The Signature Invariants of Infinite Cyclic Coverings of Closed Odd Dimensional Manifolds

Akio Kawauchi

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§0. The statement of the main result

We consider a compact oriented *topological* n-manifold M. To each $\gamma \in H^1(M; Z)$, we can consider the *infinite cyclic covering space* $\tilde{M}$ of M, which is defined, up to equivalence, to be the fibered product of the covering $\exp: R \to S^1$ defined by $\exp(x) = e^{2\pi i x}$ and a map $f_\gamma: M \to S^1$ with $f_\gamma{}^*([S^1]) = \gamma$, where R is the real number field. The covering transformation group is an infinite cyclic group with a generator t, specified by the transformation $R \to R$ sending x to $x+1$, and denoted by $\langle t \rangle$. *The coefficients of homology and cohomology will be taken in R, unless otherwise specified.* Then the homology $H_*(\tilde{M}, \partial\tilde{M})$ forms a finitely generated $R\langle t \rangle$-module (cf. §1). Let $T_*(\tilde{M}, \partial\tilde{M}) = \mathrm{Tor}_{R\langle t \rangle} H_*(\tilde{M}, \partial\tilde{M})$ and $T^*(\tilde{M}, \partial\tilde{M}) = \mathrm{Hom}_R[T_*(\tilde{M}, \partial\tilde{M}), R]$. The signature invariants on $\tilde{M}$ are defined on the basis of the following two properties (cf. §1):

Property 0.1. The orientation of M and γ determine a unique t-invariant homomorphism $\tilde{\mu}: T^{n-1}(\tilde{M}, \partial\tilde{M}) \to R$.

Property 0.2. By the natural epimorphism $H^*(\tilde{M}, \partial M) \to T^*(\tilde{M}, \partial\tilde{M})$, the cup product pairing

$$\cup: H^q(\tilde{M}, \partial\tilde{M}) \times H^{n-q-1}(\tilde{M}, \partial\tilde{M}) \longrightarrow H^{n-1}(\tilde{M}, \partial\tilde{M})$$

induces a pairing (also denoted by $\cup$)

$$T^q(\tilde{M}, \partial\tilde{M}) \times T^{n-q-1}(\tilde{M}, \partial\tilde{M}) \longrightarrow T^{n-1}(\tilde{M}, \partial\tilde{M}).$$

When $n = 2m+1$, the pairing

$$\cup: T^m(\tilde{M}, \partial\tilde{M}) \times T^m(\tilde{M}, \partial\tilde{M}) \longrightarrow T^{2m}(\tilde{M}, \partial\tilde{M})$$

is $\varepsilon(m)$-symmetric, where $\varepsilon(m) = (-1)^m$. We define a t-isometric symmetric bilinear form (called the *quadratic form* of $\tilde{M}$ or (M, γ))

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$$\tilde{b}\colon T^m(\tilde{M}, \partial\tilde{M}) \times T^m(\tilde{M}, \partial\tilde{M}) \longrightarrow R$$

by the identity $\tilde{b}(u, v) = \tilde{\mu}(u \cup (t - t^{-1})v)$ (if $\varepsilon(m) = -1$) or $\tilde{\mu}(u \cup v)$ (if $\varepsilon(m) = 1$) (cf. [K1], [K2], [K3], [Mi], [N]). The signature of $\tilde{b}$ is called the *signature* of $\tilde{M}$ or (M, γ) and denoted by $\sigma^\gamma(M)$. For each $a \in (-1, 1)$, let $p_a(t) = t^2 - 2at + 1$, which is irreducible in $R\langle t\rangle$. Let $p_1(t) = t - 1$ and $p_{-1}(t) = t + 1$. Let $T^m(\tilde{M}, \partial\tilde{M})_a$ be the $p_a(t)$-component of $T^m(\tilde{M}, \partial\tilde{M})$, so that

$$T^m(\tilde{M}, \partial\tilde{M}) = \bigoplus_{a \in [-1,1]} T^m(\tilde{M}, \partial\tilde{M})_a \oplus T^m(\tilde{M}, \partial\tilde{M})_*\,,$$

where $T^m(\tilde{M}, \partial\tilde{M})_*$ has no non-trivial $p_a(t)$-torsion elements. The signature of $\tilde{b}\,|\,T^m(\tilde{M}, \partial\tilde{M})_a$ is called the *local signature* of $\tilde{M}$ or (M,γ) at a and denoted by $\sigma_a^\gamma(M)$. Then $\sigma_a^\gamma(M) = 0$ except a finite number of a and

$$\sigma^\gamma(M) = \textstyle\sum_{a \in [-1,1]} \sigma_a^\gamma(M)$$

(cf. § 1). When $\varepsilon(m) = 1$, we denote the signature of $\tilde{b}\,|\,(t-1)T^m(\tilde{M}, \partial\tilde{M})_1$ by $\hat{\sigma}_1^\gamma(M)$. The purpose of this paper is to give a computation of the invariants $\sigma^\gamma(M)$, $\sigma_a^\gamma(M)$ and $\hat{\sigma}_1^\gamma(M)$ in the case when $\partial M = \emptyset$. To state the result, we assume that there is a bicollared proper oriented $(n-1)$-submanifold V of M representing the Poincaré dual of γ in $H_{n-1}(M, \partial M; Z)$. We call V a *leaf* of γ. We can obtain a leaf for any γ by using the transversality on a map $f_\gamma\colon M \to S^1$ except at most when $n = 4$, 5, 6 (cf. Moise [Mo], Kirby/Siebenmann [K/S]). Let $n = 2m + 1$. Let $K_m(V)$ be the kernel of the natural homomorphism $i_*\colon H_m(V) \to H_m(M)$. The *linking form* L^+ (or L^-, resp.): $K_m(V) \times K_m(V) \to R$ can be defined by the identity $L^+(x, y) = \mathrm{Link}_M(c_x^+, c_y)$ (or $L^-(x, y) = \mathrm{Link}_M(c_x^-, c_y)$, resp.) for $x = \{c_x\}$ and $y = \{c_y\}$ in $K_m(V)$, where c_x^+ (or c_x^-, resp.) denotes a cycle obtained by translating the cycle c_x off in the positive (or negative, resp.) normal direction (cf. Appendix A for "Link_M"). The linking forms L^+ and L^- were considered by Cooper [C] when $m = 1$. A *linking matrix* on $K_m(V)$ is a matrix representing the form L^+. We construct an $R\langle t\rangle$-matrix $A^{\varepsilon(m)}(t)$ from a linking matrix A on $K_m(V)$ by the identity

$$A^{\varepsilon(m)}(t) = [(1 - t^{-1}) - \varepsilon(m)(1 - t)][(1 - t)A - \varepsilon(m)(1 - t^{-1})A']\,.$$

Since $A^{\varepsilon(m)}(t)$ is t-Hermitian, i.e., $A^{\varepsilon(m)}(t^{-1})' = A^{\varepsilon(m)}(t)$, $A^{\varepsilon(m)}(\omega)$ is Hermitian for all $\omega \in S^1$. For $x \in [-1, 1]$, let $\omega_x = x + (1 - x^2)^{1/2} i \in S^1$. For $a \in [-1, 1]$ we define $\sigma_a^{\varepsilon(m)}(A)$ as follows (see § 5):

$$\sigma_a^{\varepsilon(m)}(A) = \varepsilon(m)\Big(\lim_{x \to a+0} \operatorname{sign} A^{\varepsilon(m)}(\omega_x) - \lim_{x \to a-0} \operatorname{sign} A^{\varepsilon(m)}(\omega_x)\Big)$$

for $a \neq \pm 1$ and

$$\sigma_{\varepsilon(m)}^{\varepsilon(m)}(A)=\operatorname{sign}(A+A')-\lim_{x\to\varepsilon(m)-\varepsilon(m)0}\operatorname{sign}A^{\varepsilon(m)}(\omega_x)$$

$$\sigma_{-\varepsilon(m)}^{\varepsilon(m)}(A)=\lim_{x\to-\varepsilon(m)+\varepsilon(m)0}\operatorname{sign}A^{\varepsilon(m)}(\omega_x)\,.$$

It will be seen in §5 that $\sigma_a^{\varepsilon(m)}(A)=0$ except a finite number of a and $\sum_{a\in[-1,1]}\sigma_a^{\varepsilon(m)}(A)=\operatorname{sign}(A+A')$. Note that when $\varepsilon(m)=1$, the usual signature sign V of V can be considered. Our main result is then stated as follows:

Main Theorem. *Assume that $\partial M=\emptyset$. Then for all $a\in[-1,1)$, we have $\sigma_a^{\gamma}(M)=\sigma_a^{\varepsilon(m)}(A)$. Moreover, when $\varepsilon(m)=-1$, $\sigma_1^{\gamma}(M)=\sigma_1^{-1}(A)$, so that $\sigma^{\gamma}(M)=\operatorname{sign}(A+A')$. When $\varepsilon(m)=1$, $\hat{\sigma}_1^{\gamma}(M)=\sigma_1^1(A)$ and $\sigma^{\gamma}(M)=\operatorname{sign}V$, so that*

$$\textstyle\sum_{a\in[-1,1)}\sigma_a^{\gamma}(M)+\hat{\sigma}_1^{\gamma}(M)=\operatorname{sign}(A+A')$$

and $\sigma_1^{\gamma}(M)=\operatorname{sign}V-\sum_{a\in[-1,1)}\sigma_a^1(A)$.

Remark 0.3. When $\varepsilon(m)=1$, $\hat{\sigma}_1^{\gamma}(M)\neq\sigma_1^{\gamma}(M)$ in general. For example, we take $M=CP^2\times S^1$ and γ to be a generator of $H^1(M;Z)\cong Z$ and $V=CP^2\times 1$. Then $\sigma^{\gamma}(M)=\operatorname{sign}CP^2=1$, but $K_2(V)=0$.

Remark 0.4. In case $\gamma\in H^1(M;Z)$ has no leaf (then $m=2$), we consider the product $M_P=M\times CP^2$ and $\gamma_P\in H^1(M_P;Z)$, corresponding to γ by the natural isomorphism $H^1(M_P;Z)\cong H^1(M;Z)$. By [K/S], γ_P has a leaf V_P, for $\dim M_p=9$. Let A_P be a linking matrix on $K_4(V_P)$. Since we can see that $\sigma_a^{\gamma}(M)=\sigma_a^{\gamma_P}(M_P)$ and $\hat{\sigma}_1^{\gamma}(M)=\hat{\sigma}_1^{\gamma_P}(M_P)$ (cf. §1), it follows from the Main Theorem that $\sigma_a^{\gamma}(M)=\sigma_a^1(A_P)$ for $a\neq 1$ and $\sigma^{\gamma}(M)=\operatorname{sign}V_P$ and $\hat{\sigma}_1^{\gamma}(M)=\sigma_1^1(A_P)$.

