(Z_2, S')$. Let $k \subset \underset{\sim}{S}$, $k' \subset \underset{\sim}{S}'$ be knots of the slope 1/2. Let (m, ℓ) be an $m.\ \ell.$ pair of $T(k)$ such that $[K(k)] = [m] + 2[\ell]$. Let $-P^3 = -L(2, 1) = T_0 \cup_h T_1$ for an orientation-reversing homeomorphism $h\colon \partial T_0 \to \partial T_1$ such that $h_*[m_0] = [m_1] + 2[\ell_1]$, where T_i are solid tori with $m.\ \ell.$ pair (m_i, ℓ_i), $i = 0, 1$. Construct $W_0 = \underset{\sim}{S} \times [-1, 1] \cup (-P^3) \times [-1, 1]$, identifying $T \times 1$ with $T_1 \times (-1)$ so that $m \times 1 = m_1 \times (-1)$, $\ell \times 1 = \ell_1 \times (-1)$. Then $\partial W_0 = -\underset{\sim}{S} \cup M \cup -P$, where M is the Z_2-homology handle of $k \subset \underset{\sim}{S}$ and $P \cong P^3$. Similarly, construct $W_0' = \underset{\sim}{S}' \times [-1, 1] \cup (-P^3) \times [-1, 1]$ with $\partial W_0' = -\underset{\sim}{S}' \cup M' \cup -P'$, where M' is the Z_2-homology handle of $k' \subset \underset{\sim}{S}'$ and $P' \cong P^3$. Since $\delta(Z_2, S) = \delta(Z_2, S')$, we have $\varepsilon(M) = \varepsilon(M')$ (cf. Lemma 4.2 of [18] and Theorem 2.5.). Construct $W_1 = W_0 \cup -W_0'$ identifying $-P$ with P'. W_1 is an oriented spin 4-manifold such that $\partial W_1 = -\underset{\sim}{S} \cup M \cup \underset{\sim}{S}' \cup -M'$ and the natural homomorphisms from $H_1(\underset{\sim}{S}; Z_2)$, $H_1(M; Z_2)$, $H_1(\underset{\sim}{S}'; Z_2)$ and $H_1(M'; Z_2)$ to $H_1(W_1; Z_2)$ are isomorphisms. [To see that W_1 is spin, note that W_0, W_0' are spin and $H_2(W_1; Z_2)$ is generated by Z_2-cycles in W_0 and W_0'.] Let k^* be a knot in M representing a generator of $H_1(M; Z)/(\text{odd torsion}) = Z$. Let k'^* be a similar knot in M'. Let c be a Z_2-chain in W_1 with $\partial c = -k^* + k'^*$. Let W_2 be a 4-manifold obtained from W_1 by identifying $T(k^*)$ in M with $T(k'^*)$ in $-M'$ by an orientation-reversing homeomorphism $(T(k^*), k^*) \to (T(k'^*), k'^*)$ so that the Z_2-cycle c^* induced from c has $[c^*]^2 = 0$. W_2 is spin, since $H_2(W_2; Z_2)$ is generated by c^* and Z_2-cycles in W_1. Note that $\partial W_2 - (\underset{\sim}{S}' \cup -\underset{\sim}{S}) = M''$ is a Z_2-homology handle. $\varepsilon(M) = \varepsilon(M')$ implies $\varepsilon(M'') = 0$. [In fact, the product of the Alexander polynomials of M and M' is the Alexander polynomial of M''.] By Theorem 4.1 of [18], there is a compact connected spin 4-manifold W'' with $\partial W'' = M''$ such that $H_1(M''; Z)/(\text{odd torsion}) \overset{\cong}{\to} H_1(W''; Z)/(\text{odd torsion})\ (\cong Z)$. Construct $W_3 = W_2 \cup -W''$ identifying two copies of M''. W_3 is spin, since $H_2(W_3; Z_2)$ is generated by Z_2-cycles in W_2 and W''. $\partial W_3 = \underset{\sim}{S}' \cup -\underset{\sim}{S}$. $H_1(W_3; Z_2) = Z_2 \oplus Z_2$. By a surgery, we can obtain from W_3 a compact connected oriented spin 4-manifold $\underset{\sim}{W}$ with $\partial \underset{\sim}{W} = \underset{\sim}{S}' \cup -\underset{\sim}{S}$ such that the natural homomorphisms from $H_1(\underset{\sim}{S}; Z_2)$ and $H_1(\underset{\sim}{S}'; Z_2)$ to $H_1(\underset{\sim}{W}; Z_2)$ are isomorphisms. We have $H_1(\underset{\sim}{W}; Z)/(\text{odd torsion}) = Z_2$. Then the desired 4-manifold W is a double covering space of $\underset{\sim}{W}$. [To check that $H_1(W; Z_2) = 0$, use the Thom-Gysin exact sequence corresponding to the double covering $W \to \underset{\sim}{W}$.] This completes the proof.

§ 8. The case of a non-free involution

Let S be a Z_2-homology 3-sphere with non-free Z_2-action. Since the action is assumed to be orientation-preserving, it follows from Smith theory that $k = F(Z_2, S)$ is a knot. Let $\underset{\sim}{S} = S/Z_2$ and $\underset{\sim}{k} = k/Z_2$. Let $pr\colon S \to \underset{\sim}{S}$ be the natural projection. Using that $pr|S-k: S-k \to \underset{\sim}{S}-\underset{\sim}{k}$ is an unbranched covering, we see that $pr_\#: \pi_1(S, x) \to \pi_1(\underset{\sim}{S}, prx)$ is onto for $x \in k$, so that $pr_*: H_1(S; Z) \to H_1(\underset{\sim}{S}; Z)$ is onto. Thus, $\underset{\sim}{S}$ is also a Z_2-homology 3-sphere. We shalll show the following:

THEOREM 8.1. *For any normal presentation a/b of the slope $s(k)$ of the knot $k=F(Z_2, S)$ in S such that $ab \equiv 1$ (mod 4), we have*

$$\mu(S)=\alpha(Z_2, S)/16+2\mu(\underset{\sim}{S})+(ab+a/b+a^2-3)/16$$

in Q/Z.

