Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link

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ABSTRACT

There is a question asking whether a handle-irreducible summand of every stable-ribbon surface-link is a unique ribbon surface-link. This question for the case of a trivial surface-link is affirmatively answered. That is, a handle-irreducible summand of every stably trivial surface-link is only a trivial 2-link. By combining this result with an old result of F. Hosowaka and the author that every surface-knot with infinite cyclic fundamental group is a stably trivial surface-knot, it is concluded that every surface-knot with infinite cyclic fundamental group is a trivial (i.e., an unknotted) surface-knot.

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1 Introduction

A *surface-link* is a closed oriented (possibly disconnected) surface F embedded in the 4-space $\mathbf{R}^4$ by a smooth (or a piecewise-linear locally flat) embedding. When a (possibly disconnected) closed surface $\mathbf{F}$ is fixed, it is also called an $\mathbf{F}$-*link*. If $\mathbf{F}$ is the disjoint union of some copies of the 2-sphere S^2, then it is also called a 2-*link*. When $\mathbf{F}$ is connected, it is also called a *surface-knot*, and a 2-*knot* for $\mathbf{F} = S^2$.

Two surface-links F and F' are *equivalent* by an *equivalence* f if F is sent to F' orientation-preservingly by an orientation-preserving diffeomorphism (or piecewise-linear homeomorphism) $f : \mathbf{R}^4 \to \mathbf{R}^4$. The notation $F \cong F'$ is used for equivalent surface-links F, F'. A *trivial* surface-link is a surface-link F which is the boundary of the union of mutually disjoint handlebodies smoothly embedded in $\mathbf{R}^4$, where a handlebody is a 3-manifold which is a 3-ball, a solid torus or a boundary-disk sum of some number of solid tori. A trivial surface-knot is also called an *unknotted* surface-knot. A trivial disconnected surface-link is also called an *unknotted and unlinked* surface-link. For any given closed oriented (possibly disconnected) surface $\mathbf{F}$, a trivial $\mathbf{F}$-link exists uniquely up to equivalences (see [6]). A *ribbon* surface-link is a surface-link F which is obtained from a trivial 2-link O by the surgery along an embedded 1-handle system (see [10, 11, 12, 13], [16, II]). A *stabilization* of a surface-link F is a connected sum $F^{\#sT} = F\#_{k=1}^{s} T_k$ of F and a system of trivial torus-knots $T_k\,(k = 1, 2, \ldots, s)$. By granting $s = 0$, we understand that a surface-link F itself is a stabilization of F. The trivial torus-knot system $T_k\,(k = 1, 2, ..., s)$ is called the *stabilizer* on the stabilization $F^{\#sT}$ of F.

A *stable-ribbon* surface-link is a surface-link F such that a stabilization $F^{\#sT}$ of F is a ribbon surface-link. For every surface-link F, there is a surface-link F^* with minimal total genus such that F is equivalent to a stabilization of F^*. The surface-link F^* is called a *handle-irreducible summand* of F. The following question is a central question.

Question 1.0. A handle-irreducible summand of every stable-ribbon surface-link is a ribbon surface-link which is unique up to equivalences ?

A *stably trivial* surface-link is a surface-link F such that a stabilization of F is a trivial surface-link.

In this paper, the following theorem is shown answering affirmatively this question for the case of a stably trivial surface-link. This question in the general case will be answered affirmatively in [15].

Theorem 1.1. Any handle-irreducible summand of every stably trivial surface-link is a trivial 2-link.

The following corollary is directly obtained from Theorem 1.1:

Corollary 1.2. Every stably trivial surface-link is a trivial surface-link.

If a surface-knot F has an infinite cyclic fundamental group, then F is a TOP-trivial surface-knot, which was shown by Freedman for a 2-knot and by [3, 9] for a higher genus surface-knot. In the case of a piecewise linear surface-knot(equivalent to a smooth surface-knot), it is known by [6] that a stabilization of the surface-knot F is a trivial surface-knot, namely the surface-knot F is a stably trivial surface-knot. Hence the following corollary is directly obtained from Corollary 1.2 answering the problem [17, Problem 1.55(A)] on unknotting of a 2-knot positively (see [14] for the surface-link version):

Corollary 1.3. A surface-knot F is a trivial surface-knot if the fundamental group $\pi_1(\mathbf{R}^4 \setminus F)$ is an infinite cyclic group.

The *exterior* of a surface-knot F is the 4-manifold $E = \mathrm{cl}(\mathbf{R}^4 \setminus N(F))$ for a tubular neighborhood $N(F)$ of F in $\mathbf{R}^4$. Then the boundary ∂E is a trivial circle bundle over F. A surface-knot F is *of Dehn's type* if there is a section F' of F in the bundle ∂E such that the inclusion $F' \to E$ is homotopic to a constant map. By [3, Corollary 4.2], the fundamental group $\pi_1(\mathbf{R}^4 \setminus F)$ of a surface-knot F of Dehn's type is an infinite cyclic group. Thus, we have the following corollary(answering the problem [17, Problem 1.51)] on unknotting of a 2-knot of Dehn's type positively):

Corollary 1.4. A surface-knot of Dehn's type is a trivial surface-knot.

Unknotting Conjecture asks whether an n-knot K^n(i.e., a smooth embedding image of the n-sphere S^n in the(n+2)-sphere S^{n+2}) is unknotted (i.e., bounds a smooth $(n+1)$-ball in S^{n+2}) if and only if the complement $S^{n+2} \backslash K^n$ is homotopy equivalent to S^1 (see [8] for example). This conjecture was previously known to be true for $n > 3$ by [18], for $n = 3$ by [20] and for $n = 1$ by [5, 19]. The conjecture for $n = 2$ was known only in the TOP category by [1](see also [2]). Corollary 1.3 answers this finally remained smooth unknotting conjecture affirmatively and hence Unknotting Conjecture can be changed into the following:

Unknotting Theorem. A smooth S^n-knot K^n in S^{n+2} is unknotted if and only if the complement $S^{n+2} \backslash K^n$ is homotopy equivalent to S^1 for every $n \geq 1$.

A main idea in our argument is to use the surgery of a surface-link on an orthogonal 2-handle pair, which is much different from the surgery of a surface-link on a single 2-handle. It is known that every surface-link F in $\mathbf{R}^4$ is obtained from a higher genus trivial surface-knot F' by the surgery of F' on a system of mutually disjoint 2-handles, because a handlebody in $\mathbf{R}^4$ is obtained from a connected Seifert hypersurface of F by removing mutually disjoint 1-handles (see [6]). Thus, for example, every 2-twist spun 2-bridge knot in [21] is obtained from a trivial torus-knot T in $\mathbf{R}^4$ by the surgery of T on a single 2-handle, because it bounds a once-punctured lens space as a Seifert hypersurface.

In Section 2, it is shown that every stably trivial surface-link is a trivial surface-link if and only if the uniqueness of an orthogonal 2-handle pair on every trivial surface-link holds. In Section 3, the uniqueness of every orthogonal 2-handle pair on every surface-link is shown, by which Theorem 1.1 is obtained.

2 A triviality condition on a stably trivial surface-link

A 2-*handle* on a surface-link F in $\mathbf{R}^4$ is an embedded 2-handle $D \times I$ on F with D a core disk such that $D \times I \cap F = \partial D \times I$, where I denotes a closed interval containing 0 and $D \times 0$ is identified with D. If D is an immersed disk, then call it an *immersed 2-handle*. Two (possibly immersed) 2-handles $D \times I$ and $E \times I$ on F are *equivalent* if there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself such that the restriction $f|_F : F \to F$ is the identity map and $f(D \times I) = E \times I$.

An *orthogonal 2-handle pair* (or simply, an *O2-handle pair*) on F is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$, $D' \times I$ on F such that

$$D \times I \cap D' \times I = \partial D \times I \cap \partial D' \times I$$

and $\partial D \times I$ and $\partial D' \times I$ *meet orthogonally* on F, that is, the boundary circles ∂D and $\partial D'$ meet transversely at one point p and the intersection $\partial D \times I \cap \partial D' \times I$ is homeomorphic to the square $Q = p \times I \times I$ (see Fig. 1).