In a special case that $H_m(M;Z)\cong H_m(S^1;Z)$, the Main Theorem is deduced from a combination of methods of Erle [E] and Matumoto [Ma]. When $\partial M\neq\emptyset$, the Main Theorem does not hold in general and the details will be discussed somewhere else.

In §1 we remark several properties on the theory of infinite cyclic coverings of manifolds. In §2 a splitting of the middle homology of a leaf is given. In §3 a normal form of a linking matrix is given. In §4 we establish relationship between a homology module and a linking matrix. In §5 we discuss the signature invariants of a real matrix. In §6 the Main Theorem is proved. In Appendix A, we discuss the definitions and some properties of the intersection and linking numbers of singular chains in a topological manifold. In Appendix B, the Duality Theorem, stated in § 1 is proved. In Appendex C, we describe the Blanchfield duality for the Betti modules of infinite cyclic coverings of topological manifolds.

§ 1. Several properties on the theory of infinite cyclic coverings of manifolds

Since every compact topological manifolds is homotopy equivalent to a finite complex (cf. [K/S]), we see that $H_*(\tilde{M})$ is finitely generated over $R\langle t\rangle$. Let $\tilde{S}$ be the lift to $\tilde{M}$ of a compact submanifold $S\subset M$. Since $H^*(\tilde{S})$ is also finitely generated over $R\langle t\rangle$, it follows from the homology exact sequence of $(\tilde{M}, \tilde{S})$ that $H_*(\tilde{M}, \tilde{S})$ is finitely generated over $R\langle t\rangle$. Let

$$T_*(\tilde{M}, \tilde{S})=\operatorname{Tor}_{R\langle t\rangle}H_*(\tilde{M}, \tilde{S}),$$

$$B_*(\tilde{M}, \tilde{S})=H_*(\tilde{M}, \tilde{S})/T_*(\tilde{M}, \tilde{S}),$$

$$T^*(\tilde{M}, \tilde{S})=\operatorname{Hom}_R[T_*(\tilde{M}, \tilde{S}), R] \quad \text{and} \quad B^*(\tilde{M}, \tilde{S})=\operatorname{Hom}_R[B_*(\tilde{M}, \tilde{S}), R].$$

There are natural $R\langle t\rangle$-*split* exact sequences

$$0\longrightarrow T_*(\tilde{M}, \tilde{S})\longrightarrow H_*(\tilde{M}, \partial\tilde{S})\longrightarrow B_*(\tilde{M}, \tilde{S})\longrightarrow 0$$

and

$$0\longrightarrow B^*(\tilde{M}, \tilde{S})\longrightarrow H^*(\tilde{M}, \tilde{S})\longrightarrow T^*(\tilde{M}, \tilde{S})\longrightarrow 0.$$

There is one and only one element (called *the fundamental class of the covering* $\tilde{M}\to M$) μ in $T_{n-1}(\tilde{M}, \partial\tilde{M})$ such that

(i) $(t-1)\mu=0$ and

(ii) The natural map $H_{n-1}(\tilde{M}, \partial\tilde{M})\to H_{n-1}(M, \partial M)$ sends μ to the Poincaré dual of $\gamma\otimes 1\in H^1(M; Z)\otimes R=H^1(M)$.

The proof is given in Appendix B (though it is implicity known in [K2]). Let $\bar{\mu}\colon T^{n-1}(\tilde{M}, \partial\tilde{M})\to R$ be a homomorphism corresponding to μ by the natural isomorphism $T_{n-1}(\tilde{M}, \partial\tilde{M})\cong\operatorname{Hom}_R[T^{n-1}(\tilde{M}, \partial\tilde{M}), R]$ (which was called λ in [K2]). Property 0.1 is thus obtained. Assume that ∂M is a disjoint union $\partial_1 M+\partial_2 M$, where $\partial_i M$ may be empty. Then $\partial\tilde{M}=\partial_1\tilde{M}+\partial_2\tilde{M}$ for the lifts $\partial_i\tilde{M}$ of $\partial_i M$. The following Duality Theorem is obtained by reexamining a result of [K1] and proved in Appendix B:

Duality Theorem. (D1) *The cap product*

$$\cap\mu\colon H^q(\tilde{M}, \partial_1\tilde{M})\longrightarrow H_{n-q-1}(\tilde{M}, \partial_2\tilde{M})$$

induces an R-isomorphism

$$\cap \mu : T^q(\tilde{M}, \partial_1 \tilde{M}) \cong T_{n-q-1}(\tilde{M}, \partial_2 \tilde{M}),$$

(D2) *The cup product pairing*

$$\cup : H^q(\tilde{M}, \partial_1 \tilde{M}) \times H^{n-q-1}(\tilde{M}, \partial_2 \tilde{M}) \longrightarrow H^{n-1}(\tilde{M}, \partial \tilde{M})$$

induces a pairing (also denoted by $\cup$)

$$T^q(\tilde{M}, \partial_1 \tilde{M}) \times T^{n-q-1}(\tilde{M}, \partial_2 \tilde{M}) \longrightarrow T^{n-1}(\tilde{M}, \partial \tilde{M})$$

so that the composite

$$T^q(\tilde{M}, \partial_1 \tilde{M}) \times T^{n-1}(\tilde{M}, \partial_2 \tilde{M}) \xrightarrow{\cup} T^{n-1}(\tilde{M}, \partial \tilde{M}) \xrightarrow{\tilde{\mu}} R$$

is non-singular.

By applying the natural map $j^* : T^q(\tilde{M}, \partial \tilde{M}) \to T^q(\tilde{M})$ to (D2) in the case $\partial_1 M = \emptyset$, we obtain Property 0.2. Thus, when $n = 2m+1$, the quadratic form $\tilde{b}$ of $\tilde{M}$ and its signature invariants $\sigma^\gamma(M)$, $\sigma^\gamma_a(M)$ and $\hat{\sigma}^\gamma_1(M)$ are defined.

Lemma 1.1. *Except a finite number of a, $\sigma^\gamma_a(M) = 0$ and $\sigma^\gamma(M) = \sum_{a \in [-1,1]} \sigma^\gamma_a(M)$.*

Proof. Since $T^m(\tilde{M}, \partial \tilde{M})$ is finitely generated over $R\langle t \rangle$, we see that $\sigma^\gamma_a(M) = 0$ except a finite number of a. We use the identity $\tilde{b}(f(t)x, y) = \tilde{b}(x, f(t^{-1})y)$ for all x, y in $T^m(\tilde{M}, \partial \tilde{M})$ and all $f(t)$ in $R\langle t \rangle$. Then $T^m(\tilde{M}, \partial \tilde{M})_a \perp T^m(\tilde{M}, \partial \tilde{M})_{a_1}$ if $a \neq a_1$ and $T^m(\tilde{M}, \partial \tilde{M})_a \perp T^m(\tilde{M}, \partial \tilde{M})_*$. Moreover, $T^m(\tilde{M}, \partial \tilde{M})_*$ is a direct sum of the $(t-r)$-components $T^m(\tilde{M}, \partial \tilde{M})_{(t-r)}$ for all r in R with $r \neq 0$, ± 1, and $T^m(\tilde{M}, \partial \tilde{M})_{(t-r)} \perp T^m(\tilde{M}, \partial \tilde{M})_{(t-r_1)}$ if $r \neq r_1^{-1}$ (in particular, if $r = r_1$). So, $\operatorname{sign}(\tilde{b} \mid T^m(\tilde{M}, \partial \tilde{M})_*) = 0$. The result follows. Cf. [Mi, p. 129], [K2, p. 100].

Lemma 1.2. *Let $M_P = M \times CP^2$ and $\gamma_P \in H^1(M_P; Z)$ correspond to γ. Then we have $\sigma^{\gamma_P}_a(M_P) = \sigma^\gamma_a(M)$ for all a and $\hat{\sigma}^{\gamma_P}_1(M_P) = \hat{\sigma}^\gamma_1(M)$.*

Proof.

$$T^{m+2}(\tilde{M}_P, \partial \tilde{M}_P) = [T^{m+2}(\tilde{M}, \partial \tilde{M}) \otimes H^0(CP^2)]$$
$$\oplus [T^m(\tilde{M}, \partial \tilde{M}) \otimes H^2(CP^2)] \oplus [T^{m-2}(\tilde{M}, \partial \tilde{M}) \otimes H^4(CP^2)]$$

and the quadratic form $\tilde{b}_P$ of $\tilde{M}_P$ vanishes on each of the first and third summands and the second summand is orthogonal to the first and third. The restriction $\tilde{b}_P \mid T^m(\tilde{M}, \partial \tilde{M}) \otimes H^2(CP^2)$ is clearly isomorphic to the quadratic form $\tilde{b}$ of $\tilde{M}$. The result follows.

We define a product $x*y\in R$ of $x\in T_{n-q-1}(\tilde{M})$ and $y\in T_q(\tilde{M})$ by the identity $x*y=\tilde{\mu}(u\cup v)$ for $u\in T^q(\tilde{M},\partial\tilde{M})$ and $v\in T^{n-q-1}(\tilde{M},\partial\tilde{M})$ with $u\cap\mu=x$ and $v\cap\mu=y$. Since $(tu)\cap\mu=t^{-1}(u\cap\mu)$, the duality $\cap\mu\colon T^q(\tilde{M},\partial\tilde{M})\cong T_{n-q-1}(\tilde{M})$ is a t-anti-isomorphism and we have $tx*ty=x*y$. When $n=2m+1$, we define a form

$$b\colon T_m(\tilde{M})\times T_m(\tilde{M})\longrightarrow R$$

by $b(x,y)=\tilde{b}(u,v)$ for $x,y\in T_m(\tilde{M})$ and $u,v\in T^m(\tilde{M},\partial\tilde{M})$ with $u\cap\mu=x$, $v\cap\mu=y$. Clearly, the form b is t-isometric symmetric bilinear forms and $b(x,y)=x*(t^{-1}-t)y$ (if $\varepsilon(m)=-1$) or $x*y$ (if $\varepsilon(m)=1$). Further, we have $\operatorname{sign} b=\sigma^\gamma(M)$ and $\operatorname{sign}(b\,|\,T_m(\tilde{M})_a)=\sigma_a^\gamma(M)$ for all a and $\operatorname{sign}(b\,|\,(t-1)T_m(\tilde{M})_1)=\hat{\sigma}_1^\gamma(M)$, where $T_m(\tilde{M})_a$ is the $p_a(t)$-component of $T_m(\tilde{M})$. We assume that there is a leaf V of $\gamma\in H^1(M;Z)$. Let M' be a compact oriented manifold obtained from M by splitting it along V. Let $\partial' M'$ be the manifold resulting from ∂M by splitting it along ∂V. The $\operatorname{cl}(\partial M'-\partial' M')$ is the topological sum $V^+ + V^-$ with $V^\pm\cong\pm V$ by orientation preserving homeomorphisms. The infinite cyclic covering space $\tilde{M}$ is constructed from the topological sum $+_i M_i'$ of copies M_i', $i\in Z$, of M' by pasting V_{i-1}^+ to V_i^-, so that t translates each M_i' to M_{i+1}'. By identifying V with V_{-1}^+, we regard $V\subset\tilde{M}$. Let $\partial_i V=V\cap\partial_i\tilde{M}$. Then $\partial V=\partial_1 V+\partial_2 V$. Let $I_i\colon (V,\partial_i V)\subset(\tilde{M},\partial_i\tilde{M})$ be the inclusion. Let $\tilde{T}^q(\tilde{M},\partial_i\tilde{M})$ be an $R\langle t\rangle$-submodule of $H^q(\tilde{M},\partial_i\tilde{M})$ such that the natural map $H^q(\tilde{M},\partial_i\tilde{M})\to T^q(\tilde{M},\partial_i\tilde{M})$ induces an isomorphism $\tilde{T}^q(\tilde{M},\partial_i\tilde{M})\cong T^q(\tilde{M},\partial_i\tilde{M})$. Let $\tilde{I}_i^*$ be the restriction of $I_i^*\colon H^q(\tilde{M},\partial_i\tilde{M})\to H^q(V,\partial_i V)$ to $\tilde{T}^q(\tilde{M},\partial_i\tilde{M})$.