The following is a special case of Theorem 8.1, but is a key to proving Theorem 6.9 (which we shall prove in this section) and Theorem 8.1.

LEMMA 8.2. *If $s(k)=0$, then*

$$\mu(S)=\alpha(Z_2, S)/16+2\mu(\underset{\sim}{S}).$$

PROOF. By Lemma 4.5, $\underset{\sim}{k}$ is flat in $\underset{\sim}{S}$. Let $\underset{\sim}{M}$ be a Z_2-homology handle of $\underset{\sim}{k} \subset \underset{\sim}{S}$. Suppose $\varepsilon(\underset{\sim}{M})=0$. Then $\underset{\sim}{M}$ bounds a compact connected oriented spin 4-manifold $\underset{\sim}{W}$ such that $H_1(\underset{\sim}{M}; Z)/(\text{odd torsion}) \overset{\cong}{\to} H_1(\underset{\sim}{W}; Z)/(\text{odd torsion})$ $(\cong Z)$ by Theorem 4.1 of [18]. Let $(\underset{\sim}{m}, \underset{\sim}{\ell})$ be an m. ℓ. pair of $T(\underset{\sim}{k})$ with $\underset{\sim}{\ell}=K(\underset{\sim}{k})$. Construct $\underset{\sim}{W}^*=\underset{\sim}{S}\times[-1, 1] \cup D^2\times D^2$, identifying $T(\underset{\sim}{k})\times 1$ with $\partial D^2\times D^2$ so that $\underset{\sim}{m}\times 1=p\times\partial D^2$, $\underset{\sim}{\ell}\times 1=\partial D^2\times q$, $p, q \in \partial D^2$. Clearly, $\partial \underset{\sim}{W}^*-\underset{\sim}{S}\times(-1)=\underset{\sim}{M}$. Then $\overline{\underset{\sim}{W}}=\underset{\sim}{W}^* \cup -\underset{\sim}{W}$ is a spin 4-manifold with boundary $\underset{\sim}{S}\times(-1)$. Let $\overline{W}$ be the double branched covering space of $\overline{\underset{\sim}{W}}$, branched over the disk $\overline{\underset{\sim}{D}}=k\times [-1, 1] \cup D^2\times 0$. We have $\overline{W}=W^* \cup -W$, where $W^*=S\times[-1, 1] \cup D^2\times D^2$ and $\partial W^*-S\times(-1)=-\partial W=M$ is a Z_2-homology handle of the flat knot k in S, and W is a double covering of $\underset{\sim}{W}$. Since W is spin and $H_1(M; Z)/(\text{odd torsion}) \overset{\cong}{\to} H_1(W; Z)/(\text{odd torsion})$ (cf. [18], Corollary 4.1), it follows that $\overline{W}$ is spin. Note that the disk $\overline{D}=k\times[-1, 1] \cup D^2\times 0$ has $[\overline{D}_Q^+]^2=0$ in $W^* \subset \overline{W}$, since $H_1(M; Q)=Q$. Then

$$-\alpha(Z_2, S)=\alpha(Z_2, S\times(-1))=-\operatorname{sign}\overline{W}+2\operatorname{sign}\overline{\underset{\sim}{W}},$$

so that

$$-\alpha(Z_2, S)/16=\mu(S\times(-1))-2\mu(\underset{\sim}{S}\times(-1))=-\mu(S)+2\mu(\underset{\sim}{S}),$$

i.e.,

$$\mu(S)=\alpha(Z_2, S)/16+2\mu(\underset{\sim}{S}).$$

Next, suppose $\varepsilon(\underset{\sim}{M})=1$. Let $k_0 \subset S^3$ be a classical knot with $\delta(k_0)=1/2$. Let $\underset{\sim}{M}'$ be the Z_2-homology handle of the knot sum $k\# k_0 \subset \underset{\sim}{S}\# S^3=\underset{\sim}{S}$. We have

$\varepsilon(\underset{\sim}{M}')=0$ (cf. Theorem 2.5). Let $S^3(k_0)_2$ be the double branched covering space of S^3, branched over k_0. It is well-known that $S^3(k_0)_2$ is a Z_2-homology 3-sphere. Since $S\# S^3(k_0)_2$ is a double branched covering space of $\underset{\sim}{S}$, branched over the knot $\underset{\sim}{k}\# k_0$, the above argument shows that

$$\mu(S\# S^3(k_0)_2)=\alpha(Z_2,\ S\# S^3(k_0)_2)/16+2\mu(\underset{\sim}{S}).$$

Clearly, $\mu(S\# S^3(k_0)_2)=\mu(S)+\mu(S^3(k_0)_2)$. By Lemma 4.10, $\alpha(Z_2,\ S\# S^3(k_0)_2)=\alpha(Z_2,\ S)+\alpha(Z_2,\ S^3(k_0)_2)$. To complete the proof, it remains only to prove that $\mu(S^3(k_0)_2)=\alpha(Z_2,\ S^3(k_0)_2)/16$ in Q/Z. Let F be a connected Seifert surface of k_0 in S^3. Push Int F into Int D^4. Let F' be the resulting proper surface in D^4. Let $D^4(F')_2$ be a double branched covering space of D^4, branched over F'. From construction, $[\tilde{F}'^{+}_{Q}]^2=0$ for the lift $\tilde{F}'$ of F'. $D^4(F')_2$ is spin. [This is well-known, but here is a quick proof of it. The double of $D^4(F')_2$ is a double branched covering space of S^4, branched over an orientable unknotted surface, and is homeomorphic to S^4 or a connected sum of copies of $S^2\times S^2$, which is spin. So, $D^4(F')_2$ is spin.] Hence $\alpha(Z_2,\ S^3(k_0)_2)/16=-\operatorname{sign} D^4(F')_2=\mu(S^3(k_0)_2)$. This completes the proof of Lemma 8.2.