Let $(D \times I, D' \times I)$ be an O2-handle pair on a surface-link F. Let $F(D \times I)$ and $F(D' \times I)$ be the surface-links obtained from F by the surgeries along $D \times I$ and $D' \times I$, respectively. Let $F(D \times I, D' \times I)$ be the surface-link which is the union of the plumbed disk

$$\delta = \delta_{D \times I, D' \times I} = D \times \partial I \cup Q \cup D' \times partial I$$

and the surface

$$F^c_\delta = \mathrm{cl}(F \setminus (\partial D \times I \cup \partial D' \times I).$$

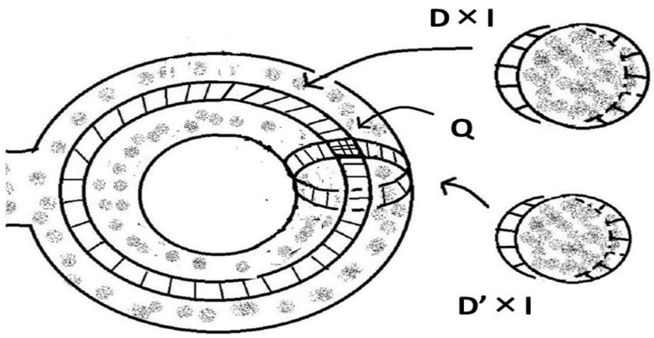

Figure 1: An orthogonal 2-handle pair(=: an O2-handle pair)

A once-punctured torus T^o in a 3-ball B is *trivial* if T^o is smoothly and properly embedded in B which splits B into two solid tori. A *bump* of a surface-link F is a 3-ball B in $\mathbf{R}^4$ with $F \cap B = T^o$ a trivial once-punctured torus in B. Let $F(B)$ be a surface-link $F^c_B \cup \delta_B$ for the surface $F^c_B = \mathrm{cl}(F \setminus T^o)$ and a disk δ_B in ∂B with $\partial\delta_B = \partial T^o$, where note that $F(B)$ is uniquely determined up to cellular moves on δ_B keeping F^c_B fixed. Here, a *cellular move* of a surface P in $\mathbf{R}^4$ is a surface $\tilde{P}$ in $\mathbf{R}^4$ such that the complements $d = \mathrm{cl}(P \setminus P_0)$ and $\tilde{d} = \mathrm{cl}(\tilde{P} \setminus P_0)$ of the intersection $P_0 = P \cap P'$ are disks in the interiors of P and $\tilde{P}$, respectively and the union $d \cup \tilde{d}$ is a 2-sphere bounding a 3-ball smoothly embedded in $\mathbf{R}^4$ and not meeting $P_0 \setminus \partial d = P_0 \setminus \partial\tilde{d}$.

For an O2-handle pair $(D \times I, D' \times I)$ on a surface-link F, let $\Delta = D \times I \cup D' \times I$ is a 3-ball in $\mathbf{R}^4$ called the *2-handle union*. Consider the 3-ball Δ as a Seifert hypersurface of the trivial S^2-knot $K = \partial\Delta$ in $\mathbf{R}^4$ to construct a 3-ball B_Δ obtained from Δ by adding an outer boundary collar. This 3-ball B_Δ is a bump of F, which we call the *associated bump* of the O2-handle pair $(D \times I, D' \times I)$. When the 3-ball Δ and a boundary collar of F^c_δ are deformed into the 3-space $\mathbf{R}^3$, this associated bump B_Δ is also considered as a regular neighborhood of Δ in $\mathbf{R}^3$ (see Fig. 2).

The following lemma shows that giving an O2-handle unordered pair on a surface-link F is the same as giving a bump of F.

Lemma 2.1. An O2-handle unordered pair $(D \times I, D' \times I)$ on a surface-link F is uniquely constructed from any given bump B of F in $\mathbf{R}^4$ with $F(D \times I, D' \times I) \cong F(B)$.

Proof of Lemma 2.1. For a bump B of F, the set of two solid tori bounded by $T^o = F \cap B$ is unique, whose meridian-longitude disk pair is an O2-handle pair. □

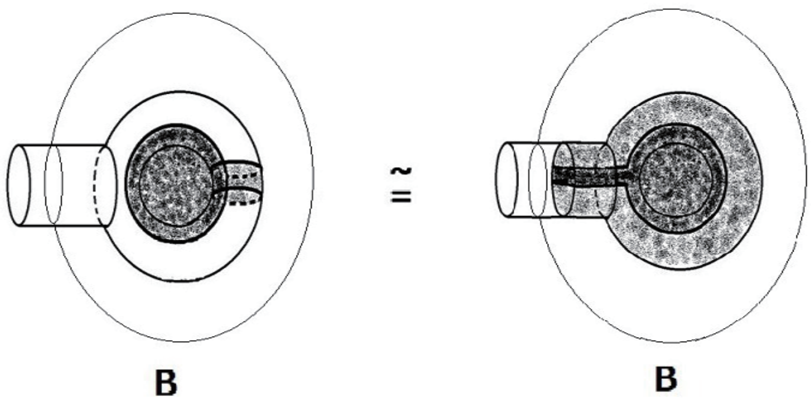

Figure 2: An associated bump B of a 2-handle union

The following lemma shows the uniqueness of the surgery of a surface-link F by an O2-handle pair.

Lemma 2.2. For any O2-handle pair $(D \times I, D' \times I)$ on any surface-link F and the associated bump B, there are equivalences

$$F(B) \cong F(D \times I, D' \times I) \cong F(D \times I) \cong F(D' \times I).$$

Further, these equivalences are attained by cellular moves keeping F_δ^c fixed.

Proof of Lemma 2.2. By definition, we have $F(B) \cong F(D \times I, D' \times I)$. The surface-link $F(D \times I, D' \times I)$ is equivalent to $F(D \times I)$ and $F(D' \times I)$ by cellular moves on the 3-balls $D' \times I$ and $D \times I$, respectively. □

Two O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on a surface-link F with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$ are *equivalent* if there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself such that the restriction $f|_F : F \to F$ is the identity map and $f(D \times I) = E \times I$ and $f(D' \times I) = E' \times I$.

The following characterization of equivalent O2-handle pairs is useful.

Lemma 2.3. Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be O2-handle pairs on a surface-link F with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. Let

$$F(D \times I, D' \times I) = F^c_\delta \cup \delta_{D\times I, D'\times I} \quad \text{and} \quad F(E \times I, E' \times I) = F^c_\delta \cup \delta_{E\times I, E'\times I}$$

for the plumbed disks $\delta_{D\times I, D'\times I}$ and $\delta_{E\times I, E'\times I}$. Then the O2-handle pairs $(D\times I, D'\times I)$ and $(E\times I, E'\times I)$ are equivalent if and only if there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from $F(D \times I, D' \times I)$ to $F(E \times I, E' \times I)$ such that the restriction $f|_{F^c_\delta} : F^c_\delta \to F^c_\delta$ is the identity map and $f(\delta_{D\times I, D'\times I}) = \delta_{E\times I, E'\times I}$.

Proof of Lemma 2.3. It suffices to show the "if"part since the "only if"part is obtained from the definition of equivalent O2-handle pairs. Assume that there is an equivalence f from $F(D \times I, D' \times I)$ to $F(E \times I, E' \times I)$ such that the restriction $f|_{F^c_\delta} : F^c_\delta \to F^c_\delta$ is the identity map and $f(\delta_{D\times I, D'\times I}) = \delta_{E\times I, E'\times I}$. The map f is isotopic to a diffeomorphism $f' : \mathbf{R}^4 \to \mathbf{R}^4$ sending the associated bump $B_{\Delta(D\times I, D'\times I)}$ of $(D\times I, D'\times I)$ to the associated bump $B_{\Delta(E\times I, E'\times I)}$ of $(E\times I, E'\times I)$ by regarding $B_{\Delta(D\times I, D'\times I)}$ and $B_{\Delta(E\times I, E'\times I)}$ as collars of $\delta_{D\times I, D'\times I}$ and $\delta_{E\times I, E'\times I}$, respectively. The diffeomorphism $f' : \mathbf{R}^4 \to \mathbf{R}^4$ is modified into an equivalence $f'' : \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself such that the restriction $f''|_F : F \to F$ is the identity map and $f''(D\times I) = E\times I$ and $f''(D'\times I) = E'\times I$. Thus, the O2-handle pairs $(D\times I, D'\times I)$ and $(E\times I, E'\times I)$ are equivalent. □

The following corollary is a concrete application of Lemma 2.3.

Corollary 2.4. Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be O2-handle pairs on a surface-link F with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. If the surface-link $F(D\times I, D'\times I)$ is obtained from the surface-link $F(E\times I, E'\times I)$ by a finite number of cellular moves on $D \times I$, $D' \times I$, $E \times I$ and $E' \times I$ keeping F^c_δ fixed, then the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ are equivalent.

Proof of Corollary 2.4. By the assumption, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from $F(D\times I, D'\times I)$ to $F(E\times I, E'\times I)$ such that the restriction $f|_{F^c_\delta} : F^c_\delta \to F^c_\delta$ is the identity map and $f(\delta_{D\times I, D'\times I}) = \delta_{E\times I, E'\times I}$. By Lemma 2.3, the result is obtained. □

A surface-link F has *only unique O2-handle pair* if any two O2-handle pairs on F with the same attaching part are equivalent. A surface-link not admitting any O2-handle pair is understood as a surface-link with only unique O2-handle pair.