Definition 1.3. For a fixed $\tilde{T}^q(\tilde{M},\partial_1\tilde{M})$, we define a homomorphism $\pi_2\colon T_{n-q-1}(\tilde{M},\partial_2\tilde{M})\to H_{n-q-1}(V,\partial_2 V)$ so that the following square is commutative:

$$\begin{array}{ccc}
\tilde{T}^q(\tilde{M},\partial_1\tilde{M}) & \xrightarrow{\tilde{I}_1^*} & H^q(V,\partial_1 V)\\
\cong\big\downarrow\cap\mu & & \cong\big\downarrow\cap[V]\\
T_{n-q-1}(\tilde{M},\partial_2\tilde{M}) & \xrightarrow{\pi_2} & H_{n-q-1}(V,\partial_2 V),
\end{array}$$

where $[V]$ denotes the fundamental class of V.

According to if $\partial_1\tilde{M}=\emptyset$ or $\partial\tilde{M}$, we denote I_1 by I or $\bar{I}$ (and I_2 by $\bar{I}$ or I) and π_2 by $\bar{\pi}$ or π, respectively. By the uniqueness of μ, note that $\mu=\bar{I}_*([V])$. In fact, $tV-V$ respresents the same homology class as $\partial M_1'$ in $H_{n-1}(\tilde{M},\partial\tilde{M})$, so that $(t-1)\bar{I}_*([V])=0$. By definition, V represents the Poincaré dual of $\gamma\otimes 1\in H^1(M)$.

Lemma 1.4. *The composite*

$$I_{2*}\pi_2 : T_{n-q-1}(\tilde{M}, \partial_2\tilde{M}) \longrightarrow H_{n-q-1}(\tilde{M}, \partial_2\tilde{M})$$

is the inclusion map.

Proof. For $u \in \tilde{T}^q(\tilde{M}, \partial_1\tilde{M})$ with $u \cap \mu = x \in T_{n-q-1}(\tilde{M}, \partial_2\tilde{M})$,

$$I_{2*}\pi_2(x) = I_{2*}(I_1{}^*(u) \cap [V]) = u \cap \bar{I}_*([V]) = u \cap \mu = x\,,$$

completing the proof.

Lemma 1.5. *For* $x \in T_{n-q-1}(\tilde{M})$ *and* $y \in T_q(\tilde{M})$, *we have* $x * y = \mathrm{Int}_V(\pi(x), \pi(y))$, *where* Int_V *denotes the intersection pairing on* V (cf. Appendix A).

Proof. For $u \in \tilde{T}^q(\tilde{M}, \partial\tilde{M})$ and $v \in \tilde{T}^{n-q-1}(\tilde{M}, \partial\tilde{M})$ with $u \cap \mu = x$ and $v \cap \mu = y$, we have $\pi(x) = \bar{I}^*(u) \cap [V]$ and $\pi(y) = \bar{I}^*(v) \cap [V]$, so that

$$\mathrm{Int}_V(\pi(x), \pi(y)) = \varepsilon_V[(\bar{I}^*(u) \cup \bar{I}^*(v)) \cap [V]] = \varepsilon_V(\bar{I}^*(u \cup v) \cap [V])$$

$$= \varepsilon_{\tilde{M}}((u \cup v) \cap \mu) = \tilde{\mu}(u \cup v) = x * y\,,$$

where ε_X denotes the augmentation map $H_0(X) \to R$. This completes the proof.

§ 2. Splitting the middle homology of a leaf

Let $n = 2m+1$. By Lemma 1.4, the composite

$$T_m(\tilde{M}) \xrightarrow{\ \pi\ } H_m(V) \xrightarrow{\ I_*\ } H_m(\tilde{M})$$

is the inclusion $T_m(\tilde{M}) \subset H_m(\tilde{M})$. Let $\bar{T}_m(V) = I_*^{-1}T_m(\tilde{M})$, $T = \mathrm{Im}\,\pi$ and $K = \mathrm{Ker}\, I_*$. Then we have $\bar{T}_m(V) = T \oplus K$.

Lemma 2.1. $T \perp K$ *with respect to the pairing* $\mathrm{Int}_V : H_m(V) \times H_m(V) \to R$.

Proof. For $x = \pi(u \cap \mu) = \bar{I}^*(u) \cap [V] \in T$ and $y = v \cap [V] \in K$,

$$\mathrm{Int}_V(x, y) = \varepsilon_V((\bar{I}^*(u) \cup v) \cap [V]) = \varepsilon_V(\bar{I}^*(u) \cap y) = \varepsilon_{\tilde{M}}(u \cap I_*(y)) = 0\,,$$

since $I_*(y) = 0$. This completes the proof.

Since V splits $\tilde{M}$ into the submanifolds $\tilde{M}^+ = M'_0 \cup M'_1 \cup \cdots$ and $\tilde{M}^- = M'_{-1} \cup M'_{-2} \cup \cdots$, the following three boundary homomorphisms are considered: $\partial : H_{m+1}(\tilde{M}, V) \to H_m(V)$ and $\partial^\pm : H_{m+1}(\tilde{M}^\pm, V) \to H_m(V)$. Since by excision, $H_{m+1}(\tilde{M}, V) \cong H_{m+1}(\tilde{M}^+, V) \oplus H_{m+1}(\tilde{M}^-, V)$, it follows that $K = \operatorname{Ker} I_* = \operatorname{Im} \partial = K^+ + K^-$, where $K^\pm = \operatorname{Im}\partial^\pm$.

Lemma 2.2. $\operatorname{Int}_V | K^\pm = 0.$

Proof. Let $x, y \in K^+$ and $x = \partial^+ \tilde{x}$.

$$\operatorname{Int}_V(x, y) = \operatorname{Int}_V(\partial^+ \tilde{x}, y) = -\operatorname{Int}_{\tilde{M}^+}(\tilde{x}, \tilde{I}^+_*(y)),$$

where $\operatorname{Int}_{\tilde{M}^+}$ denotes the intersection pairing $H_{m+1}(\tilde{M}^+, V) \times H_m(\tilde{M}^+) \to R$ and $\tilde{I}^+_*$ denotes the natural map $H_m(V) \to H_m(\tilde{M}^+)$. Since $\tilde{I}^+_*(y) = 0$, we have $\operatorname{Int}_V(x, y) = 0$. Similarly, $\operatorname{Int}_V | K^- = 0$. This completes the proof.

From now on we will assume that $\partial M = \varnothing$. Then $\partial V = \varnothing$. Let $K^0 = K^+ \cap K^-$.

Lemma 2.3. *The null space of* $\operatorname{Int}_V | K$ *is* K^0. *Letting* $K = K^0 \oplus K_{(*)}$, *we have a subspace* $B \subset H_m(V)$ *such that* $H_m(V)$ *has an orthogonal splitting* $T \perp K_{(*)} \perp (K^0 \oplus B)$ *with respect to* Int_V.

Proof. Since $T^m(\tilde{M}) \times T^m(\tilde{M}) \xrightarrow{\cup} T^{2m}(\tilde{M}) \xrightarrow{\beta} R$ is non-singular by the Duality Theorem (D2), we see from Lemma 1.5 that $\operatorname{Int}_V | T$ is non-singular. By Lemma 2.1, we can write $H_m(V) = T \perp (K \oplus B_{(1)})$ for some $B_{(1)}$. Let $K = K_{(*)} \oplus (\text{Null})$, where (Null) is the null space of $\operatorname{Int}_V | K$. By Lemma 2.2, $(\text{Null}) \supset K^0$. We have $H_m(V) = T \perp K_{(*)} \perp [(\text{Null}) \oplus B]$ for some B. For each non-zero $x \in B$, $I_*(x)$ is no $R\langle t \rangle$-torsion element of $H_m(\tilde{M})$. So we see from the Blanchfield duality for $R\langle t \rangle$-Betti modules (cf. Appendix C) that there is an element $y \in H_{m+1}(\tilde{M})$ such that $\sum_{i \in \mathbf{Z}} \operatorname{Int}_{\tilde{M}}(I_*(x), t^i y) t^{-i} \neq 0$ in $R\langle t \rangle$. In particular, we find $y \in H_{m+1}(\tilde{M})$ such that $\operatorname{Int}_{\tilde{M}}(I_*(x), y) \neq 0$. Let $e^\pm$ be the following composite

$$H_{m+1}(\tilde{M}) \longrightarrow H_{m+1}(\tilde{M}, V) = H_{m+1}(\tilde{M}^+, V) \oplus H_{m+1}(\tilde{M}^-, V) \xrightarrow{\text{projection}} H_{m+1}(\tilde{M}^\pm, V).$$

Let $y_V = \partial^+ e^+(y) \in H_m(V)$. Then $-y_V = \partial^- e^-(y)$ and $y_V \in K^0$. Further,

$$\operatorname{Int}_V(y_V, x) = -\operatorname{Int}_{\tilde{M}^+}(e^+(y), \tilde{I}^+_*(x)) = -\operatorname{Int}_{\tilde{M}}(y, I_*(x)) \neq 0.$$

This implies that there is some $K^0_1 \subset K^0$ such that Int_V induces an isomorphism $B \cong \operatorname{Hom}_R(K^0_1, R)$. Then by Lemma 2.2, $\operatorname{Int} | K^0_1 \oplus B$ is non-singular. We have $H_m(V) = T \perp K_{(*)} \perp (K^0_1 \oplus B) \perp N_1$ for some $N_1 \subset (\text{Null})$. Int_V is non-singular. So, $N_1 = 0$ and $K^0 \subset (\text{Null}) = K^0_1 \oplus N_1 = K^0_1$, i.e., $(\text{Null}) = K^0_1 = K^0$. This completes the proof.