PROOF OF THEOREM 6.9. Let S be a Z_2-homology 3-sphere with free Z_2-action. Let k be a Z_2-invariant flat knot in S. Let $\underset{\sim}{S}=S/Z_2$ and $\underset{\sim}{k}=k/Z_2$. Let $(\underset{\sim}{m},\ \underset{\sim}{\ell})$ be an $m.\ \ell.$ pair of $T(\underset{\sim}{k})$ with $[K(\underset{\sim}{k})]=[\underset{\sim}{m}]+2[\underset{\sim}{\ell}]$. Construct $\underset{\sim}{W}=\underset{\sim}{S}\times[-1,1]\cup D^2\times D^2$, identifying $T(\underset{\sim}{k})\times 1$ with $\partial D^2\times D^2$ such that $\underset{\sim}{m}\times 1=p\times\partial D^2$, $\underset{\sim}{\ell}\times 1=\partial D^2\times q$ $(p,\ q\in\partial D^2)$. Let W be the double branched covering space of $\underset{\sim}{W}$, branched over the disk $\underset{\sim}{D}'=0\times D^2\subset\underset{\sim}{W}$. Note that $\partial W=-S\cup S'$, where S' is the double branched covering space of $\underset{\sim}{S}(\underset{\sim}{k};\ -2/1)$, branched over the dual flat knot $\bar{\underset{\sim}{k}}=\partial\underset{\sim}{D}'$ in $\underset{\sim}{S}(\underset{\sim}{k};\ -2/1)$ of $\underset{\sim}{k}$ in $\underset{\sim}{S}$ with respect to the normal presentation $1/2$. By Lemma 8.2,

$$\mu(S')=\alpha(Z_2,\ S')/16+2\mu(\underset{\sim}{S}(\underset{\sim}{k};\ -2/1)).$$

The lift $D'\subset W$ of $\underset{\sim}{D}'\subset\underset{\sim}{W}$ is admissible for the lift $\bar{k}\subset S'$ of $\bar{\underset{\sim}{k}}\subset\underset{\sim}{S}(\underset{\sim}{k};\ -2/1)$. Note that $[D'^{+}_{Q}]=\operatorname{sign} W=-1$. So,

$$\delta(\bar{k}\subset S')=-\mu(\partial W)=\mu(S)-\mu(S').$$

But, since $\bar{k}\subset S'$ and $k\subset S$ are flat and have the same Z_2-homology handle, we see from Theorem 2.5 that $\delta(Z_2,\ S)=\delta(k\subset S)=\delta(\bar{k}\subset S')$, so that $\mu(S)=\mu(S')+\delta(Z_2,\ S)$. By definition, $\mu(Z_2,\ S)=\mu(\underset{\sim}{S}(\underset{\sim}{k};\ -2/1))$. To complete the proof, it suffices to check that $\alpha(Z_2,\ S')=\alpha(Z_2,\ S)$. In fact,

$$\begin{aligned}\alpha(Z_2,\ S')-\alpha(Z_2,\ S)&=-\operatorname{sign} W+2\operatorname{sign}\underset{\sim}{W}-[D'^{+}_{Q}]^2\\&=-(-1)+2(-1)-(-1)=0.\end{aligned}$$

This completes the proof.

PROOF OF THEOREM 8.1. Construct $W=S\times[-1,1]\cup D^2\times D^2$ identifying $T(k)\times 1$ with $\partial D^2\times D^2$ such that $m\times 1=p\times\partial D^2$, $\ell\times 1=\partial D^2\times q$ $(p, q\in\partial D^2)$, where (m, ℓ) is an $m.\ \ell.$ pair of a Z_2-invariant $T(k)$ such that m is Z_2-invariant and $\ell\cap t\ell=\phi$ $(t(\neq 1)\in Z_2)$ and $[K(k)]=a[m]+b[\ell]$. Let $S^*=\partial W-S\times(-1)$. The Z_2-action on S extends to a Z_2-action on W with $F(Z_2, W)=\bar{D}$, $\bar{D}=k\times[-1,1]\cup D^2\times 0$, so that Z_2 acts on S^* freely. Since a and b are odd, S^* is a Z_2-homology 3-sphere. By Theorem 6.9,

$$\mu(S^*)=\alpha(Z_2, S^*)/16+2\mu(Z_2, S^*)+\delta(Z_2, S^*).$$

Let k^* be a core of $D^2\times\partial D^2(\subset S^*)$, which is Z_2-invariant. Note that $s(k^*\subset S^*)=-b/a$ and $s(\underset{\sim}{k}^*\subset\underset{\sim}{S}^*)=-b/2a$. By Corollary 3.6, $\delta(K(k)^*\subset S^*)=\delta(K(k)\subset S)$, so that $\delta(Z_2, S^*)=\delta(K(k)\subset S)+(a^2-1)/16$. Since $b/2a$ is a normal presentation of $s(k^*\subset -S^*)$ and $ab\equiv 1 \pmod 4$, we see that $\mu(Z_2, -S^*)=\mu(-\underset{\sim}{S})=-\mu(\underset{\sim}{S})$. By Proposition 6.7, $2\mu(Z_2, S^*)=-2\mu(Z_2, -S^*)=2\mu(\underset{\sim}{S})$. Note that $[\bar{D}_Q^+]^2=-a/b$ and $\operatorname{sign} W=\operatorname{sign}\underset{\sim}{W}=-1$ by Lemma 2.6. We have $\alpha(Z_2, S^*)=\alpha(Z_2, S)+a/b-1$ and $-\delta(k\subset S)=\mu(S)-\mu(S^*)+(-a/b+1)/16$. By Theorem 2.7, $\delta(k\subset S)=\delta(K(k)\subset S)+(a/b-ab)/16$. Therefore,

$$\begin{aligned}\mu(S)&=-\delta(K(k)\subset S)+(ab-1)/16+\mu(S^*)\\&=\alpha(Z_2, S)/16+2\mu(\underset{\sim}{S})+(ab+a/b+a^2-3)/16.\end{aligned}$$

This completes the proof.