We have the following characterization on a stably trivial surface-link.

Lemma 2.5. The following (1)-(3) are mutually equivalent.

(1) If a connected sum $F\#T$ of a surface-link F and a trivial torus-knot T is a trivial surface-link, then F is a trivial surface-link.

(2) If F is a trivial surface-link and $(D\times I, D'\times I)$ is an O2-handle pair on F, then $F(D\times I, D'\times I)$ is a trivial surface-link.

(3) Any trivial surface-link has only unique O2-handle pair.

Proof of Lemma 2.5. (1) $\Rightarrow$ (2): Let B be the associated bump of the O2-handle pair $(D\times I, D'\times I)$. A 4-ball A obtained by taking a bi-collar $c(B\times[-1,1])$ of B in $\mathbf{R}^4$ with $c(B\times 0)=B$ gives a connected sum decomposition $F\cong F(D\times I, D'\times I)\#T$. By (1), $F(D\times I, D'\times I)$ is a trivial surface-link.

(2) $\Rightarrow$ (3): Let $(D\times I, D'\times I)$ and $(E\times I, E'\times I)$ be O2-handle pairs with $\partial D\times I=\partial E\times I$ and $\partial D'\times I=\partial E'\times I$. Let $F(D\times I, D'\times I)=F^c_\delta\cup\delta_{D\times I,D'\times I}$ and $F(E\times I, E'\times I)=F^c_\delta\cup\delta_{E\times I,E'\times I}$ be trivial surface-links for disks $\delta_{D\times I,D'\times I}$ and $\delta_{E\times I,E'\times I}$ in the boundaries $\partial\Delta(D\times I, D'\times I)$ and $\partial\Delta(E\times I, E'\times I)$ of the 2-handle unions $\Delta(D\times I, D'\times I)$ and $\Delta(E\times I, E'\times I)$, respectively. Let $F(D\times I, D'\times I)_0$ and $F(E\times I, E'\times I)_0$ be the components of $F(D\times I, D'\times I)$ and $F(E\times I, E'\times I)$ containing the loop $\partial\delta_{D\times I,D'\times I}=\partial\delta_{E\times I,E'\times I}$, respectively, which are made split from the other components in $\mathbf{R}^4$ because all the components of every trivial surface-link are split in $\mathbf{R}^4$. Since $F(D\times I, D'\times I)_0$ and $F(E\times I, E'\times I)_0$ are trivial surface-knots of the same genus, there is an equivalence $f:\mathbf{R}^4\to\mathbf{R}^4$ sending $F(D\times I, D'\times I)_0$ to $F(E\times I, E'\times I)_0$ orientation-preservingly and the other components identically. By a cellular move of $\delta_{D\times I,D'\times I}$ in $F(D\times I, D'\times I)_0$, this map f is modified to have $f(\delta_{D\times I,D'\times I})=\delta_{E\times I,E'\times I}$. Further, this map f is modified to send $F^c_\delta\cup\delta_{D\times I,D'\times I}$ to $F^c_\delta\cup\delta_{E\times I,E'\times I}$ by sending all the components except for $F(D\times I, D'\times I)_0$ and $F(E\times I, E'\times I)_0$ identically. Thus, we have an equivalence f with $f(F^c_\delta)=F^c_\delta$ and $f(\delta_{D\times I,D'\times I})=\delta_{E\times I,E'\times I}$. By Lemma 2.3, the O2-handle pairs $(D\times I, D'\times I)$ and $(E\times I, E'\times I)$ are equivalent.

(3) $\Rightarrow$ (1): Let $F_i\,(i=0,1,\dots,r)$ be the components of F, and $F\#T=F_0\#T\cup F_1\cup\dots\cup F_r$ a trivial surface-link. Let V be the disjoint union of handlebodies $V_i\,(i=0,1,\dots,r)$ in $\mathbf{R}^4$ such that $\partial V_0=F_0\#T$ and $\partial V_i=F_i\,(i=1,2,\dots,r)$.

A *loop basis* of $F_0\#T$ of genus $g+1$ is a system of oriented simple loop pairs $(e_j,e'_j)\,(j=0,1,2,\dots,g)$ on $F_0\#T$ representing a basis for $H_1(F_0\#T;\mathbb{Z})$ such that $e_j\cap e_{j'}=e'_j\cap e'_{j'}=e_j\cap e'_{j'}=\emptyset$ for all distinct j,j' and $e_j\cap e'_j$ is one point with the intersection number $\mathrm{Int}(e_j,e'_j)=+1$ in $F_0\#T$ for all j. A loop basis $(e_j,e'_j)\,(j=0,1,2,\dots,g)$ of $F_0\#T$ is *spin* if the $\mathbb{Z}_2$-quadratic function $q:H_1(F_0\#T;\mathbb{Z}_2)\to\mathbb{Z}_2$ associated with the surface-knot $F_0\#T$ has $q(e_j)=q(e'_j)=0$ for all j. The following result is obtained from [3, Lemma 2.2] where a non-oriented spin loop basis

$(e_j, e'_j)\,(j = 0, 1, 2, \ldots, g)$ of $F_0\#T$ is constructed.

(2.5.1) For a surface-knot $F_0\#T$ of genus $g+1$ in $\mathbf{R}^4$, there is a spin loop basis $(e_j, e'_j)\,(j = 0, 1, 2, \ldots, g)$ of $F_0\#T$. In particular, for a trivial surface-knot $F_0\#T$ bounded by a handlebody V_0 in $\mathbf{R}^4$, every loop basis $(e_j, e'_j)\,(j = 0, 1, 2, \ldots, g)$ on ∂V_0 with $e'_j\,(j = 0, 1, 2, \ldots, g)$ a meridian loop system of V_0 has $q(e'_j) = 0$ and either $q(e_j) = 0$ or $q(e_j + e'_j) = 0$ for all j, where $e_j + e'_j$ denotes a Dehn twist of e_j along e'_j.

The following result is obtained from [4]:

(2.5.2) For any two loop bases $(e_j, e'_j)\,(j = 0, 1, 2, \ldots, g)$ and $(\tilde{e}_j, \tilde{e}'_j)\,(j = 0, 1, 2, \ldots, g)$ on a trivial genus g surface-knot $F_0\#T$ with $q(e_j) = q(\tilde{e}_j)$ and $q(e'_j) = q(\tilde{e}'_j)$ for all j, there is an orientation-preserving diffeomorphism $f : \mathbf{R}^4 \to \mathbf{R}^4$ with $f(F_0\#T) = F_0\#T$ such that $f(e_j) = \tilde{e}_j$ and $f(e'_j) = \tilde{e}'_j$ for all j.

Let $(D \times I, D' \times I)$ be an O2-handle pair on $F\#T$ in $\mathbf{R}^4$ attached to T^o such that $(F\#T)(D \times I, D' \times I) \cong F$. By (2.5.1), there is a spin loop basis for $F_0\#T$ containing the pair $(\partial D, \partial D')$. Also, let $(e_i, e'_i)\,(i = 0, 1, 2, \ldots, g)$ be a spin loop basis for $F_0\#T$ such that e_0 bounds a disk d in $\mathbf{R}^4$ with $d \cap V = e_0$ and e'_0 bounds a meridian disk d' of V_0. Since the handlebodies $V_i\,(i = 0, 1, \ldots, r)$ are splittable in $\mathbf{R}^4$ by [6], we see from (2.5.2) that there is an orientation-preserving diffeomorphism $f : \mathbf{R}^4 \to \mathbf{R}^4$ with $f(F_0\#T) = F_0\#T$ and $f|_{V_i} = 1\,(i = 1, 2, \ldots, r)$ such that $f(\partial D) = e_0$ and $f(\partial D') = e'_0$. A thickening pair $(d \times I, d' \times I)$ of the disk pair (d, d') is an O2-handle pair with $(F\#T)(d \times I, d' \times I)$ is a trivial surface-knot. Since $(f(D) \times I, f(D') \times I)$ is an O2-handle pair on $F\#T$, we obtain from (3) that

$$
\begin{aligned}
F &\cong (F\#T)(D \times I, D' \times I) \\
&\cong (F\#T)(f(D) \times I, f(D') \times I) \\
&\cong (F\#T)(d \times I, d' \times I).
\end{aligned}
$$

Thus, F is a trivial surface-link. □

3 Uniqueness of an orthogonal 2-handle pair

The following theorem is our main result.

Theorem 3.1. Any (not necessarily trivial) surface-link has only unique O2-handle pair.

Theorem 1.1 is proved by Theorem 3.1 and Lemma 2.5, which is done as follows:

Proof of Theorem 1.1. Let F be a stably trivial link. That is, assume that a stabilization $F^{\#sT} = F\#_{k=1}^{s} T_k$ of F is a trivial link for some $s \geq 1$. By Theorem 3.1 and Lemma 2.5, $F\#_{k=1}^{s-1} T_k$ is a trivial surface-link. Inductively, F is a surface-link, so that any handle-irreducible summand F^* of F is a trivial S^2-link. □

The following lemma is a key lemma to Theorem 3.1.