We regard T as an $R\langle t \rangle$-module by the identity $t \cdot x = \pi(ty)$ for all $x = \pi(y) \in T$, so that the following diagram is commutative:

$$\begin{array}{ccccc} T_m(\tilde{M}) & \xrightarrow[\cong]{\pi} & T & \xrightarrow[\cong]{I_*} & T_m(\tilde{M}) \\ \downarrow t & & \downarrow t & & \downarrow t \\ T_m(\tilde{M}) & \xrightarrow[\cong]{\pi} & T & \xrightarrow[\cong]{I_*} & T_m(\tilde{M}) . \end{array}$$

Then by Lemma 1.5, $\mathrm{Int}_V | T$ is t-isometric. Let $T_{(a)}$ be the $p_a(t)$-component of T and write $T=\oplus_a T_{(a)} \oplus T_{(*)}$, where $T_{(*)}$ has no non-trivial $p_a(t)$-torsion elements. In other words, $T_{(a)}=\pi(T_m(\tilde{M})_a)$ and $T_{(*)}=\pi(T_m(\tilde{M})_*)$, writing $T_m(\tilde{M})=\oplus_a T_m(\tilde{M})_a \oplus T_m(\tilde{M})_*$. Note that $T_{(a)} \perp T_{(a_1)}$ if $a \neq a_1$ and $T_{(a)} \perp T_{(*)}$ with respect to Int_V. To consider a splitting of $K_m(V)$, we note that $K_m(V)= I_*^{-1}[(t-1)H_m(\tilde{M})]$. This follows from the Wang exact sequence (cf. [Mi]). Then by Lemma 2.3, we have

$$K_m(V)= \underset{a \neq 1}{\perp} T_{(a)} \perp T_{(*)} \perp K_{(*)} \perp [K^0 \oplus (t-1) \cdot T_{(1)} \oplus B^{(1)}]$$

with respect to Int_V, where $B^{(1)} \subset T_{(1)} \oplus B$ and the natural map $B^{(1)} \to B_m(\tilde{M})$ is injective. By the proof of Lemma 2.3, we can find some $K_B^{(1)} \subset K^0$ so that Int_V induces $B^{(1)} \cong \mathrm{Hom}_R(K_B^{(1)}, R)$ and $\mathrm{Int}_V | K_B^{(1)} \oplus B^{(1)}$ is non-singular. Then we have

$$K_m(V)= \underset{a \neq 1}{\perp} T_{(a)} \perp T_{(*)} \perp K_{(*)} \perp (K_B^{(1)} \oplus B^{(1)}) \perp \bar{T}^{(1)}$$

for some $\bar{T}^{(1)} \subset K^0 \oplus (t-1) \cdot T_{(1)}$ with $I_*(\bar{T}^{(1)})=(t-1)T_m(\tilde{M})_1$. Let $\bar{T}^{(1)}= T^{(1)} \oplus K_T^{(1)}$ so that $I_* | T^{(1)}: T^{(1)} \cong (t-1)T_m(\tilde{M})_1$ and $K_T^{(1)} \subset K^0$. Clearly, $K^0= K_B^{(1)} \oplus K_T^{(1)}$. By the isomorphism $K_{(*)} \cong (K^+/K^0) \oplus (K^-/K^0)$, we can write $K_{(*)}=K_{(*)}^+ \oplus K_{(*)}^-$ for some $K_{(*)}^{\pm} \subset K^{\pm}$. Then we have obtained the following:

Lemma 2.4. *$K_m(V)$ has an orthogonal splitting*

$$\underset{a \neq 1}{\perp} T_{(a)} \perp T_{(*)} \perp (K_{(*)}^+ \oplus K_{(*)}^-) \perp (K_B^{(1)} \oplus B^{(1)}) \perp T^{(1)} \perp K_T^{(1)}$$

with respect to Int_V.

§ 3. A normal form of a linking matrix

Lemma 3.1. $L^{\pm} | K^+ = L^{\pm} | K^- = 0.$

Proof. Let $x=\{c_x\}$, $y=\{c_y\} \in K^+$ and $\tilde{x}=\{\tilde{c}_x\} \in H_{m+1}(\tilde{M}^+, V)$ so that $\partial^+ \tilde{x}=x$. Regard the translation c_y^- of c_y on $V \subset M$ as a translation of c_y on $V \subset \tilde{M}$. Then

$$L^+(x, y) = \mathrm{Link}_M(c_x, c_y^-) = \sum_{i>0} \mathrm{Int}_{\tilde{M}}(\tilde{c}_x, t^i c_y^-)$$
$$= \sum_{i>0} \mathrm{Int}_{\tilde{M}}(t^{-i}\tilde{c}_x, c_y^-) = -\sum_{i>0} \mathrm{Int}_V(c_{x,i}, c_y),$$

where $c_{x,i}$ is a cycle representing the image of $\{t^{-i}\tilde{c}_x\}$ under the following composite

$$H_{m+1}(t^{-i}\tilde{M}^+, t^{-i}V) \longrightarrow H_{m+1}(\tilde{M}, V \cup t^{-i}V)$$
$$= H_{m+1}(\tilde{M}^+, V) \oplus H_{m+1}(\tilde{M}^-, V \cup t^{-i}V)$$
$$\xrightarrow{\text{projection}} H_{m+1}(\tilde{M}^+, V) \xrightarrow{\partial^+} H_m(V).$$

Since $\{c_{x,i}\} \in K^+$, we see from Lemma 2.2 that $L^+(x, y) = 0$. By the identity $L^+(x, y) - L^-(x, y) = \mathrm{Int}_V(x, y)$ $(=0)$, we also have $L^-(x, y) = 0$. Similarly, $L^\pm \mid K^- = 0$. This completes the proof.

Note that $(t-1) \cdot T = \pi((t-1)T_m(\tilde{M})) = K_m(V) \cap T$.

Lemma 3.2. *For* $x = (t-1) \cdot x_1 = t \cdot x_1 - x_1 \in (t-1) \cdot T$ *and* $y \in K_m(V)$ *such that* $\mathrm{Int}_V(t \cdot x_1, y) = \mathrm{int}_V(x_1, y) = 0$, *we have* $L^\pm(x, y) = 0$.

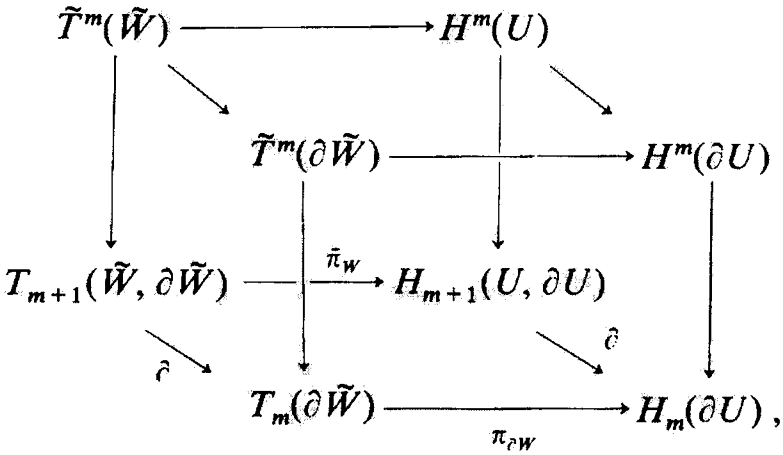
Proof. Let $W = M \times [-1, 1]$ and $\tilde{\gamma} \in H^1(W; Z)$ correspond to γ by the natural isomorphism $H^1(W; Z) \cong H^1(M; Z)$. We construct a leaf U of $\tilde{\gamma}$ by using a leaf V of γ so that

$$(U, U \cap M \times (-1), U \cap M \times 1) = (U, V \times (-1), tV \times 1) \cong (M', V^-, V^+).$$

Let $\tilde{T}^m(\tilde{M} \times (\pm 1))$ be isomorphic to $\tilde{T}^m(\tilde{M})$ by the obvious maps and $\tilde{T}^m(\tilde{W})$ be isomorphic to $\tilde{T}^m(\tilde{M} \times 1)$ by the natural isomorphism $H^m(\tilde{W}) \cong H^m(\tilde{M} \times 1)$. Let $\tilde{T}^m(\partial\tilde{W}) = \tilde{T}^m(\tilde{M} \times (-1)) \oplus \tilde{T}^m(\tilde{M} \times 1)$. Then the natural map $H^m(\tilde{W}) \to H^m(\partial\tilde{W})$ induces a map $\tilde{T}^m(\tilde{W}) \to \tilde{T}^m(\partial\tilde{W})$. By Definition 1.4 we have the following square:

$$\begin{array}{ccc} T_{m+1}(\tilde{W}, \partial\tilde{W}) & \xrightarrow{\pi_W} & H_{m+1}(U, \partial U) \\ \partial \downarrow & & \downarrow \partial \\ T_m(\partial\tilde{W}) & \xrightarrow{\pi_{\partial W}} & H_m(\partial U). \end{array}$$

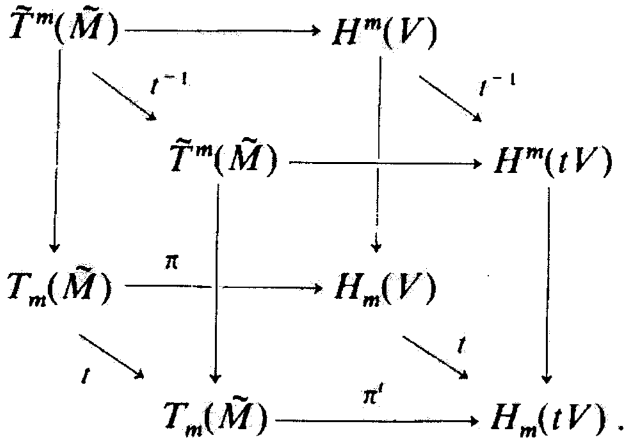
It commutes, which can be seen by examining the following diagram:



where the vertical maps are the duality maps and the others are natural except $\tilde{\pi}_W$ and $\pi_{\partial W}$. The map $\pi_{\partial W}: T_m(\partial\tilde{W})\to H_m(\partial U)$ is identical with the map

$$\pi\times(-1)+\pi'\times 1: T_m(\tilde{M}\times(-1))\oplus T_m(\tilde{M}\times 1)$$
$$\longrightarrow H_m(V\times(-1))\oplus H_m(tV\times 1),$$

where $\pi'=t\pi t^{-1}$, which follows by checking the following diagram (with vertical duality maps):