COROLLARY 8.3. *Let S be a Z_2-homology 3-sphere with semi-free Z_{2^n}-action such that $k=F(Z_{2^n}, S)$ is a knot. Let b be the order (≥ 1) of the slope $s(k)$ in Q/Z. We have*

$$b\mu(S)=b\alpha(Z_{2^n}, S)/16+b2^n\mu(\underset{\sim}{S})\ (\underset{\sim}{S}=S/Z_{2^n}).$$

Here, $b=1$ if k is flat and $2^n\mu(\underset{\sim}{S})=0$ if $n\geq 3$.

PROOF. By Lemma 4.5, b is the order of $s(\underset{\sim}{k}\subset\underset{\sim}{S})$ in Q/Z. By Corollary 1.13, the knot sum $\#^b\underset{\sim}{k}\subset\#^b\underset{\sim}{S}$ of b copies of $\underset{\sim}{k}\subset\underset{\sim}{S}$ is flat. The branched Z_{2^n}-covering space of this knot sum is the space of the knot sum $\#^b k\subset\#^b S$ of b copies of $k\subset S$. Note that $F(Z_{2^n}, \#^b S)=\#^b k$ and this is flat in $\#^b S$. Since the α-invariant is additive on the knot sum operation by Lemma 4.10, it suffices to check that the above congruence is true when $b=1$ (i.e., k is flat). The case $n=1$ is true by Theorem 8.1 (or Lemma 8.2). The case $n>1$ follows easily by induction on n, applying Lemma 4.9. This completes the proof.

The following generalizes a result of L. Contreras-Caballero [4] (See also, [19], [34]).

COROLLARY 8.4. *Let S be a Z_2-homology 3-sphere with Z_2-action such that*

$k=F(Z_2, S)$ is a knot. If $\underset{\sim}{k}\subset\underset{\sim}{S}$ is amphicheiral, then $\mu(S)=0$.

PROOF. By Corollary 1.11, $\underset{\sim}{k}\subset\underset{\sim}{S}$ is a flat knot. So, by Theorem 8.1 (or Lemma 8.2), $\mu(S)=\alpha(Z_2, S)/16+2\mu(\underset{\sim}{S})$. The amphicheirality implies that $\mu(\underset{\sim}{S})=\mu(-\underset{\sim}{S})=-\mu(\underset{\sim}{S})$ and $\alpha(Z_2, S)=\alpha(Z_2, -S)=-\alpha(Z_2, S)$ [Note that there is an equivariant, orientation-preserving homeomorphism $(Z_2, S)\rightarrow(Z_2, -S)$.], so that $2\mu(\underset{\sim}{S})=0$ and $\alpha(Z_2, S)=0$. This completes the proof.

§ 9. The case of a free cyclic action of order four

Let S be a Z_2-homology 3-sphere with free Z_n-action, and n be an even integer >0. The following is a generalization of Definition 6.6.

DEFINITION 9.1. $\delta(Z_n, S)=\delta(Z_2, S)$ for a subgroup Z_2 of Z_n of order 2, which is unique.

We consider the case of a free Z_4-action. Let $\underset{\sim}{S}=S/Z_4$. Let k be a knot in $\underset{\sim}{S}$ with $[k]\neq 0$ in $H_1(\underset{\sim}{S}; Z_2)=Z_2$. The slope $s(k)$ has a normal presentation of type $a/4b$ with odd a, b. Let (m, ℓ) be an $m.\ \ell.$ pair of $T(k)$ such that $[K(k)]=a[m]+4b[\ell]$. Construct $W=\underset{\sim}{S}\times[-1, 1]\cup D^2\times D^2$, identifying $T(k)\times 1$ with $\partial D^2\times D^2$ such that $m\times 1=p\times\partial D^2$, $\ell\times 1=\partial D^2\times q$ $(p, q\in\partial D^2)$. $\partial W-\underset{\sim}{S}\times(-1)$ is a Z_2-homology 3-sphere. Denote it by $\underset{\sim}{S}(k, -4b/a)$. The knot $\bar{k}=0\times\partial D^2\subset\underset{\sim}{S}(k; -4b/a)$ with orientation specified by $K(\bar{k})=K(k)$ is called the dual knot of $k\subset\underset{\sim}{S}$ with respect to the normal presentation $a/4b$ of $s(k)$. Note that $s(\bar{k})=-4b/a$. Let $k'\subset\underset{\sim}{S}$ be another knot with $[k']\neq 0$ in $H_1(\underset{\sim}{S}; Z_2)=Z_2$, and $a'/4b'$ be a normal presentation of $s(k')$. Let $r=\#P(k)$, $r'=\#P(k')$.

LEMMA 9.2. *$\mu(\underset{\sim}{S}(k'; -4b'/a'))=\mu(\underset{\sim}{S}(k; -4b/a))$ if $a'b'r'^2\equiv abr^2$ (mod 8).*

PROOF. The proof is almost parallel to that of Lemma 6.1. Let $W'=\underset{\sim}{S}\times[-1, 1]\cup D^2\times D^2$, identifying $T(k')\times 1$ with $\partial D^2\times D^2$ so that $m'\times 1=p\times\partial D^2$, $\ell'\times 1=\partial D^2\times q$ $(p, q\in\partial D^2)$ for an $m.\ \ell.$ pair (m', ℓ') of $T(k')$ with $[K(k')]=a'[m']+4b'[\ell']$. Construct $\overline{W}=W'\cup -W$ identifying two copies of $\underset{\sim}{S}\times(-1)$ in W' and W. Since sign $\overline{W}=0$, it suffices to show that $\overline{W}$ is spin. Note that brk is homologous to $\varepsilon'b'r'k'$ in $\underset{\sim}{S}$ for some $\varepsilon'=\pm 1$, since $br[k]$ and $b'r'[k']$ are both generators of the 2-primary subgroup $H_1(\underset{\sim}{S})_2=Z_4$ of $H_1(\underset{\sim}{S}; Z)$. Let c be a 2-chain in $\underset{\sim}{S}\times(-1)$ with $\partial c=brk\times(-1)-\varepsilon'b'r'k'\times(-1)$. Let $\bar{c}=\varepsilon'b'r'\overline{D}'+c-br\overline{D}$ is an integral 2-cycle in $\overline{W}$, where $\overline{D}=k\times[-1, 1]\cup D^2\times 0\subset W$, $\overline{D}'=k'\times[-1, 1]\cup D^2\times 0\subset W'$. To see that $\overline{W}$ is spin, it is sufficient to check that $[\bar{c}]^2\equiv 0$ (mod 2) (cf. Proof of Lemma 6.1). The 2-cycle $\bar{c}$ is Q-homologous to $\varepsilon'b'r'\overline{D}'^{+}_{Q}-br\overline{D}^{+}_{Q}$, since $H_2(\underset{\sim}{S}; Q)=0$. By Lemma 2.6,