Lemma 3.2. Let $(D \times I, D' \times I)$ and $(E' \times I, E' \times I)$ be O2-handle pairs on a surface-link F in $\mathbf{R}^4$ with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. Then there is a 2-handle $D'_* \times I$ on F with $\partial D'_* = \partial D'$ such that the pair $(E \times I, D'_* \times I)$ is an O2-handle pair on F and the 2-handle $D'_* \times I$ on F is equivalent to the 2-handle $D' \times I$.

By assuming Lemma 3.2, the proof of of Theorem 3.1 is done as follows:

Proof of Theorem 3.1. Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be O2-handle pairs on a surface-link F in $\mathbf{R}^4$ with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. Then there is a 2-handle $D'_* \times I$ on F be a 2-handle on F given by Lemma 3.2 such that $(E \times I, D'_* \times I)$ is an O2-handle pair on F and there is an equivalence f from F to itself such that the restriction $f|_F$ is the identity map on F and $f(D'_* \times I) = D' \times I$. By Lemma 2.2 and Corollary 2.4, the O2-handle pair $(E \times I, E' \times I)$ on F is equivalent to the O2-handle pair $(E \times I, D'_* \times I)$ on F, which is equivalent to the O2-handle pair $(f(E) \times I, D' \times I)$ on F and hence to the O2-handle pair $(D \times I, D' \times I)$ on F. Thus, the O2-handle pair $(D \times I, D' \times I)$ on F is equivalent to an O2-handle pair $(E \times I, E' \times I)$ on F. This completes the proof of Theorem 3.1. □

Throughout the remainder of this section, the proof of Lemma 3.2 is done.

Proof of Lemma 3.2. For the core disks D, D' E and E' of $D \times I$, $D' \times I$, $E \times I$ and $E' \times I$, respectively, assume the following conditions (see Fig. 3):
(a) A neighborhood $n(\partial D)$ of ∂D in D coincides with a neighborhood $n(\partial E)$ of ∂E in E and $(\partial D') \times I \cap \partial E' = \emptyset$ by slightly sliding $\partial E'$ along F,

(b) The disk interiors Int$D$, IntD', Int$E$ and IntE' meet transversely except for the part $n(\partial D) = n(\partial E)$ and $D \cap D' = \partial D \cap \partial D' = \{p_{D \cap D'}\}$ and $E \cap E' = \partial E \cap \partial E' = \{p_{E \cap E'}\}$ for distinct points $p_{D \cap D'}$ and $p_{E \cap E'}$.

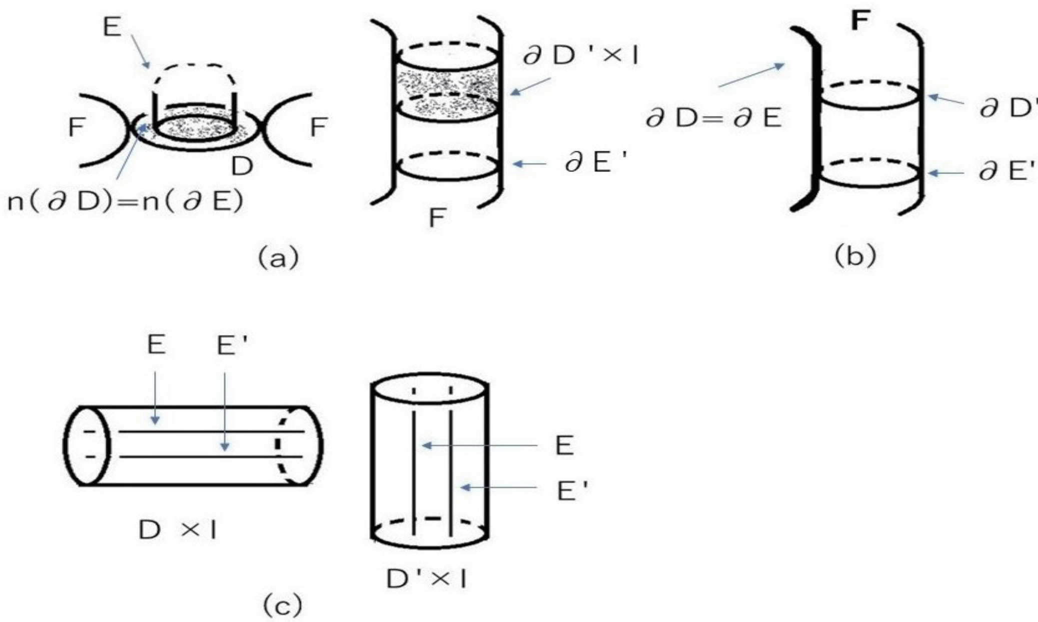

Figure 3: Positions among the core disks D, D', E and E'

(c) The disk interiors $E \backslash n(\partial E)$ and IntE' meet $D \times I$ with a finite number of mutually disjoint arcs which are parallel to a fiber I of the line bundle $D \times I$ over D. Similarly, the disk interiors Int$E$ and IntE' meet $D' \times I$ with a finite number of mutually disjoint arcs which are parallel to a fiber I of the line bundle $D' \times I$ over D'.

The following operation, called *Finger Move Canceling* eliminates an intersection point $x \in \text{Int}E \cap \text{Int}D'$ by creating a disk D'' with $\partial D'' = \partial D'$ from the disk D'.

Finger Move Canceling. Let S be a trivial S^2-knot in $\mathbf{R}^4$ such that the 2-sphere S^2 is disjoint from F and D' and meets the disk interior Int$E$ transversely in just one point x. Let y be a double point between the disk interiors Int$E$ and IntD', and L a simple arc in the disk E joining x and y not meeting the other double points between E' and D. Let V_L be a solid tube in $\mathbf{R}^4$ around the arc L such that $V_L \cap E = L$ and V_L joins a disk neighborhood d_x of x in the disk D' and a disk neighborhood d_y of y in the 2-sphere S. Then a disk D'' with $\partial D'' = \partial D'$ and $E \cap D'' = E \cap D' \setminus \{x\}$ is constructed so that

$$D'' = \text{cl}(D' \setminus d_x) \cup \text{cl}(\partial V_L \setminus (d_x \cup d_y)) \cup \text{cl}(S \setminus d_y).$$

A trivial S^2-knot S used in Finger Move Canceling is constructed as follows:

Claim 3.2.1. After an isotopic deformation of F, E and E' keeping D and D', there is a trivial S^2-knot S in $\mathbf{R}^4$ such that

(1) $S \cap D = S \cap E = \{x\}$ for a point $x \in n(\partial D) = n(\partial E)$,

(2) $S \cap (F \cup D' \times I \cup E') = \emptyset$,

(3) There is a 3-ball B^S in $\mathbf{R}^4$ with $\partial B^S = S$ such that $B^S \cap (F \cup D' \times I) = D'$.

By assuming Claim 3.2.1, let D'_1 be a disk parallel to the core disk D' of the 2-handle $D' \times I$ on the surface-link F such that $D'_1 \cap F = \partial D'_1$ and $D'_1 \cap (D' \times I) = \emptyset$. Let y be a double point between the disk interiors $\mathrm{Int}D'_1$ and $\mathrm{Int}E$. Apply Finger Move Canceling to the trivial S^2-knot S in Claim 3.2.1 along an arc c in E from the point x to the point $x \in S \cap E$ which avoids the double point set $E \cap D'_1 \setminus \{y\}$ to obtain a disk D'_2 such that

(1) $\partial D'_2 = \partial D'_1$,

(2) $E \cap D'_2 = (E \cap D'_1) \setminus \{y\}$, and

(3) $D'_2 \cap F = \partial D'_2$ and $D'_2 \cap (D' \times I) = \emptyset$.

By continuing this Finger Move Canceling on a trivial S^2-knot parallel to S, a 2-handle $D'_* \times I$ on F with $\partial D'_* = \partial D'_1$ such that $(E \times I, D'_* \times I)$ is an O2-handle pair on F is obtained. The following claim shows that this 2-handle $D'_* \times I$ on the surface-link F is a desired 2-handle in Lemma 3.2.

Claim 3.2.2. The 2-handle $D^*_1 \times I$ on F is equivalent to the 2-handle $D'_1 \times I$.

This completes the proof of Lemma 3.2 under the assumptions of Claims 3.2.1 and 3.2.2.