Let $x_1=\{c\}$ and $t\cdot x_1=\{c_t\}$ in T. Since c_t is homologous to tc in $\tilde{M}$, we have an $(m+1)$-chain $\tilde{c}'$ in $\tilde{W}$ such that $\partial\tilde{c}'=c_t\times(-1)+(tc)\times 1$. Since $\partial: H_{m+1}(\tilde{W}, \partial\tilde{W})\to H_m(\partial\tilde{W})$ is injective, and $\{c_t\times(-1)\}$ and $\{(tc)\times 1\}$ are in $T_m(\partial\tilde{W})$, we see that $\{\tilde{c}'\}\in T_{m+1}(\tilde{W}, \partial\tilde{W})$. Let $\tilde{c}''$ be an $(m+1)$-chain in U representing the element $\tilde{\pi}_W(\{\tilde{c}'\})$. Then

$$\partial\{\tilde{c}''\}=\pi_{\partial W}\partial\{\tilde{c}'\}=\pi\times(-1)\{c_t\times(-1)\}+\pi'\times 1\{(tc)\times 1\}$$
$$=\{c_t\times(-1)\}+\{(tc)\times 1\}$$

in $H_m(\partial U)$ by Lemma 1.4. Thus we have showed the following:

Assertion 3.3. *For $x_1=\{c\}$ and $t\cdot x_1=\{c_t\}$ in T, there is an $(m+1)$-chain $\bar{c}$ in $M'_0\subset\tilde{M}$ such that $\partial\bar{c}=c_t-tc$.*

For $y=\{c_y\}$ in $K_m(V)$,

$$L^+(x,y)=\mathrm{Link}_M(c_t-c,\ c_y^-)=\mathrm{Int}_{\tilde{M}}(\bar{c},\ tc_y^-)$$
$$=\mathrm{Int}_{tV}(-tc,\ tc_y)=-\mathrm{Int}_V(x_1,y)=0$$

and

$$L^-(x,y)=\mathrm{Int}_{\tilde{M}}(\bar{c},\ c_y^+)=-\mathrm{Int}_V(c_t,\ c_y)=-\mathrm{Int}_V(t\cdot x_1,\ y)=0\,.$$

This completes the proof of Lemma 3.2.

We denote by $T_{(s)}$ and $T_{(\nu)}$ the orthogonal summands $\perp_{a\neq 1}T_{(a)}\perp T_{(*)}$ and $(K^+_{(*)}\oplus K^-_{(*)})\perp(K_B^{(1)}\oplus B^{(1)})\perp T^{(1)}\perp K_T{}^{(1)}$ of $K_m(V)$ appearing in Lemma 2.4, respectively. Since $(t-1)\cdot T_{(s)}=T_{(s)}$, we obtain from Lemma 3.2 a linking matrix A on $K_m(V)$ which is the block sum of linking matrices A_s on $T_{(s)}$ and A_ν on $T_{(\nu)}$. [Note that $L^\pm(x,y)=\varepsilon(m)L^\mp(y,x)$ for all $x,y\in K_m(V)$.] Moreover, A_s is the block sum $\bigoplus_{a\neq 1}A_a\oplus A_*$ with A_a and A_* linking matrices on $T_{(a)}$ and $T_{(*)}$, respectively. Let A_1 be a linking matrix on $(t-1)\cdot T_{(1)}$. Since

$$T^{(1)}\xrightarrow[\cong]{i_*}(t-1)T_m(\tilde{M})_1\xrightarrow[\cong]{\pi}(t-1)\cdot T_{(1)}$$

and $T^{(1)}\subset K^0\oplus(t-1)\cdot T_{(1)}$, we see from Lemmas 3.1 and 3.2 that A_1 is a linking matrix on $T^{(1)}$. Further applying Lemmas 3.1 and 3.2 to the direct summands of $T_{(\nu)}$, we have the following:

Lemma 3.4. *$A=A_s\oplus A_\nu$, $A_s=\bigoplus_{a\neq 1}A_a\oplus A_*$ and A_ν is given as follows (C_{ij} are matrices undetermined):*

	$T^{(1)}$	$K^+_{(*)}$	$K^-_{(*)}$	$K_B^{(1)}$	$B^{(1)}$	$K_T^{(1)}$
$T^{(1)}$	A_1	0	0	0	C_{15}	0
$K^+_{(*)}$	0	0	C_{23}	0	C_{25}	0
$K^-_{(*)}$	0	C_{32}	0	0	C_{35}	0
$K_B^{(1)}$	0	0	0	0	C_{45}	0
$B^{(1)}$	C_{51}	C_{52}	C_{53}	C_{54}	C_{55}	C_{56}
$K_T^{(1)}$	0	0	0	0	C_{65}	0

§4. Relationship between a homology module and a linking matrix

The purpose of this section is to prove the following lemma:

Lemma 4.1. *Let A_a, A_* and $A_\natural$ are matrices appearing in Lemma 3.4. Then $tA_a+\varepsilon(m)A'_a$ $(a\neq 1)$, $tA_*+\varepsilon(m)A'_*$, $tA_1+\varepsilon(m)A'_1$ and $tA_\natural+\varepsilon(m)A'_\natural$ are $R\langle t\rangle$-presentation matrices of $T_m(\tilde{M})_a$, $T_m(\tilde{M})_*$, $(t-1)T_m(\tilde{M})_1$ and $(t-1)T_m(\tilde{M})_1\oplus$(a free $R\langle t\rangle$-module), respectively.*

Let $e_1, e_2, \cdots, e_q$ be a basis for $K_m(V)$ such that each e_i is in a direct summand of the splitting of $K_m(V)$ of *Lemma* 2.4.

Lemma 4.2. *There are elements e_1^*, e_2^*, $\cdots$, e_q^* in $H_m(V)$ with $\mathrm{Int}_V(e_i^*, e_j)=\delta_{ij}$ for all i, j such that*

(1) *If e_i is not in $T^{(1)}$ or $K_T^{(1)}$, then e_i and e_i^* are in the same orthogonal summand of the splitting of $K_m(V)$, and moreover, if e_i is in $B^{(1)}$, then e_i^* is in $K_B^{(1)}$.*

(2) *If e_i is in $T^{(1)}$, then e_i^* is in $T_{(1)}$.*

Proof. First we construct e_i^* of the case (1). The construction is easy, since Int_V is non-singular on each orthogonal summand other than $T^{(1)}$ and $K_T^{(1)}$ and induces $B^{(1)}\cong \mathrm{Hom}_R(K_B^{(1)}, R)$ and is zero on $K_B^{(1)}$. Next we construct e_i^* of the case (2). To do it, we need some preliminaries. Let $K_B^{(1)}\oplus B^{(1)}=(K_B^{(2)}\oplus B^{(2)})\perp(K_B^{(3)}\oplus B^{(3)})$ where $B^{(2)}=B^{(1)}\cap B$ with B in Lemma 2.3 and $\mathrm{Int}_V\,|\,K_B^{(2)}\oplus B^{(2)}$ is non-singular and $K_B^{(1)}=K_B^{(2)}\oplus K_B^{(3)}$.

Assertion 4.3. *The composite*

$$B^{(3)}\subset K_B^{(1)}\oplus B^{(1)}\subset K_B^{(1)}\oplus T_{(1)}\oplus B \xrightarrow{\text{projection}} B\subset H_m(V)\xrightarrow{i_*} H_m(M)$$

is injective.

Proof. If $x=x^K+x^T+x^B\in B^{(3)}$ $(x^K\in K_B^{(1)},\ x^T\in T_{(1)},\ x^B\in B)$ is non-zero and sent to 0 by the above composite, then $x^B\in B^{(2)}$ and $x^B\neq 0$. So there is an element $y\in K_B^{(2)}$ with $\mathrm{Int}_V(x^B, y)\neq 0$. Then $\mathrm{Int}_V(x, y)=\mathrm{Int}_V(x^B, y)\neq 0$, which is a contradiction. This completes the proof.

Let $x_i=x_i^K+x_i^T+x_i^B$ $(x_i^K\in K_B^{(1)},\ x_i^T\in T_{(1)},\ x_i^B\in B)$, $i=1, 2, \cdots, r$, be a basis for $B^{(3)}$.

Assertion 4.4. *The elements $x_1^T, x_2^T, \cdots, x_r^T$ are linearly independent in $T_{(1)}/(t-1)\cdot T_{(1)}$.*

Proof. If $\sum_{i=1}^r c_i x_i^T\in(t-1)\cdot T_{(1)}$, then $\sum_{i=1}^r c_i x_i^T$ is sent to 0 by the map i_*. Clearly,

$$i_*(\textstyle\sum_{i=1}^r c_i x_i)=i_*(\sum_{i=1}^r c_i x_i^K)=0\,.$$

So $i_*(\sum_{i=1}^r c_i x_i^B)=0$. By Assertion 4.3, we have $\sum_{i=1}^r c_i x_i=0$ and $c_i=0$ for all i. This completes the proof.

Assertion 4.5. *There are elements* $y_1^T, y_2^T, \cdots, y_r^T$ *in* $\mathrm{Ker}(t-1) = \mathrm{Ker}(t-1: T_{(1)} \to T_{(1)})$ *such that* $\mathrm{Int}_V(y_i^T, x_j^T) = \delta_{ij}$ *for all* i, j.

Proof. Int_V induces a non-singular pairing

$$\mathrm{Ker}(t-1) \times [T_{(1)}/(t-1) \cdot T_{(1)}] \longrightarrow R\,.$$

By Assertion 4.4 we find the desired elements, completing the proof.

To construct e_i^* of the case (2), let $e_1, e_2, \cdots, e_s$ be a basis for $T^{(1)}$. Note that there are elements $e_1^1, e_2^1, \cdots, e_s^1$ in $T_{(1)}$ such that $\mathrm{Int}_V(e_i^1, e_j) = \delta_{ij}$ and $e_1^1, e_2^1, \cdots, e_s^1$ form a basis for $T_{(1)}/\mathrm{Ker}(t-1)$, because $T^{(1)} \xrightarrow[\cong]{\pi I_*} (t-1) \cdot T_{(1)}$ and $T^{(1)} \subset K^0 \oplus (t-1) \cdot T_{(1)}$ and Int_V induces a non-singular pairing

$$[T_{(1)}/\mathrm{Ker}(t-1)] \times (t-1) \cdot T_{(1)} \longrightarrow R\,.$$

Let $c_{ij} = \mathrm{Int}_V(e_i^1, x_j^T)$ and $e_i^* = e_i^1 - \sum_{j=1}^r c_{ij} y_j^T \in T_{(1)}$. Then

$$\mathrm{Int}_V(e_i^*, e_j) = \mathrm{Int}_V(e_i^1, e_j) = \delta_{ij}\,,$$

since

$$\mathrm{Int}_V(\mathrm{Ker}(t-1), K^0 \oplus (t-1) \cdot T_{(1)}) = 0\,.$$

Moreover,

$$\begin{aligned}\mathrm{Int}_V(e_i^*, x_j) &= \mathrm{Int}_V(e_i^*, x_j^T) \\ &= \mathrm{Int}_V(e_i^1, x_j^T) - \textstyle\sum_{k=1}^r c_{ik} \mathrm{Int}_V(y_k^T, x_j^T) = c_{ij} - c_{ij} = 0\,.\end{aligned}$$

Hence e_i^* is orthogonal to $B^{(3)}$. Further using that $e_i^* \in T_{(1)}$, we see that e_i^* is orthogonal to the summands other than $T^{(1)}$. Thus, the elements e_i^* of the cases (1) and (2) are constructed. Hereafter, it is easy to construct e_i^* for the basis of $K_T^{(1)}$. This completes the proof of Lemma 4.2.