$$[\bar{c}]^2=b'^2r'^2(-a'/4b')-b^2r^2(-a/4b)=(-a'b'r'^2+abr^2)/4=0 \pmod 2).$$

This completes the proof.

Let ϕ_2 be the restriction of the non-singular linking pairing ϕ on $H_1(\underset{\sim}{S}; Z)$ to the 2-primary component $H_2(\underset{\sim}{S})_2 = Z_4$. The pairing ϕ_2 is represented by a 1×1-matrix $(-u/4)$ over Q/Z for $u=1$ or 3. The integer u $(=1$ or 3) is determined uniquely by the orientation-preserving homeomorphism type of $\underset{\sim}{S}$. Write $(\phi_2, H_1(\underset{\sim}{S})_2) \cong (-u/4)$. For a knot k in $\underset{\sim}{S}$ of order 4, we have $s(k)=u/4$ by Lemma 1.8 and $\sharp P(k)=1$.

DEFINITION 9.3. $\mu(Z_4, S)=\mu(\underset{\sim}{S}(k; -4/u))$ for any knot k in $\underset{\sim}{S}$ of order 4. Equivalently, $\mu(Z_4, S)=\mu(\underset{\sim}{S}(k'; -4b/a))$ for any knot k' in $\underset{\sim}{S}$ with $[k']\neq 0$ in $H_1(\underset{\sim}{S}; Z_2)$ and any normal presentation $a/4b$ of $s(k')$ such that $abr^2\equiv u \pmod 8$, where $r=\sharp P(k')$.

By Lemma 9.2, $\mu(Z_4, S)$ is an invariant of the equivariant, orientation-preserving homeomorphism type of (Z_4, S).

LEMMA 9.4. $-\mu(Z_4, S)=\mu(Z_4, -S)+1/8+\delta(Z_4, S)$.

PROOF. Let k be a knot in $\underset{\sim}{S}$ of order 4, so that $s(k)=u/4$ and $\sharp P(k)=1$. First, let $u=1$. Let $S'=\underset{\sim}{S}(k; -4/1)$, and k' be the dual flat knot in S' of k in $\underset{\sim}{S}$ with respect to the normal presentation 1/4. Let (m', ℓ') be an $m.\ell.$ pair of $T(k'\subset -S')$ with $\ell'=K(k')=K(k)$. By definition, $\mu(S')=\mu(Z_4, S)$. Since $s(k\subset -\underset{\sim}{S})=-1/4=3/4$, we can take an $m.\ell.$ pair (m^*, ℓ^*) of $T(k\subset -\underset{\sim}{S})$ such that $[K(k)]=3[m^*]+4[\ell^*]$. Let $S'^*=(-\underset{\sim}{S})(k; -4/3)$, and k'^* be the dual knot in S'^* of k in $-\underset{\sim}{S}$ with respect to the normal presentation 3/4. By definition, $\mu(S'^*)=\mu(Z_4, -S)$. Then we have

$$[m']=-[m'^*]+[\ell'^*],$$
$$[\ell']=[K(k')]=-4[m'^*]+3[\ell'^*]$$

for the $m.\ell.$ pair (m'^*, ℓ'^*) of k'^* in S'^*, identical with $(-\ell^*, m^*)$ on $\partial T(k'^*)$. By Theorem 3.3,

$$-\mu(Z_4, S)=\mu(Z_4, -S)+\mu(L(3, -4))+\delta(k'\subset -S').$$

$\delta(k'\subset -S')=\delta(Z_4, S)$ follows from Lemma 4.2 of [18] and Theorem 2.5 (cf. the proof of Lemma 6.3). $\mu(L(3, -4))=-\mu(L(3, 1))=1/8$. Hence

$$-\mu(Z_4, S)=\mu(Z_4, S)+1/8+\delta(Z_4, S).$$

Next, let $u=3$. In this case $s(k\subset -\underset{\sim}{S})=1/4$. The above argument shows that

$$-\mu(Z_4, -S)=\mu(Z_4, (-S))+1/8+\delta(Z_4, -S).$$

That is,

$$-\mu(Z_4, S)=\mu(Z_4, -S)+1/8+\delta(Z_4, S).$$

This completes the proof.

THEOREM 9.5. *For* $u=1$ *or* 3, *determined by* $(\phi_2, H_1(\underset{\sim}{S})_2)\cong(-u/4)$, *we have*

$$\mu(S)=\alpha(Z_4, S)/16+u/8+4\mu(Z_4, S)+\delta(Z_4, S)$$

in Q/Z.