The proof of Claim 3.2.1 is done as follows:

Proof of Claim 3.2.1. Let Δ is the handle union of the O2-handle pair $(D \times I, D' \times I)$, and $B = B_\Delta$ an associated bump of Δ (see Fig. 2). Assume that the bump B is in the 3-space $\mathbf{R}^3$ by an isotopic deformation of B. Let $T^o_B = F \cap B$ be an unknotted once-punctured torus in B. Let $F^c = \mathrm{cl}(F \setminus T^o)$. For the sub-surface $T^o_\Delta = F \cap \Delta$ of T^o, the closed complement $A(T^o) = \mathrm{cl}(T^o_B \setminus T^o_\Delta)$ is an annulus bounded by the loops $o_F = \partial T^o_B = \partial \delta_B = \partial F^c_B$ and $o_\Delta = \partial T^o_\Delta = \partial \delta_{D \times I, D' \times I}$.

Assume that the disk E meets the associated bump B with the union of the loop ∂E, a set $J^E_{D\times I}$ of trivial parallel arcs and a set $J^E_{D'\times I}$ of trivial parallel arcs such that

(i) the set $J^E_{D\times I}$ of trivial proper parallel arcs in B is obtained by extending the intersection set $\mathrm{Int}E \cap (D \times I)$ of trivial parallel arcs in $D \times I$ and

(ii) the set $J^E_{D'\times I}$ of trivial proper parallel arcs in B is obtained by extending the intersection set $\mathrm{Int}E \cap (D' \times I)$ of trivial parallel arcs in $D' \times I$.

Similarly, assume that the disk E' meets the associated bump B with the union of the loop $\partial E'$, a set $J^{E'}_{D\times I}$ of trivial proper parallel arcs in B and a set $J^{E'}_{D'\times I}$ of trivial proper parallel arcs in B such that

(i)′ the set $J^{E'}_{D\times I}$ of trivial proper parallel arcs in B is obtained by extending the intersection set $\mathrm{Int}E' \cap (D \times I)$ of trivial parallel arcs in $D \times I$ and

(ii)′ the set $J^{E'}_{D'\times I}$ of trivial proper parallel arcs in B is obtained by extending the intersection $\mathrm{Int}E' \cap (D' \times I)$ of trivial parallel arcs in $D' \times I$.

Let
$$J = J^E_{D\times I} \cup J^E_{D'\times I} \cup J^{E'}_{D\times I} \cup J^{E'}_{D'\times I}.$$
Let $o_E = \partial n(\partial E) \setminus \partial E$. Let $d(D')$ be a disk in the associated bump B containing the disk D' in the interior such that the link $o_E \cup \partial d(D')$ for the boundary loop $\partial d(D')$ is a trivial link in B and $\partial d(D')$ transversely meets the disks E and D with just one point in the interior of the part $n(\partial D) = n(\partial E)$. A situation of the intersections of the disks E and E' with the associated bump B of the O2-handle pair $(D \times I, D' \times I)$ is illustrated in Fig. 4.

Notations. For a subspace A of $\mathbf{R}^3[0]$ and a subinterval K of $\mathbf{R}$ the notation

$$AK = \{(x,t) \in \mathbf{R}^4 |\, x \in A, t \in K\}$$

is used for a subspace of $\mathbf{R}^4$ as it is used in [16]. Since the associated bump $B = B_\Delta$ of the handle union Δ of the O2-handle pair $(D \times I, D' \times I)$ is assumed to be in the 3-space $\mathbf{R}^3 = \mathbf{R}^3[0]$, the 4-ball

$$B[-1,1] \subset \mathbf{R}^3[-1,1] \subset \mathbf{R}^4$$

is a bi-collar of the associated bump of B in the 4-space $\mathbf{R}^4$. To avoid a confusion, the notation AK_B is used for the subspace AK in $B[-1,1]$ defined for a subspace A of B and a subinterval K of $[-1,1]$.

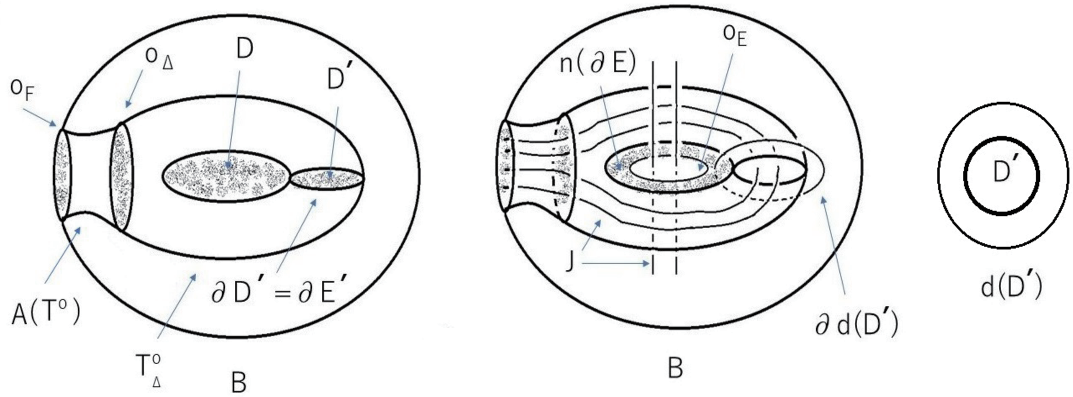

Figure 4: A situation of the intersections of the disks E and E' with the associated bump B

The following situation may be imposed on the intersection of the union $F \cup E \cup E'$ with the 4-ball $B[-1,1]$:

(3.2.1.1) The surface-knot F and the disks E and E' meet the 4-ball $B[-1,1]$ such that

$$(F \cup E \cup E') \cap B[t]_B = \begin{cases} (o_\Delta \cup J \cup o_E \cup \partial E')[t]_B, & \text{for } 0 < t \leq 1, \\ (T^o_\Delta \cup J \cup n(\partial E))[t]_B, & \text{for } t = 0, \\ J[t]_B, & \text{for } -1 \leq t < 0. \end{cases}$$

In (3.2.1.1), note that the annulus $A(T^o) \subset B$ bounded by $o_\Delta \cup o_F$ is deformed into the annulus $o_\Delta[0,1]_B \subset B[-1,1]$ identifying $o_\Delta \subset B$ with $o_\Delta[0]_B \subset B[0]_B$ and $o_F \subset B$ with $o_\Delta[1]_B \subset B[1]_B$.

Consider the 4-ball $U = \mathrm{cl}(\bar{\mathbf{R}}^4 \setminus B[-1,1])$ for the one-point-compactification $\bar{\mathbf{R}}^4$ of the 4-space $\mathbf{R}^4$ and the proper surfaces

$$\begin{aligned} R(F) &= \mathrm{cl}(F \setminus F \cap B[-1,1]), \\ R(E) &= \mathrm{cl}(E \setminus E \cap B[-1,1]), \\ R(E') &= \mathrm{cl}(E' \setminus E' \cap B[-1,1]) \end{aligned}$$

in the 4-ball U. The link

$$\mathbf{L} = \partial R(F) \cup \partial R(E) \cup \partial R(E')$$

in the 3-sphere $\partial U = B[-1]_B \cup (\partial B)[-1,1]_B \cup B[1]_B$ is illustrated in Fig. 5, where $\partial R(F)$ and $\partial R(E) \cup \partial R(E')$ are given as follows:

$$\begin{aligned} \partial R(F) &= o_\Delta[1]_B \subset \partial U, \\ \partial R(E) \cup \partial R(E') &= o_E[1]_B \cup \partial E'[1]_B \cup \mathbf{L}' \subset \partial U \\ & \qquad \text{for} \quad \mathbf{L}' = J[-1]_B \cup (\partial J)[-1,1]_B \cup J[1]_B. \end{aligned}$$