Let $\bar{e}_1, \cdots, \bar{e}_q, \bar{e}_{q+1}, \cdots, \bar{e}_N$ be a basis for $H_m(M, V)$ so that, for the boundary map $\partial: H_{m+1}(M, V) \to H_m(V)$, $\partial\bar{e}_1 = e_1, \cdots, \partial\bar{e}_q = e_q$ form a basis for $K_m(V)$ and $\partial\bar{e}_{q+1} = 0, \cdots, \partial\bar{e}_N = 0$. Let $\bar{e}_i$ be the image of $\bar{e}_i$ under the excision isomorphism $H_{m+1}(M, V) \cong H_{m+1}(M', V^+ \cup V^-)$. Let $e'_1, \cdots, e'_N$ be a basis of $H_m(M')$ such that $\mathrm{Int}_{M'}(e'_i, \bar{e}_j) = \delta_{ij}$. Let $I^\pm$ be the natural injections $V \cong V^\mp \subset M'$.

Lemma 4.6. *For* $i \le q$, $e'_i = \varepsilon(m+1)[I_*^+(e_i^*) - I_*^-(e_i^*)]$.

Proof. Let $I_*^\pm(e_i) = \{c_i^{*\pm}\}$ for cycles $c_i^{*\pm}$ in $\mathrm{Int}\, M'$. For $i, j \le q$

$$\begin{aligned}\mathrm{Int}_{M'}(I_*^+(e_i^*)-I_*^-(e_i^*),\ \bar{e}_j) &= \mathrm{Int}_{M'}(\bar{e}_j,\ I_*^+(e_i^*)-I_*^-(e_i^*))\\ &=\mathrm{Int}_M(\bar{c}_j,\ c_i^{*+}-c_i^{*-})=\mathrm{Link}_M(c_j,\ c_i^{*+}-c_i^{*-})\\ &=\varepsilon(m+1)\mathrm{Link}_M(c_i^{*+}-c_i^{*-},\ c_j)\\ &=\varepsilon(m+1)\ \mathrm{Int}_V(e_i^*,\ e_j)=\varepsilon(m+1)\delta_{ij}\,,\end{aligned}$$

where $e_j=\{c_j\}$ and $\bar{e}_j=\{\bar{c}_j\}$ for cycles c_j in V and $\bar{c}_j$ in M with $\partial\bar{c}_j=c_j$. For $i\leq q$ and $j\geq q+1$,

$$\mathrm{Int}_{M'}(I_*^+(e_i^*)-I_*^-(e_i^*),\ \bar{e}_j)=\mathrm{Int}_M(i_*(e_i^*)-i_*(e_i^*),\ \bar{e}_j)=0\,,$$

because $\bar{e}_j$ is represented by a cycle in M. Since $\mathrm{Int}_{M'}(e'_i,\ \bar{e}_j)=\delta_{ij}$ for all i, j, we have $e'_i=\varepsilon(m+1)[I_*^+(e_i^*)-I_*^-(e_i^*)]$ for $i\leq q$. This completes the proof.

From the Mayer/Vietoris sequence, we can obtain the following $R\langle t\rangle$-exact sequence (cf. Levine [L]):

$$\begin{aligned}&\longrightarrow H_p(V)\otimes R\langle t\rangle \xrightarrow{I_*^- - tI_*^+} H_p(M')\otimes R\langle t\rangle\\ &\xrightarrow{J} H_p(\tilde{M}) \xrightarrow{\partial} H_{p-1}(V)\otimes R\langle t\rangle \longrightarrow\,,\end{aligned}$$

where J is an $R\langle t\rangle$-map obtained by composing the identification map $H_p(M')\otimes R\langle t\rangle\cong H_p(+_iM'_i)$ to the map induced by the quotient map $+_iM'_i\to\tilde{M}$. Let $K_m(M')$ be the subspace of $H_m(M')$ generated by $e'_1,\cdots,e'_q$. Let $K_m(\tilde{M})=J(K_m(M')\otimes R\langle t\rangle)$.

Lemma 4.7. *The above sequence induces an exact sequence*

$$K_m(V)\otimes R\langle t\rangle \xrightarrow{I_*^- - tI_*^+} K_m(M')\otimes R\langle t\rangle \xrightarrow{J} K_m(\tilde{M}) \longrightarrow 0\,,$$

and we have $I_*^+(e_1,\cdots,e_q)=(e'_1,\cdots,e'_q)\varepsilon(m+1)A'$ *and* $I_*^-(e_1,\cdots,e_q)=(e'_1,\cdots,e'_q)A$ *and*

$$(I_*^- - tI_*^+)(e_1,\cdots,e_q)=(e'_1,\cdots,e'_q)(A+\varepsilon(m)tA')$$

for the linking matrix A on the basis $e_1,\cdots,e_q$ of $K_m(V)$.

Proof. Let $I_*^\pm(e_i)=a_{i1}^\pm e'_1+\cdots+a_{iN}^\pm e'_N$, $a_{ij}^\pm\in R$. For $j\geq q+1$,

$$a_{ij}^\pm=\mathrm{Int}_{M'}(I_*^\pm(e_i),\ \bar{e}_j)=\mathrm{Int}_M(i_*(e_i),\ \bar{e}_j)=0\,,$$

since $\bar{e}_j$ is represented by a cycle in M and $i_*(e_i)=0$. So, $I_*^\pm(K_m(V))\subset K_m(M')$ and the above sequence is semi-exact. To show the exactness, let $x\in K_m(M')\otimes R\langle t\rangle$ with $J(x)=0$. Then there is an element $y\in H_m(V)\otimes R\langle t\rangle$

such that $(I_*^- - tI_*^+)(y) = x$. Consider the following commutative square:

$$\begin{array}{ccc} H_m(V)\otimes R\langle t\rangle & \xrightarrow{I_*^- - tI_*^+} & H_m(M')\otimes R\langle t\rangle \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle j_*} \\ H_m(M)\otimes R\langle t\rangle & \xrightarrow{1-t} & H_m(M)\otimes R\langle t\rangle , \end{array}$$

where i_* and j_* are the maps induced by inclusions. By Lemma 4.6, we have $j_*(I_*^- - tI_*^+)(y) = j_*(x) = 0$. So, $(1-t)i_*(y) = 0$ and $i_*(y) = 0$, implying that $x \in K_m(M')\otimes R\langle t\rangle$. Thus, the above sequence is exact. Next, let $I_*^\pm(e_i) = \{c_i^\pm\}$, $e_i = \{c_i\}$ and $\bar{e}_i = \{\bar{c}_i\}$ for $c_i^\pm$ in Int M' and c_i in V and $\bar{c}_i$ in M with $\partial\bar{c}_i = c_i$ $(i \leq q)$. Then for $i, j \leq q$,

$$\begin{aligned} \mathrm{Link}_M(c_i^\pm, c_j) &= \varepsilon(m+1)\mathrm{Link}_M(c_j, c_i^\pm) = \varepsilon(m+1)\,\mathrm{Int}_M(\bar{c}_j, c_i^\pm) \\ &= \varepsilon(m+1)\mathrm{Int}_{M'}(\bar{e}_j, I_*^\pm(e_i)) = \varepsilon(m+1)\mathrm{Int}_{M'}(I_*^\pm(e_i), \bar{e}_j) = \varepsilon(m+1)a_{ij}^\pm . \end{aligned}$$

For $A = (a_{ij})$ with $a_{ij} = \mathrm{Link}_M(c_i^+, c_j)$, we have

$$I_*^+(e_1, \cdots, e_q) = (e'_1, \cdots, e'_q)\varepsilon(m+1)A' .$$

Further,

$$\mathrm{Link}_M(c_i^-, c_j) = \varepsilon(m+1)\mathrm{Link}_M(c_j^+, c_i) = \varepsilon(m+1)a_{ji} ,$$

which implies that $I_*^-(e_1, \cdots, e_q) = (e'_1, \cdots, e'_q)A$. This completes the proof.

If $\{e_1, \cdots, e_s\}$ is a basis for $T_{(a)}$ $(a \neq 1)$, $T_{(*)}$ or $T^{(1)}$, then we see from Lemma 4.2 that $\{(1-t)I_*(e_1^*), \cdots, (1-t)I_*(e_s^*)\}$ is a basis (over R) for $T_m(\tilde{M})_a$ $(a \neq 1)$, $T_m(\tilde{M})_*$ or $(t-1)T_m(\tilde{M})_1$, respectively.

Lemma 4.8. *If $\{e_1, \cdots, e_s\}$ is a basis for $T_{(a)}$ $(a \neq 1)$, $T_{(*)}$ or $T^{(1)}$, on which the linking matrix is A_a, A_* or A_1, then we have $I_*(e_1, \cdots, e_s) = ((1-t)I_*(e_1^*), \cdots, (1-t)I_*(e_s^*))\varepsilon(m+1)\tilde{A}$ with $\tilde{A}$ being A_a, A_* or A_1, respectively. In particular, A_a, $a \in [-1, 1]$, and A_* are non-singular.*

Proof. Let $\{e_1, \cdots, e_s\}$ be the basis of $T_{(a)}$, $a \neq 1$. By Lemmas 3.4 and 4.7, $I_*^-(e_1, \cdots, e_s) = (e'_1, \cdots, e'_s)A_a$. Noting that $JI_*^- = I_*$ and $J(I_*^+(e_i^*) - I_*^-(e_i^*)) = (1-t)I_*(e_i^*)$, we see from Lemma 4.6 that $I_*(e_1, \cdots, e_s) = ((1-t)I_*(e_1^*), \cdots, (1-t)I_*(e_s^*))\varepsilon(m+1)A_a$. I_* induces an isomorphism $T_{(a)} \cong T_m(\tilde{M})_a$. So, det $A_a \neq 0$. The same proof is applicable to A_*. Let $\{e_1, \cdots, e_s\}$ be the basis of $T^{(1)}$. By Lemma 3.4

$$I_*^-(e_1, \cdots, e_s) = (e'_1, \cdots, e'_s, e'_{s+1}, \cdots, e'_r)\begin{pmatrix} A_1 \\ C_{51} \end{pmatrix}$$

for some matrix C_{51}, where $\{e'_{s+1}, \cdots, e'_r\}$ corresponds to a basis

$\{e_{s+1}, \cdots, e_r\}$ of $B^{(1)}$. By Lemmas 4.2(1) and 4.6, we have

$$I_*(e_1, \cdots, e_s) = JI_*^-(e_1, \cdots, e_s)$$
$$= ((1-t)I_*(e_1^*), \cdots, (1-t)I_*(e_s^*))\varepsilon(m+1)A_1 .$$

Since I_* induces an isomorphism $T^{(1)} \cong (1-t)T_m(\tilde{M})_1$, we have det $A_1 \neq 0$. This completes the proof.