PROOF. Let $\underset{\sim}{k}$ be a knot in $\underset{\sim}{S}$ of order 4, so that $s(\underset{\sim}{k})=u/4$ and $\sharp P(\underset{\sim}{k})=1$. Let $u=1$. Let $(\underset{\sim}{m}, \underset{\sim}{\ell})$ be an $m.\ell.$ pair of $T(\underset{\sim}{k})$ such that $[K(\underset{\sim}{k})]=[\underset{\sim}{m}]+4[\underset{\sim}{\ell}]$. Construct $\underset{\sim}{W}=\underset{\sim}{S}\times[-1, 1]\cup D^2\times D^2$ identifying $T(\underset{\sim}{k})\times 1$ with $\partial D^2\times D^2$ such that $\underset{\sim}{m}\times 1=p\times\partial D^2$, $\underset{\sim}{\ell}\times 1=\partial D^2\times q$ $(p, q\in\partial D^2)$. Let $W=S\times[-1, 1]\cup D^2\times D^2$ be a branched Z_4-covering space of $\underset{\sim}{W}$, branched over the disk $\underset{\sim}{D}'=D^2\times 0\subset\underset{\sim}{W}$. Let D' be the lift of $\underset{\sim}{D}'$. $S'=\partial W-S\times(-1)$ is a Z_2-homology 3-sphere. Let $\underset{\sim}{S}'=S'/Z_4=\partial\underset{\sim}{W}-\underset{\sim}{S}\times(-1)$. By definition, $\mu(Z_4, S)=\mu(\underset{\sim}{S}')$. $F(Z_2, S')=\partial D'$ is a flat knot in S'. By Corollary 8.3,

$$\mu(S')=\alpha(Z_4, S')/16+4\mu(\underset{\sim}{S}').$$

Let k be a flat knot in S which is the lift of $\underset{\sim}{k}$. By Theorem 2.5, $\delta(k'\subset S')=\delta(k\subset S)=\delta(Z_4, S)$. Note that $[D_Q'^+]^2=\operatorname{sign} W=\operatorname{sign}\underset{\sim}{W}=-1$. Then $\delta(k'\subset S')=\mu(S)-\mu(S')$, and $\alpha(Z_4, S')-\alpha(Z_4, S)=-\operatorname{sign} W+4\operatorname{sign}\underset{\sim}{W}-(4^2-1)[D_Q'^+]^2/3=2$. Therefore,

$$\mu(S)=\alpha(Z_4, S)/16+1/8+4\mu(Z_4, S)+\delta(Z_4, S).$$

Next, let $u=3$. We have $s(\underset{\sim}{k}\subset-\underset{\sim}{S})=1/4$. By the above argument,

$$\mu(-S)=\alpha(Z_4, -S)/16+1/8+4\mu(Z_4, -S)+\delta(Z_4, -S).$$

By Lemma 9.4,

$$\mu(S)=\alpha(Z_4, S)/16+3/8+4\mu(Z_4, S)+\delta(Z_4, S).$$

This completes the proof.

§ 10. The case of a free cyclic action of an order which is a power of 2 greater than four

Let S be a Z_2-homology 3-sphere with free Z_{2^n}-action, and $n\geq 3$. Let $\underset{\sim}{S}=S/Z_{2^n}$. Let ϕ_2 be the restriction of the linking pairing ϕ on $H_1(\underset{\sim}{S}; Z)$ to the 2-primary component $H_1(\underset{\sim}{S})_2=Z_{2^n}$. The pairing ϕ_2 is represented by a 1×1-matrix $(-u/2^n)$ over Q/Z for $u=\pm 1, \pm 3$ (cf. Wall [36]). The integer u $(=\pm 1, \pm 3)$ is determined uniquely by the orientation-preserving homeomorphism type of $\underset{\sim}{S}$. Write $(\phi_2, H_1(\underset{\sim}{S})_2)\cong(-u/2^n)$.

THEOREM 10.1. *For $u=\pm1, \pm3$, determined by $(\phi_2, H_1(\underset{\sim}{S})_2)\cong(-u/2^n)$, we have*

$$\mu(S)=\alpha(Z_{2^n}, S)/16+\nu(n, u)/8+\delta(Z_{2^n}, S)$$

in Q/Z, where

$$\nu(n, u)=\begin{cases} -u & (n=3, u=\pm1) \\ u/|u| & (n=3, u=\pm3) \\ 3u & (n>3, u=\pm1) \\ -u & (n>3, u=\pm3). \end{cases}$$

PROOF. Considering $-S$ in place of S, it is sufficient to prove this for $u=1$ or 3. Let $\underset{\sim}{k}$ be a knot in $\underset{\sim}{S}$ with $s(\underset{\sim}{k})=u/2^n$. Let $(\underset{\sim}{m}, \underset{\sim}{\ell})$ be an $m.\ \ell.$ pair of $T(\underset{\sim}{k})$ such that $[K(\underset{\sim}{k})]=u[\underset{\sim}{m}]+2^n[\underset{\sim}{\ell}]$. Let k be a flat knot in S which is the lift of $\underset{\sim}{k}$. Construct $\underset{\sim}{W}=\underset{\sim}{S}\times[-1, 1]\cup D^2\times D^2$, identifying $T(\underset{\sim}{k})\times 1$ with $\partial D^2\times D^2$ so that $\underset{\sim}{m}\times 1=p\times\partial D^2$, $\underset{\sim}{\ell}\times 1=\partial D^2\times q$ $(p, q\in\partial D^2)$. Let $W=S\times[-1, 1]\cup D^2\times D^2$ be a branched Z_{2^n}-covering space of $\underset{\sim}{W}$, branched over the disk $\underset{\sim}{D}'=D^2\times 0\subset\underset{\sim}{W}$. Let D' be the lift of $\underset{\sim}{D}'$. Note that $S'=\partial W-S\times(-1)$ and $\underset{\sim}{S}'=\partial\underset{\sim}{W}-\underset{\sim}{S}\times(-1)$ are Z_2-homology 3-spheres. Let $k'=\partial D'\subset S'$ and $\underset{\sim}{k}'=\partial\underset{\sim}{D}'\subset\underset{\sim}{S}'$. Let $u=1$. The knot k' is flat in S' and $k'=F(Z_{2^n}, S')$. By Corollary 8.4, $\mu(S')=\alpha(Z_{2^n}, S')/16$. By Theorem 2.5, $\delta(k'\subset S')=\delta(k\subset S)=\delta(Z_{2^n}, S)$. Note that $[D_Q'^+]^2=\text{sign } W=\text{sign }\underset{\sim}{W}=-1$. Then $\delta(k'\subset S')=\mu(S)-\mu(S')$ and $\alpha(Z_{2^n}, S')-\alpha(Z_{2^n}, S)=-\text{sign } W+2^n\text{ sign }\underset{\sim}{W}-\{(2^n)^2-1\}[D_Q'^+]^2/3=1-2^n+(4^n-1)/3$. So,