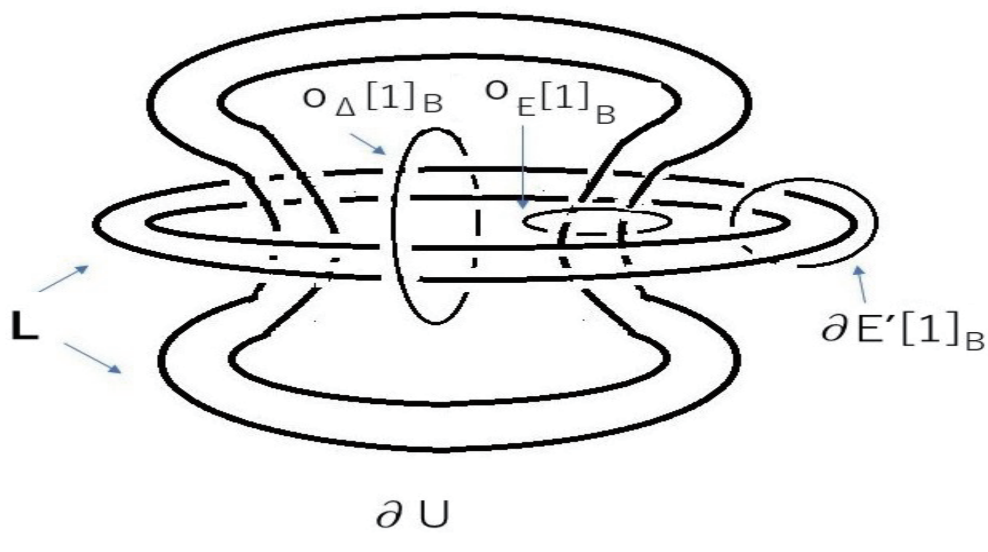

Figure 5: The link $\mathbf{L}$ in the 3-sphere ∂U

Consider the pair $(U, \partial U)$ as the one-point-compactification of the pair of the upper-half 4-space

$$\mathbf{R}^4_+ = \{(x,t) \in \mathbf{R}^3 \times \mathbf{R} |\, x \in \mathbf{R}^3, t \in \mathbf{R}\}$$

and the boundary 3-space $\partial \mathbf{R}^4_+ = \mathbf{R}^3 = \mathbf{R}^3[0]$. The same notations for the proper surface $R(F) \cup R(E) \cup R(E')$ in the 4-ball U and the link $\mathbf{L} = o_\Delta[1]_B \cup o_E[1]_B \cup \partial E'[1]_B \cup L'$ in the boundary 3-sphere ∂U are used for the corresponding proper surface in $\mathbf{R}^4_+$ and the corresponding link in the boundary 3-space $\mathbf{R}^3 = \mathbf{R}^3[0]$.

By an argument of [16], a normal form of the surface $R(F) \cup R(E) \cup R(E')$ in $\mathbf{R}^4_+$ is considered to obtain the following surface G from the surface $R(F) \cup R(E) \cup R(E')$ by an ambient isotopy of $\mathbf{R}^4_+$ keeping the boundary $\mathbf{R}^3 = \mathbf{R}^3[0]$ fixed:

(3.2.1.2) The surface G in $\mathbf{R}^4_+$ is given by

$$G \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 3 \\ \mathbf{d}(\mathbf{O})[t], & \text{for } t = 3, \\ \mathbf{O}[t], & \text{for } 2 < t < 3, \\ (\mathbf{L} \cup \mathbf{o} \cup \mathbf{b})[t], & \text{for } t = 2, \\ (\mathbf{L} \cup \mathbf{o})[t], & \text{for } 1 < t < 2, \\ (\mathbf{L} \cup \mathbf{d})[t], & \text{for } t = 1, \\ \mathbf{L}[t], & \text{for } 0 \leq t < 1, \end{cases}$$

where

- $\mathbf{d}$ is a disk system in $\mathbf{R}^3$ disjoint from the link L and $\mathbf{o} = \partial\mathbf{d}$, a trivial link,
- $\mathbf{b}$ is a band system in $\mathbf{R}^3$ spanning the link $L \cup \mathbf{o}$,
- $\mathbf{O}$ is a trivial link obtained from the link $\mathbf{L} \cup \mathbf{o}$ by the surgery along $\mathbf{b}$ and $\mathbf{d}(\mathbf{O})$ is a disk system bounding the trivial link $\mathbf{O}$.

Let $(d(D')[0], D'[0])$ be the disk pair in $\mathbf{R}^3[0]$ corresponding to the disk pair $(d(D')[1]_B, D'[1]_B)$ in the 3-ball $B[1]_B \subset \partial U$ obtained from the disk pair $(d(D'), D')$ in B. Let $(\iota \cdot d(D')[0], \iota \cdot D'[0])$ be the disk pair in $\mathbf{R}^3[0]$ corresponding to the disk pair $(d(D')[-1]_B, D'[-1]_B)$ in the 3-ball $B[-1]_B \subset \partial U$ obtained from the disk pair $(d(D'), D')$ in B, where note that the disk pair $(d(D')[-1]_B, D'[-1]_B)$ is the image of the disk pair $(d(D')[1]_B, D'[1]_B)$ by the reflection ιin $B[-1, 1]$ sending the point (x, t) to the point $(x, -t)$ for $x \in B$ and $t \in [-1, 1]$.

By a replacement to a narrow band and a band slide on the band system $\mathbf{b}[2]$ in (3.2.1.2), the following condition cab be imposed:

(3.2.1.3) The band system $\mathbf{b}[2]$ does not meet the disks $d(D')[2]$ and $\iota \cdot d(D')[2]$. Thus, for every t with $0 \leq t \leq 3$, we have:

$$\begin{aligned} d(D')[3] \cap G &= d(D')[3] \cap \mathbf{d}(\mathbf{O})[3], \\ d(D')[t] \cap G &= (D' \cap \mathbf{L})[t], \quad \text{for } 0 \leq t < 3; \\ \iota \cdot d(D')[3] \cap G &= \iota \cdot d(D')[3] \cap \mathbf{d}(\mathbf{O})[3], \\ \iota \cdot d(D')[t] \cap G &= (\iota \cdot D' \cap \mathbf{L})[t], \quad \text{for } 0 \leq t < 3. \end{aligned}$$

Let $\mathbf{p} = D' \cap \mathbf{L}$ be the point system in B, and $\mathbf{p}[0]$ the point system in $\mathbf{R}^3[0]$ representing the point system $\mathbf{p}[1]_B$ in 3-ball $B[1]_B \subset \partial U$. Similarly, let $\iota \cdot \mathbf{p}[0]$ be the point system in $\mathbf{R}^3[0]$ representing the point system $\mathbf{p}[-1]_B$ in 3-ball $B[-1]_B \subset \partial U$ which is ι-reflection image of the point system $\mathbf{p}[1]_B$.

In (3.2.1.3), the intersection $d(D')[3]\cap\mathbf{d}(\mathbf{O})[3]$ is the disjoint union of an improper arc system $\boldsymbol{\alpha}[3]$ joining the point system $\mathbf{p}[3]$ with a point system $\mathbf{p}^d[3]$ in the loop $\partial d(D')[3]$ and a proper arc system $\boldsymbol{\beta}[3]$ in the disk $d(D')[3]$.

Similarly, the intersection $\iota\cdot d(D')[3]\cap\mathbf{d}(\mathbf{O})[3]$ is the disjoint union of an improper arc system $\iota\cdot\boldsymbol{\alpha}[3]$ joining the point system $\iota\cdot\mathbf{p}[3]$ with a point system $\iota\cdot\mathbf{p}^d[3]$ in the loop $\partial\iota\cdot d(D')[3]$ and a proper arc system $\iota\cdot\boldsymbol{\beta}[3]$ in the disk $\iota\cdot d(D')[3]$.

Let $\boldsymbol{\beta}^+[3]$ and $\iota\cdot\boldsymbol{\beta}^+[3]$ be slightly extended arc systems of the arc systems $\boldsymbol{\beta}[3]$ and $\iota\cdot\boldsymbol{\beta}[3]$ in $\mathbf{d}(\mathbf{O})[3]$, respectively. Let $\boldsymbol{\gamma}$ and $\iota\cdot\boldsymbol{\gamma}$ be the arc systems in $\mathbf{R}^3[3,4]$ obtained respectively by deforming the extended arc systems $\boldsymbol{\beta}^+[3]$ and $\iota\cdot\boldsymbol{\beta}^+[3]$ as follows:

(3.2.1.4) For every t with $3\leq t\leq 4$, the arc systems $\boldsymbol{\gamma}$ and $\iota\cdot\boldsymbol{\gamma}$ in $\mathbf{R}^3[3,4]$ are given by

$$\boldsymbol{\gamma}\cap\mathbf{R}^3[t]=\begin{cases}\boldsymbol{\beta}\sqcap^+[t], & \text{for } t=4,\\ \partial\boldsymbol{\beta}^+[t], & \text{for } 3\leq t<4,\end{cases}$$

where $\boldsymbol{\beta}\sqcap^+[4]$ is an arc system which is deformed from the arc system $\boldsymbol{\beta}^+[4]$ with $\partial\boldsymbol{\beta}\sqcap^+[4]=\partial\boldsymbol{\beta}^+[4]$ and $\boldsymbol{\beta}\sqcap^+[4]\cap d(D')[4]=\emptyset$ (see Fig. 6), and

$$\iota\cdot\boldsymbol{\gamma}\cap\mathbf{R}^3[t]=\begin{cases}\iota\cdot\boldsymbol{\beta}\sqcap^+[t], & \text{for } t=4,\\ \partial\iota\cdot\boldsymbol{\beta}^+[t], & \text{for } 3\leq t<4,\end{cases}$$

where $\iota\cdot\boldsymbol{\beta}\sqcap^+[4]$ is an arc system which is deformed from the arc system $\iota\cdot\boldsymbol{\beta}^+[4]$ with $\partial\iota\cdot\boldsymbol{\beta}\sqcap^+[4]=\partial\iota\cdot\boldsymbol{\beta}^+[4]$ and $\iota\cdot\boldsymbol{\beta}\sqcap^+[4]\cap\iota\cdot d(D')[4]=\emptyset$ (see Fig. 6).