Proof of Lemma 4.1. *Let* $K_m(M')_a$ $(a \neq 1)$, $K_m(M')_*$, $K_m(M')_\natural$, $K_m(M')_1$ and $K_m(M')_B^1$ be the subspaces of $K_m(M')$ which correspond to $T_{(a)}$, $T_{(*)}$, $T_{(\natural)}$, $T^{(1)}$ and $B^{(1)}$ by the correspondence $K_m(V) \to K_m(M')$ sending e_i to e_i'. By Lemmas 3.4 and 4.7,

$$T_{(a)} \otimes R\langle t\rangle \xrightarrow{I_*^- - tI_*^+} K_m(M')_a \otimes R\langle t\rangle \xrightarrow{J} T_m(\tilde{M})_a \longrightarrow 0$$

is exact for $a \neq 1$, which shows that $A_a + \varepsilon(m)tA_a'$ is an $R\langle t\rangle$-presentation matrix of $T_m(\tilde{M})_a$, $a \neq 1$. Similarly, the sequence

$$T_{(*)} \otimes R\langle t\rangle \xrightarrow{I_*^- - tI_*^+} K_m(M')_* \otimes R\langle t\rangle \xrightarrow{J} T_m(\tilde{M})_* \longrightarrow 0$$

is exact and $A_* + \varepsilon(m)tA_*'$ is an $R\langle t\rangle$-presentation matrix of $T_m(\tilde{M})_*$. The same reason also implies that the sequence

$$T_{(\natural)} \otimes R\langle t\rangle \xrightarrow{I_*^- - tI_*^+} K_m(M')_\natural \otimes R\langle t\rangle \xrightarrow{J} K_m(\tilde{M})_\natural \longrightarrow 0$$

is exact, where $K_m(\tilde{M})_\natural = J(K_m(M')_\natural \otimes R\langle t\rangle)$. Noting that $\mathrm{Tor}_{R\langle t\rangle} K_m(\tilde{M})_\natural = (t-1)T_m(\tilde{M})_1$, we see that $A_\natural + \varepsilon(m)tA_\natural'$ is an $R\langle t\rangle$-presentation matrix of a direct sum of $(t-1)T_m(\tilde{M})_1$ and a free $R\langle t\rangle$-module. By Lemmas 3.4 and 4.7 we have $I_*^\pm(T^{(1)}) \subset K_m(M')_1 \oplus K_m(M')_B^1$. Let $(I_*^- - tI_*^+)_1$ be the composite

$$T^{(1)} \otimes R\langle t\rangle \xrightarrow{I_*^- - tI_*^+} [K_m(M')_1 \oplus K_m(M')_B^1] \otimes R\langle t\rangle$$
$$\xrightarrow{\text{projection}} K_m(M')_1 \otimes R\langle t\rangle .$$

By Lemmas 4.2(1) and 4.6, $J(K_m(M')_B^1 \otimes R\langle t\rangle) = 0$. So we have a semi-exact sequence

$$T^{(1)} \otimes R\langle t\rangle \xrightarrow{(I_*^- - tI_*^+)_1} K_m(M')_1 \otimes R\langle t\rangle \xrightarrow{J} (t-1)T_m(\tilde{M})_1 \longrightarrow 0 .$$

By Lemmas 3.4 and 4.7 we have

$$(I_*^- - tI_*^+)_1(e_1, \cdots, e_s) = (e_1', \cdots, e_s')(A_1 + \varepsilon(m)tA_1') .$$

By Lemma 4.8, det $A_1 \neq 0$ and hence $(I_*^- - tI_*^+)_1$ is injective. This implies that $\dim_R \mathrm{Coker}(I_*^- - tI_*^+)_4 =$ the size of $A_1 = s$. Since $\dim_R (1-t)T_m(\tilde{M})_1 = \dim_R T^{(1)} = s$ and J sends $\mathrm{Coker}(I_*^- - tI_*^+)_1$ onto $(t-1)T_m(\tilde{M})_1$, we see that J sends $\mathrm{Coker}(I_*^- - tI_*^+)_1$ isomorphically onto $(t-1)T_m(\tilde{M})_1$. So the above semi-exact sequence is actually exact. Thus, $A_1 + \varepsilon(m)tA_1'$ is an $R\langle t\rangle$-presentation matrix of $(t-1)T_m(\tilde{M})_1$. This completes the proof of Lemma 4.1.

§5. The signature invariants of a real matrix

Let $A(x)=(a_{ij}(x))$ be a Hermitian matrix such that $a_{ij}(x)$ are continuously variable on x in a space X. Let $r=\max_{x\in X}\mathrm{rank}_C A(x)\geq 1$, C being the complex number field.

Lemma 5.1. *For any $x_0\in X$ with $\mathrm{rank}_C A(x_0)=r$, there is a neighborhood $N(x_0)$ of x_0 in X such that $\mathrm{sign}\, A(x)=\mathrm{sign}\, A(x_0)$ for all $x\in N(x_0)$.*

The following is direct from Lemma 5.1:

Corollary 5.2. *Assume that X is connected and $\mathrm{rank}_C A(x)$ is constant on X. Then $\mathrm{sign}\, A(x)$ is constant on X.*

Proof of Lemma 5.1. We use a standard argument on a relationship between the signature and the principal minors (cf. Iyanaga/Kodaira [I/K]). We can take principal submatrices $A_1(x)\subset A_2(x)\subset\cdots\subset A_r(x)\subset A(x)$ ($\subset$ means a principal submatrix) so that

(1) $A_i(x)$ is of size i,
(2) Letting $F_i(x)=\det A_i(x)$, we have $F_r(x_0)\neq 0$,
(3) If $F_i(x_0)=0$ for some i, then $F_{i-1}(x_0)F_{i+1}(x_0)<0$,

where we let $F_0(x)=1$ for all x.

Then we have $\mathrm{sign}\, A(x_0)=\sum_{i=1}^r \mathrm{sign}(F_{i-1}(x_0)F_i(x_0))$, where $\mathrm{sign}\, c=c/|c|$ (if $c\in R-\{0\}$) or 0 (if $c=0$). Let $N(x_0)$ be a neighborhood of x_0 in X so that $\mathrm{sign}\, F_i(x)=\mathrm{sign}\, F_i(x_0)$ for all $x\in N(x_0)$ and i with $F_i(x_0)\neq 0$. If $F_i(x_1)=0$ for some $x_1\in N(x_0)$ and some i, then $F_i(x_0)=0$, so that $F_{i-1}(x_1)F_{i+1}(x_1)<0$. Thus, we have $\mathrm{sign}\, A(x)=\sum_{i=1}^r \mathrm{sign}(F_{i-1}(x)F_i(x))$ for all $x\in N(x_0)$. If $F_i(x_0)=0$, then $F_{i-1}(x)F_{i+1}(x)<0$ and hence $\mathrm{sign}(F_{i-1}(x)F_i(x))+\mathrm{sign}(F_i(x)F_{i+1}(x))=0$ for all $x\in N(x_0)$. This implies that $\mathrm{sign}\, A(x)=\mathrm{sign}\, A(x_0)$ for all $x\in N(x_0)$, completing the proof.

Let $A(t)$ be a t-Hermitian $R\langle t\rangle$-matrix. Then $A(\omega)$ is Hermitian for all $\omega\in S^1$ and by Lemma 5.1 $\mathrm{sign}\, A(\omega)$ is locally constant except a finite number of ω, for $\mathrm{rank}_C A(\omega)=\mathrm{rank}_{R\langle t\rangle} A(t)$ except a finite number of ω. Recall the notation $\omega_x=x+(1-x^2)^{1/2}i\in S^1$ for $x\in[-1,1]$.

Definition 5.3. For $a\in[-1,1)$, $\tau_{a+0}(A(t))=\lim_{x\to a+0}\mathrm{sign}\, A(\omega_x)$ and for $a\in(-1, 1]$, $\tau_{a-0}(A(t))=\lim_{x\to a-0}\operatorname{sign} A(\omega_x)$.

Note that $\tau_{a\pm 0}(A(t))$ are locally constant on a except a finite number of a. For a real square matrix A and $\varepsilon=\pm 1$, we define a t-Hermitian $R\langle t\rangle$-matrix $A^{\varepsilon}(t)$ by

$$A^{\varepsilon}(t)=[(1-t^{-1})-\varepsilon(1-t)][(1-t)A-\varepsilon(1-t^{-1})A'] .$$

Definition 5.4. For $a\in(-1, 1)$, $\sigma_a^{\varepsilon}(A)=\varepsilon[\tau_{a+0}(A^{\varepsilon}(t))-\tau_{a-0}(A^{\varepsilon}(t))]$ and $\sigma_{\varepsilon}^{\varepsilon}(A)=\operatorname{sign}(A+A')-\tau_{\varepsilon-\varepsilon 0}(A^{\varepsilon}(t))$ and $\sigma_{-\varepsilon}^{\varepsilon}(A)=\tau_{-\varepsilon+\varepsilon 0}(A^{\varepsilon}(t))$.

Using that $\tau_{a\pm 0}(A^{\varepsilon}(t))$ are locally constant on a except a finite number of a, one can easily check that $\sigma_a^{\varepsilon}(A)=0$ except a finite number of a and $\sum_{a\in[-1,1]}\sigma_a^{\varepsilon}(A)=\operatorname{sign}(A+A')$.