$$\mu(S)=\alpha(Z_{2^n}, S)/16+\{1-2^n+(4^n-1)/3\}/16+\delta(Z_{2^n}, S).$$

Let $\omega(n)=\{1-2^n+(4^n-1)/3\}/16 \pmod 1$. Clearly, $\omega(3)=-1/8$. For $n\geq4$, $2^n\equiv0 \pmod{16}$ and $4^n/3\equiv4^2/3 \pmod{16}$, for $4^n/3=(3+1)4^{n-1}/3=4^{n-1}+4^n/3$. Thus, for $n\geq4$ $\omega(n)\equiv3/8 \pmod 1$. The case $u=\pm1$ was obtained. Next, let $u=3$. Letting $\underset{\sim}{m}'=-\underset{\sim}{\ell}$ and $\underset{\sim}{\ell}'=\underset{\sim}{m}$, the pair $(\underset{\sim}{m}', \underset{\sim}{\ell}')$ is an $m.\ \ell.$ pair of $T(\underset{\sim}{k}'\subset\underset{\sim}{S}')$ such that $[K(\underset{\sim}{k}')]=-2^n[\underset{\sim}{m}']+3[\underset{\sim}{\ell}']$. Let (m', ℓ') be an $m.\ \ell.$ pair of $T(k'\subset S')$, obtained from the lift of the pair $(\underset{\sim}{m}', \underset{\sim}{\ell}')$ of $T(\underset{\sim}{k}'\subset\underset{\sim}{S}')$. We have $[K(k')]=-[m']+3[\ell']$, so that $[D_Q'^+]^2=-1/3$ and sign $W=\text{sign }\underset{\sim}{W}=-1$. Then $\delta(k'\subset S')=\{(-1/3)+1\}/16-\mu(S')+\mu(S)$, and $\alpha(Z_{2^n}, S')-\alpha(Z_{2^n}, S)=-\text{sign } W+2^n\text{sign }\underset{\sim}{W}-\{(2^n)^2-1\}[D_Q'^+]^2/3=1-2^n+(4^n-1)/9$. By Theorem 2.7, $\delta(k'\subset S')=\delta(K(k')\subset S')+(5/3-15)/16$, since $s(k'\subset S')=-1/3=5/3$. By Corollary 3.6, $\delta(K(k')\subset S')=\delta(k\subset S)=\delta(Z_{2^n}, S)$. So, $\mu(S)=\mu(S')+1/8+\delta(Z_{2^n}, S)$. By Corollary 8.5, $3\mu(S')=3\alpha(Z_{2^n}, S')/16$. Therefore,

$$3\mu(S)=3\alpha(Z_{2^n}, S)/16+\omega'(n)+3\delta(Z_{2^n}, S),$$

where $\omega'(n)=\{3-3.2^n+(4^n-1)/3\}/16+3/8 \pmod 1$. For $n\geq 4$, $3.2^n\equiv 0 \pmod{16}$ and $4^n/3\equiv 4^2/3 \pmod{16}$, so that $\omega'(n)\equiv 1/2+3/8\equiv 3(-3/8)\equiv 3\nu(n, 3)/8 \pmod 1$. Clearly, $\omega'(3)=3/8=3\nu(3, 3)/8$. Because $\{(2^n)^2-1\}/3$ is an integer, we see from the definition of the α-invariant that $\alpha(Z_{2^n}, S)$ is an integer. Therefore,

$$\mu(S)=\alpha(Z_{2^n}, S)/16+\nu(n, 3)/8+\delta(Z_{2^n}, S).$$

The case $u=\pm 3$ was obtained. This completes the proof.

§ 11. The case of a free cyclic action of odd-prime order

Let S be a Z_2-homology 3-sphere with free Z_p-action for an odd-prime number p. Let $\underset{\sim}{S}=S/Z_p$. Clearly, $\underset{\sim}{S}$ is a Z_2-homology 3-sphere.

THEOREM 11.1. *In Q/Z,*

$$\mu(S)=\begin{cases} 9\alpha(Z_3, S)/16+3\mu(\underset{\sim}{S}) & (p=3) \\ \alpha(Z_p, S)/16+p\mu(\underset{\sim}{S}) & (p>3). \end{cases}$$

PROOF. Let $\underset{\sim}{W}$ be a simply connected spin 4-manifold with $\partial \underset{\sim}{W}=\underset{\sim}{S}$. By [5], the projection $pr\colon S\to \underset{\sim}{S}$ extends to a branched Z_p-covering $W\to \underset{\sim}{W}$ with $F(Z_p, W)=F$, a locally flat, closed orientable surface. By Lemma 5.1, W is spin. By definition,

$$\alpha(Z_p, S)=-\operatorname{sign} W+p\operatorname{sign}\underset{\sim}{W}-(p^2-1)[F]^2/3.$$

For $p>3$, we show that $(p^2-1)[F]^2/3\equiv 0 \pmod{16}$. Note that $[F]^2$ is an even integer since W is spin. We have $(p^2-1)/3\equiv 0 \pmod 8$. In fact, letting $p=4n+\varepsilon$ for $\varepsilon=\pm 1$ and $n\geq 1$, $(p^2-1)/3=8n(2n+\varepsilon)/3\equiv 0 \pmod 8$ if $n\equiv 0$ or $\varepsilon \pmod 3$. If $n\equiv -\varepsilon \pmod 3$, then by letting $n=3n'-\varepsilon$, we have $p=3(4n'-\varepsilon)$, which is impossible since p is a prime number >3. Hence $(p^2-1)/3\equiv 0 \pmod 8$ and $(p^2-1)[F]^2/3\equiv 0 \pmod{16}$ for $p>3$. Thus, $\alpha(Z_p, S)/16=\mu(S)-p\mu(\underset{\sim}{S})$ for $p>3$. When $p=3$, $9\alpha(Z_3, S)=-9\operatorname{sign} W+9.3\operatorname{sign}\underset{\sim}{W}-3.8[F]^2$. Using that $8\mu(S)=8\mu(\underset{\sim}{S})=0$ and $8[F]^2\equiv 0 \pmod{16}$, we have that

$$9\alpha(Z_3, S)/16=\mu(S)-3\mu(\underset{\sim}{S}).$$

This completes the proof.