The deformation from the extended arc systems $\boldsymbol{\beta}^+[3]$ and $\iota\cdot\boldsymbol{\beta}^+[3]$ into the arc systems $\boldsymbol{\gamma}$ and $\iota\cdot\boldsymbol{\gamma}$ in (3.2.1.4) turns the disk system $\mathbf{d}(\mathbf{O})[3]$ into a disk system $\mathbf{d}'(\mathbf{O})\subset\mathbf{R}^3[3,4]$ with the intersection

$$\mathbf{d}''(\mathbf{O})[3]=\mathbf{d}'(\mathbf{O})\cap\mathbf{R}^3[3]$$

a compact multi-punctured disk system such that

$$d(D')[3,4]\cap\mathbf{d}'(\mathbf{O})=\boldsymbol{\alpha}[3]\quad\text{and}\quad\iota\cdot d(D')[3,4]\cap\mathbf{d}'(\mathbf{O})=\iota\cdot\boldsymbol{\alpha}[3].$$

Let $\mathbf{q}$ be a point system in the arc system $J^{E'}_{D\times I}\cup J^{E'}_{D'\times I}$ in B which is not in the 2-handle union Δ. Let $\mathbf{a}$ be an arc system in the link $\mathbf{L}$ in B joining the point system $\mathbf{p}$ with the point system $\mathbf{q}$. Let $\mathbf{a}[0]$ and $\iota\cdot\mathbf{a}[0]$ be the arc systems in $\mathbf{R}^3[0]$ representing the arc system $\mathbf{a}[1]_B$ in $B[1]_B$ and the arc system $\mathbf{a}[-1]_B$ in $\iota(B[1]_B)=B[-1]_B$, respectively. By a replacement to a narrow band on the band system $\mathbf{b}[2]$ and a band slide, assume that the band system $\mathbf{b}[2]$ does not attach to the arc systems $\mathbf{a}[2]$ and

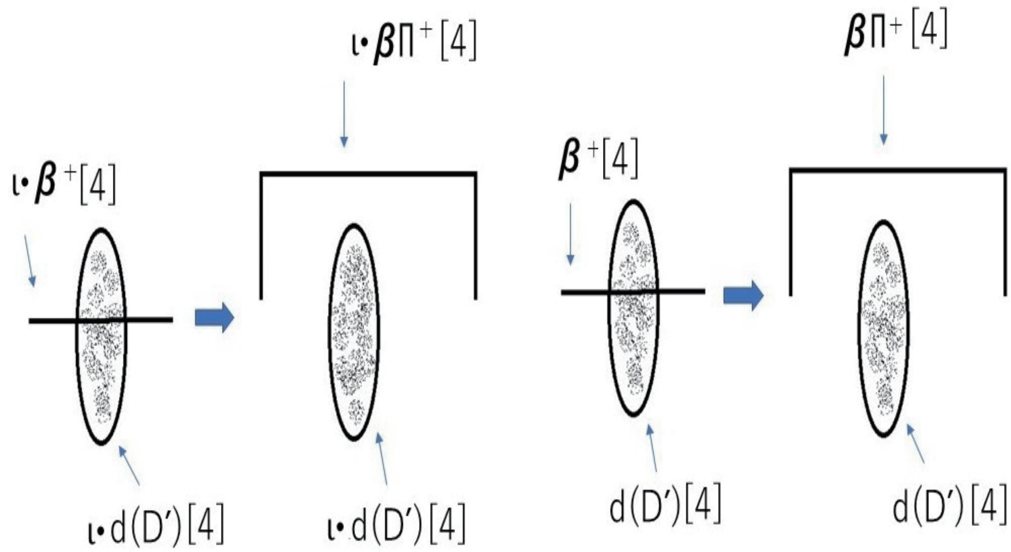

Figure 6: The arc systems $\boldsymbol{\beta}\sqcap^+[4]$ and $\iota\cdot\boldsymbol{\beta}\sqcap^+[4]$ deformed from $\boldsymbol{\beta}^+[4]$ and $\iota\cdot\boldsymbol{\beta}^+[4]$

$\iota\cdot\mathbf{a}[2]$. Then the arc systems $\mathbf{a}[3]$ and $\iota\cdot\mathbf{a}[3]$ are in the boundary of the multi-punctured disk system $\mathbf{d}''(\mathbf{O})[3]$ with $\partial\mathbf{a}[3]=\mathbf{p}[3]\cup\mathbf{q}[3]$ and $\partial\iota\cdot\mathbf{a}[3]=\iota\cdot\mathbf{p}[3]\cup\iota\cdot\mathbf{q}[3]$.

Let $\mathbf{a}^d[3]$ and $\iota\cdot\mathbf{a}^d[3]$ be arc systems in the multi-punctured disk system $\mathbf{d}''(\mathbf{O})[3]$ such that $\partial\mathbf{a}^d[3]=\mathbf{p}^d[3]\cup\mathbf{q}[3]$ and $\partial\iota\cdot\mathbf{a}^d[3]=\iota\cdot\mathbf{p}^d[3]\cup\iota\cdot\mathbf{q}[3]$. See Fig. 7 for this situation where $T^0_\Delta[3]$ and $\iota\cdot T^0_\Delta[3]$ denote the copies of $T^0_\Delta\subset B$ in $\mathbf{R}^3[3]$ via the copy in $B[1]$ and the reflection image in $\iota(B[1])=B[-1]$ for the reflection ι in $B[-1,1]$, respectively.

Let $n(\mathbf{a}^d)[3]$ and $n(\iota\cdot\mathbf{a}^d)[3]$ be regular neighborhood disk systems of the arc systems $\mathbf{a}^d[3]$ and $\iota\cdot\mathbf{a}^d[3]$ in the multi-punctured disk system $\mathbf{d}''(\mathbf{O})[3]$.

Let $\mathbf{d}^*(\mathbf{O})=\mathrm{cl}(\mathbf{d}'(\mathbf{O})\setminus(n(\mathbf{a}^d)[3]\cup n(\iota\cdot\mathbf{a}^d)[3]))$, and $\mathbf{O}^*[t]$ the trivial link obtained from the trivial link $\mathbf{O}[t]$ by the surgery along the disk systems $n(\mathbf{a}^d)[t]$ and $n(\iota\cdot\mathbf{a}^d)[t]$ for every t with $2<t<3$. Also, let $\mathbf{L}^*[t]$ be the link obtained from the link $\mathbf{L}[t]$ by surgery along the disk systems $n(\mathbf{a}^d)[t]$ and $n(\iota\cdot\mathbf{a}^d)[t]$ for every t with $1\leq t\leq 2$. Then the surface G^* in $\mathbf{R}^4_+$ which is isotopic to G by an ambient isotopy keeping

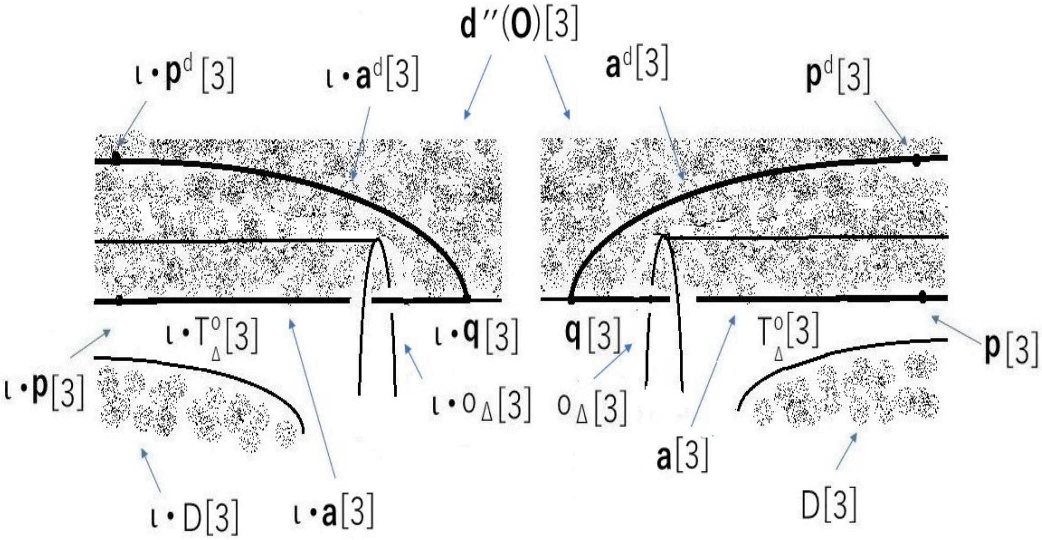

Figure 7: Arc systems $\mathbf{a}^d[3]$ and $\iota \cdot \mathbf{a}^d[3]$

$\mathbf{R}^3[0]$ fixed is given by

$$G^* \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 4 \\ \mathbf{d}'(\mathbf{O}) \cap \mathbf{R}^3[t], & \text{for } 3 < t \leq 4, \\ \mathbf{d}^*(\mathbf{O})[t], & \text{for } t = 3, \\ \mathbf{O}^*[t], & \text{for } 2 < t < 3, \\ (\mathbf{L}^* \cup \cap \mathbf{o} \cup \mathbf{b})[t], & \text{for } t = 2, \\ (\mathbf{L}^* \cup \mathbf{o})[t], & \text{for } 1 < t < 2, \\ (\mathbf{L}^* \cup \mathbf{d})[t], & \text{for } t = 1, \\ \mathbf{L}^*[t], & \text{for } 0 \leq t < 1. \end{cases}$$

Let $J^*[1]_B \cup J^*[-1]_B$ be the arc system in the 3-sphere $\partial(B[-1,1]) = \partial U$ obtained from $J[1]_B \cup J[-1]_B$ by replacing the link $\mathbf{L}[1]_B$ with the link $\mathbf{L}^*[1]_B$ in $\partial(B[-1,1]) = \partial U$.

The multi-punctured disk system $\mathbf{d}''(\mathbf{O})[3]$ is deformed in $\mathbf{R}^3[3]$ so that $T^o_\Delta[0]$ does not meet the neighborhood disk systems $n(\mathbf{a}^d)[3]$ and $n(\iota \cdot \mathbf{a}^d)[3]$. Then the arc systems $J^*[1]_B$ and $J^*[-1]_B$ extend to the disk system $J^*[-1,1]_B$ in $B[-1,1]$.

Let F^*, E^* and E'^* be the deformation results of F, E and E' using G^* and $J^*[-1,1]_B$, which are obtained by isotopic deformations on F, E and E' keeping D and D' fixed. Let D^S and $\iota \cdot D^S$ be the disks in $\mathbf{R}^3[0,4]$ defined by

$$D^S \cap \mathbf{R}^3[t] = \begin{cases} d(D')[t], & \text{for } t = 4, \\ \partial d(D')[t], & \text{for } 0 \leq t < 4, \end{cases}$$

$$\iota \cdot D^S \cap \mathbf{R}^3[t] = \begin{cases} \iota \cdot d(D')[t], & \text{for } t = 4, \\ \partial\iota \cdot d(D')[t], & \text{for } 0 \leq t < 4, \end{cases}$$

Let S be the 2-sphere obtained from the disks D^S and $\iota \cdot D^S$ by connecting the tube $\partial d(D')[-1,1]_B$ in the 4-ball $B[-1,1]$ bounded by the loops ∂D^S and $\partial\iota \cdot D^S$. By construction, this 2-sphere S does not meet the surface-link F^* and the disks D', E'^* and meets the disks D and E^* with just one point in the part $n(\partial D) = n(\partial E^*)$. By construction, there is a 3-ball B^S in $\mathbf{R}^4$ with $\partial B^S = S$ such that $B^S \cap (F^* \cup D' \times I) = D'$. Thus, S is a desired 2-sphere. This completes the proof of Claim 3.2.1. □

The proof of Claim 3.2.2 is done as follows:

Proof of Claim 3.2.2. Let S be a trivial 2-knot in Claim 3.2.1. Let $D'_1 \times I$ be a 2-handle on F with core disk D'_1 which is disjoint from $D' \times I$.

Let D'_2 be the disk obtained from the disk D'_1 and the 2-sphere S by taking the surgery along a 1-handle h joining a disk d' in D' and a disk d in the S^2-knot S and not meeting the interior of the 3-ball B^3. Let $D'_2 \times I$ be the 2-handle on F with D'_2 a core disk and with $\partial D'_2 \times I = \partial D'_1 \times I$ which is obtained from the 2-handle $D'_1 \times I$ and a collaring $S \times I$ of the trivial S^2-knot S and a collaring $h \times I$ of the 1-handle h. For the bounded surface $F_1^c = \mathrm{cl}(F \setminus \partial D'_1 \times I)$, the surface-links $F(D'_1 \times I)$ and $F(D'_2 \times I)$ are given as follows:

$$\begin{aligned} F(D'_1 \times I) &= F_1^c \cup D'_1 \times \partial I, \\ F(D'_2 \times I) &= F_1^c \cup D'_2 \times \partial I. \end{aligned}$$

The disk union $D'_2 \times \partial I$ is obtained from the disk union $D'_1 \times \partial I$ by the surgery along the 1-handle union $h \times \partial I$. In Fig 8, it is shown that one 1-handle of the 1-handle union $h \times \partial I$ is a self-intersecting 1-handle connecting one disk of the disk union $D'_1 \times \partial I$ and one 3-ball in the 3-ball unions $B^3 \times \partial I$ for a collaring $B^3 \times I$ of B^3. This implies that the disk union $D'_2 \times \partial I$ is deformed into the disk union $D'_1 \times \partial I$ by an ambient isotopy of $\mathbf{R}^4$ keeping the surface F_1^c fixed. Thus, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from $F(D'_2 \times I)$ to $F(D'_1 \times I)$ keeping the surface F_1^c identically.

The 2-handle $D'_* \times I$ on F constructed by continuing this operation has the property that the pair $(E \times I, D'_* \times I)$ is an O2-handle pair on F and there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from $F(D'_* \times I)$ to $F(D'_1 \times I)$ keeping the surface F_1^c identically.

Let $a' = \partial D \cap D'_1 \times I = \partial E \cap D'_* \times I$ be the arc parallel to a fiber I of the line bundle $\partial D'_1 \times I = \partial D'_* \times I$ over the circle $\partial D'_1 = \partial D'_*$. The arc a' attaching to $F(D'_1 \times I)$ is ∂-relatively isotopic to an arc parallel to F_1^c through the disk D. Similarly, the arc a' attaching to $F(D'_* \times I)$ is also ∂-relatively isotopic to an arc parallel to F_1^c through the disk E. This means that the equivalence f is isotopically deformed into

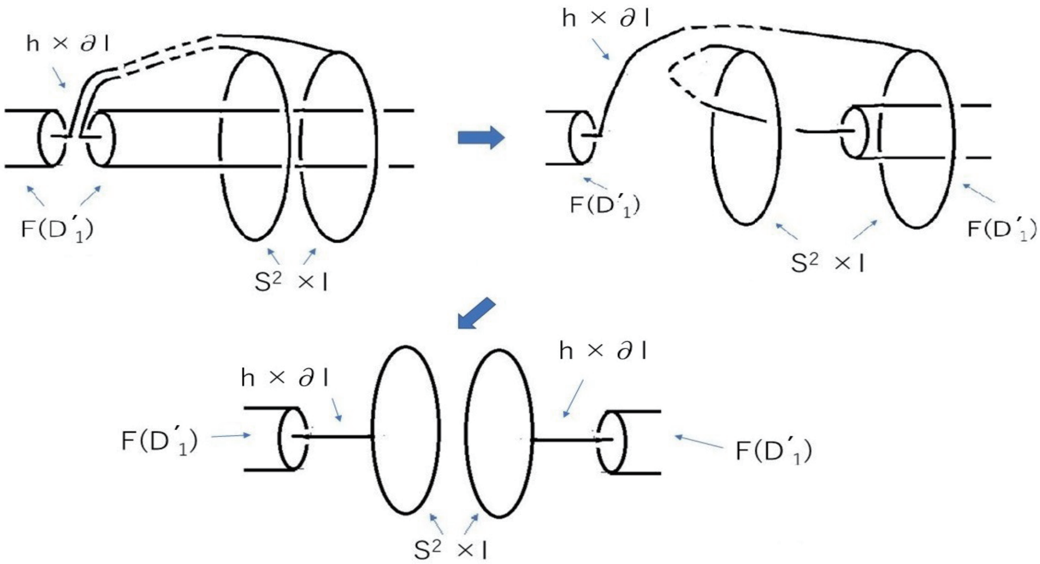

Figure 8: An equivalence from the disk D_2' to the disk D_1'

an equivalence f' from $F(D_*' \times I)$ to $F(D_1' \times I)$ keeping the surface F_1^c fixed such that $f'(a') = a'$. Since the arc a' is regarded as a core of the 1-handle $D_*' \times I$ on $F(D_*' \times I)$ and a core of the 1-handle $D_1' \times I$ on $F(D_1' \times I)$, the equivalence f' is isotopically deformed into an equivalence f'' from F to itself such that the restriction $f'|_F$ is the identity and $f''(D_*' \times I) = D_1' \times I$ (see [6]). This completes the proof of Claim 3.2.2. □

This completes the proof of Lemma 3.2. □

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