§ 12. The case of a non-free cyclic action of odd-prime order

Let S be a Z_2-homology 3-sphere with non-free Z_p-action for an odd-prime number p. $F(Z_p, S)=L$ is a link and $\underset{\sim}{S}=S/Z_p$ is a Z_2-homology 3-sphere. Note that L is not connected in general, though if $H_1(S; Z_p)=0$, then L is a knot by Smith theory. Many examples of a Z_2-homology 3-sphere having a Z_p-action with disconnected fixed point set can be obtained by taking branched Z_p-coverings of a 3-sphere, branched over many classical links (cf. Hosokawa-Kinoshita [15], Sakuma [32]). Note that $(p^2-1)/24$ is an integer for $p>3$ (cf. the proof of Theorem 11.1).

THEOREM 12.1. *Let $k_1, \cdots, k_r$ be the components of $L=F(Z_p, S)$. For a normal presentation $2a_i/b_i$ of the slope $s(k_i)$, $i=1, 2, \cdots, r$, we have*

$$\mu(S)=\begin{cases}9\alpha(Z_3, S)/16+3\mu(\underset{\sim}{S})+3\sum_{i=1}^r a_i/b_i & (p=3)\\ \alpha(Z_p, S)/16+p\mu(\underset{\sim}{S})+\{(p^2-1)/24\}\sum_{i=1}^r a_i/b_i & (p>3)\end{cases}$$

in Q/Z.

PROOF. Let $2\underset{\sim}{a}_i/\underset{\sim}{b}_i$ be a normal presentation of $s(\underset{\sim}{k}_i\subset\underset{\sim}{S})$ and $(\underset{\sim}{m}_i, \underset{\sim}{\ell}_i)$ be an $m.\ell.$ pair of $T(\underset{\sim}{k}_i)$ with $[K(\underset{\sim}{k}_i)]=2\underset{\sim}{a}_i[\underset{\sim}{m}_i]+\underset{\sim}{b}_i[\underset{\sim}{\ell}_i]$ such that the lift of $\underset{\sim}{\ell}_i$ has p components. Construct $\underset{\sim}{W}=\underset{\sim}{S}\times[-1, 1]\cup D^2\times D_1^2\cup\cdots\cup D^2\times D_r^2$ identifying $T(\underset{\sim}{k}_i)\times 1$ with $\partial D^2\times D_i^2$ so that $\underset{\sim}{m}_i\times 1=\underset{\sim}{p}\times\partial D_i^2$, $\underset{\sim}{\ell}_i\times 1=\partial D^2\times\underset{\sim}{q}_i$ $(\underset{\sim}{p}, \underset{\sim}{q}\in\partial D^2)$. Note that $\underset{\sim}{W}$ is spin. Let $\underset{\sim}{M}=\partial\underset{\sim}{W}-\underset{\sim}{S}\times(-1)$. By Milnor [24] or Kaplan [16], we find a simply connected spin 4-manifold $\underset{\sim}{W}_1$ with $\partial\underset{\sim}{W}_1=-\underset{\sim}{M}$ such that $\overline{\underset{\sim}{W}}=\underset{\sim}{W}\cup\underset{\sim}{W}_1$ is spin. By [5], the projection $pr\colon S\to\underset{\sim}{S}$ extends to a branched Z_p-covering $\overline{W}\to\overline{\underset{\sim}{W}}$ such that $F(Z_p, \overline{W})=\bar{D}_1\cup\cdots\cup\bar{D}_r\cup F_1$, where $\bar{D}_i$ is the lift of the disk $\underset{\sim}{k}_i\times[-1, 1]\cup D^2\times 0_i$ and F_1 is a locally flat closed orientable surface. By Lemma 5.1, $\overline{W}$ is spin. By definition, $-\alpha(Z_p, S)=-\operatorname{sign}\overline{W}+p\operatorname{sign}\overline{\underset{\sim}{W}}-(p^2-1)(\sum_{i=1}^r[\bar{D}_{iQ}^+]^2+[F_1]^2)/3$. By Lemma 2.6 and the proof of Lemma 4.5, $[\bar{D}_{iQ}^+]^2=-2\underset{\sim}{a}_i/p\underset{\sim}{b}_i\equiv-s(k_i\subset S)\equiv-2a_i/b_i \pmod 1$, so that $[\bar{D}_{iQ}^+]^2\equiv-2a_i/b_i \pmod 2$, for b_i is odd. First, let $p>3$. Since $(p^2-1)/3\equiv 0 \pmod 8$ and $[F_1]^2$ is even, we have

$$-\alpha(Z_p, S)/16=-\mu(S)+p\mu(\underset{\sim}{S})+\{(p^2-1)/24\}\sum_{i=1}^r a_i/b_i$$

in Q/Z. Next, let $p=3$. Since $-9\alpha(Z_3, S)=-9\operatorname{sign}\overline{W}+9.3\operatorname{sign}\overline{\underset{\sim}{W}}-3.8$ $(\sum_{i=1}^r[\bar{D}_{iQ}^+]^2+[F_1]^2)$ and $8\mu(S)=8\mu(\underset{\sim}{S})=0$, we see that

$$-9\alpha(Z_3, S)/16=-\mu(S)+3\mu(\underset{\sim}{S})+3\sum_{i=1}^r a_i/b_i$$

in Q/Z. This completes the proof.

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