Lemma 5.5. *Let A be a real square matrix, $\varepsilon=\pm 1$ and $a_1\in[-1, 1]$. We assume that $A+\varepsilon tA'$ is an $R\langle t\rangle$-presentation matrix of an $R\langle t\rangle$-module whose $p_a(t)$-components are trivial except when $a=a_1$. Then $\tau_{x\pm 0}(A^{\varepsilon}(t))=\operatorname{sign}(A+A')$ (if $\varepsilon x>\varepsilon a_1$) or 0 (if $\varepsilon x<\varepsilon a_1$).*

Proof. Let $\varepsilon=1$. Then $A^1(t)=(t-t^{-1})(1-t)(A+t^{-1}A')$ and $\operatorname{rank}_C(A+\omega_x^{-1}A')$ is constant on $x\in[-1, 1]$ with $x\neq a_1$, since $p_{a_1}(\omega_x^{-1})\neq 0$. Note that $A^1(\omega_x)=2(1-x)[(1+x)+(1-x^2)^{1/2}i](A+\omega_x^{-1}A')$. Let $x<a_1$. Then $a_1\neq -1$. By Corollary 5.2,

$$\tau_{x\pm 0}(A^1(t))=\lim_{x\to -1+0}\operatorname{sign}[(1+x)^{-1/2}A^1(\omega_x)]=\operatorname{sign}[i(A-A')] ,$$

which is 0 [To see this, note that $\operatorname{sign} C_1=\operatorname{sign}\bar{C}_1$ for a Hermitian matrix C_1 and its conjugate $\bar{C}_1$. Since A is real, $\operatorname{sign}[i(A-A')]=\operatorname{sign}[-i(A-A')]$ and $\operatorname{sign}[i(A-A')]=0$]. Let $x>a_1$. Then $a_1\neq 1$. By Corollary 5.2,

$$\tau_{x\pm 0}(A^1(t))=\lim_{x\to 1-0}\operatorname{sign} A^1(\omega_x)=\lim_{x\to 1-0}\operatorname{sign}[(1-x^2)^{-1}A^1(\omega_x)]$$
$$=\lim_{x\to 1-0}\operatorname{sign}[\{1+(1-x)^{1/2}(1+x)/^{1/2}i\}(A+\omega_x^{-1}A')]=\operatorname{sign}(A+A') .$$

Next, let $\varepsilon=-1$. Then

$$A^{-1}(t)=(1-t)^2(1-t^{-1})(A-t^{-1}A')$$

and $\operatorname{rank}_C(A-\omega_x^{-1}A')$ is constant on $x\in[-1, 1]$ with $x\neq a_1$, since $p_{a_1}(\omega_x^{-1})\neq 0$. Let $x<a_1$. Then $a_1\neq -1$. By Corollary 5.2,

$$\tau_{x\pm 0}(A^{-1}(t))=\operatorname{sign} A^{-1}(-1)=\operatorname{sign}(A+A') .$$

Let $x>a_1$. Then $a_1\neq 1$. By Corollary 5.2,

$$\tau_{x\pm 0}(A^{-1}(t))=\lim_{x\to 1-0}\operatorname{sign} A^{-1}(\omega_x)$$
$$=\lim_{x\to 1-0}\operatorname{sign}[(1-x^2)^{-1/2}(1-\omega_x)(A-\omega_x^{-1}A')]$$
$$=\operatorname{sign}[-i(A-A')]=0\,.$$

This completes the proof.

The above method of proof is similar to Matumoto's one [Ma], obtaining similar results when A is non-singular and $\varepsilon=-1$.

§6. Proof of the Main Theorem

Lemma 6.1. *For A_a, $a\in[-1,1]$, appearing in Lemma 3.4, we have $\sigma_a^\gamma(M)=\operatorname{sign}(A_a+A_a')$ except when $\varepsilon(m)=1$ and $a=1$. When $\varepsilon(m)=1$, $\sigma^\gamma(M)=\operatorname{sign} V$ and $\hat{\sigma}_1^\gamma(M)=\operatorname{sign}(A_1+A_1')$.*

Proof. Let $\varepsilon(m)=1$. Then note that A_a+A_a' $(a\neq 1)$ and A_1+A_1' are intersection matrices on $T_{(a)}$ and $(t-1)\cdot T_{(1)}$, respectively, since

$$\operatorname{Int}_V(x,y)=L^+(x,y)-L^-(x,y)=L^+(x,y)+\varepsilon(m)L^+(y,x)$$

for $x, y\in K_m(V)$. So, by Lemma 1.5

$$\operatorname{sign}(A_a+A_a')=\operatorname{sign}(\operatorname{Int}_V|T_{(a)})=\sigma_a^\gamma(M)$$

for $a\neq 1$ and

$$\operatorname{sign}(A_1+A_1')=\operatorname{sign}(\operatorname{Int}_V|(t-1)\cdot T_{(1)})=\hat{\sigma}_1^\gamma(M)\,.$$

By Lemmas 1.5, 2.2 and 2.3, we also have $\sigma^\gamma(M)=\operatorname{sign}(\operatorname{Int}_V|T)=\operatorname{sign} V$. Next let $\varepsilon(m)=-1$. By Lemma 4.8,

$$I_*(e_1,\cdots,e_s)=((1-t)I_*(e_1^*),\cdots,(1-t)I_*(e_s^*))A_a$$

for a basis $e_1,\cdots,e_s$ of $T_{(a)}$ $(a\neq 1)$ or $T^{(1)}$ and its dual basis $e_1^*,\cdots,e_s^*$ for $T_{(a)}$ $(a\neq 1)$ or $T_{(1)}/\operatorname{Ker}(t-1)$, respectively. By using a non-singular pairing $[T_m(\tilde{M})/\operatorname{Ker}(t-1)]\times(1-t)T_m(\tilde{M})\to R$ induced by $*$, we can take a basis $\tilde{e}_1^*,\cdots,\tilde{e}_s^*$ for $T_m(\tilde{M})_a$ $(a\neq 1)$ or $T_m(\tilde{M})_1/\operatorname{Ker}(t-1)$ so that

$$\tilde{e}_i^* * (1-t)I(e_j^*)=\delta_{ij}\,.$$

Assertion 6.2. *$\pi(\tilde{e}_1^*,\cdots,\tilde{e}_s^*)=(e_1^*,\cdots,e_s^*)A_a'$, where when $a=1$, we regard π as the isomorphism $T_m(\tilde{M})_1/\operatorname{Ker}(t-1)\cong T_{(1)}/\operatorname{Ker}(t-1)$.*

Proof. Let $\pi(\tilde{e}_i^*)=a_{i1}^*e_1^*+\cdots+a_{is}^*e_s^*$. Then

$$a^*_{ij}=\mathrm{Int}_V(\pi(\tilde{e}^*_i), e_j)=\mathrm{Int}_V(\pi(\tilde{e}^*_i), \pi I_*(e_j))=\tilde{e}^*_i * I_*(e_j).$$

Let $A_a=(a_{ij})$. Then

$$I_*(e_j)=a_{1j}(1-t)I_*(e^*_1)+\cdots+a_{sj}(1-t)I_*(e^*_s),$$

so that $a^*_{ij}=\tilde{e}^*_i * I_*(e_j)=a_{ij}$ and $\pi(\tilde{e}^*_1, \cdots, \tilde{e}^*_s)=(e^*_1, \cdots, e^*_s)A'_a$, as desired.

Let b_a be the restriction of the form b to $T_m(\tilde{M})_a$. Note that the form b_1 induces a form (also denoted by b_1)

$$[T_m(\tilde{M})_1/\mathrm{Ker}(t-1)]\times[T_m(\tilde{M})_1/\mathrm{Ker}(t-1)]\to R.$$

By Assertion 6.2, we have

$$((1-t)\tilde{e}^*_1, \cdots,(1-t)\tilde{e}^*_s)=((1-t)I_*(e^*_1), \cdots, (1-t)I_*(e^*_s))A'_a.$$

That is, $(1-t)\tilde{e}^*_i=a_{i1}(1-t)I_*(e^*_1)+\cdots+a_{is}(1-t)I_*(e^*_s)$. Then

$$b_a(\tilde{e}^*_i, \tilde{e}^*_j)=\tilde{e}^*_i * (t^{-1}-t)\tilde{e}^*_j=\tilde{e}^*_i * (1-t)\tilde{e}^*_j+\tilde{e}^*_j * (1-t)\tilde{e}^*_i=a_{ji}+a_{ij}.$$

Thus, the form b_a is represented by the matrix $A_a+A'_a$, and $\sigma^\gamma_a(M)=\mathrm{sign}\, b_a=\mathrm{sign}(A_a+A'_a)$ for all a. This completes the proof of Lemma 6.1.

When $\varepsilon(m)=-1$, the proof of Lemma 6.1 suggests a simpler proof of Erle's result [E]. The identity $\sigma^\gamma(M)=\mathrm{sign}\, V$ in the case $\varepsilon(m)=1$ was also observed by Neumann [N] in connection with the first higher Novikov signature.

6.3. Proof of the Main Theorem. We use the splitting of A appearing in Lemma 3.4. Note that $A^{\varepsilon(m)}(t)=A^{\varepsilon(m)}_\sharp(t)\oplus A^{\varepsilon(m)}_\natural(t)$

$$A^{\varepsilon(m)}_\sharp(t)=\oplus_{a\neq 1}A^{\varepsilon(m)}_a(t)\oplus A^{\varepsilon(m)}_*(t).$$

We show that $\tau_{x\pm 0}(A^{\varepsilon(m)}_\natural(t))=\tau_{x\pm 0}(A^{\varepsilon(m)}_1(t))$ for all x. $A^{\varepsilon(m)}_\natural(t)$ is given as follows:

$$\begin{array}{c|cccccc} & T^{(1)} & K^+_{(*)} & K^-_{(*)} & K^{(1)}_B & B^{(1)} & K^{(1)}_T \\ \hline T^{(1)} & A^{\varepsilon(m)}_1(t) & 0 & 0 & 0 & D_{15}(t) & 0 \\ K^+_{(*)} & 0 & 0 & D_{23}(t) & 0 & D_{25}(t) & 0 \\ K^-_{(*)} & 0 & D_{32}(t) & 0 & 0 & D_{35}(t) & 0 \\ K^{(1)}_B & 0 & 0 & 0 & 0 & D_{45}(t) & 0 \\ B^{(1)} & D_{51}(t) & D_{52}(t) & D_{53}(t) & D_{54}(t) & C^{\varepsilon(m)}_{55}(t) & D_{56}(t) \\ K^{(1)}_T & 0 & 0 & 0 & 0 & D_{65}(t) & 0 \end{array}$$

where for $i<j$ $D_{ij}(t)=[(1-t^{-1})-\varepsilon(m)(1-t)][(1-t)C_{ij}-\varepsilon(m)(1-t^{-1})C'_{ji}]$ and $D_{ji}(t)=D_{ij}(t^{-1})'$. By the identity $\mathrm{Int}_V(x, y)=L^+(x, y)+\varepsilon(m)L^+(y, x)$ for $x, y\in K_m(V)$, the matrices

$$\begin{pmatrix} 0 & C_{23}+\varepsilon(m)C'_{32} \\ C_{32}+\varepsilon(m)C'_{23} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & C_{45}+\varepsilon(m)C'_{54} \\ C_{54}+\varepsilon(m)C'_{45} & C_{55}+\varepsilon(m)C'_{55} \end{pmatrix}$$

are intersection matrices on $K^+_{(*)}\oplus K^-_{(*)}$ and $K^{(1)}_B\oplus B^{(1)}$, respectively and hence non-singular. So, $\det D_{23}(t)\neq 0$ and $\det D_{45}(t)\neq 0$. Since $\det A_1\neq 0$ by Lemma 4.8, we have $\det A_1^{\varepsilon(m)}(t)\neq 0$. These imply that except a finite number of $\omega\in S^1$, $\det A_1^{\varepsilon(m)}(\omega)\neq 0$ and $\det D_{23}(\omega)\neq 0$ and $\det D_{45}(\omega)\neq 0$. For any such ω, we can see that $A_\natural^{\varepsilon(m)}(\omega)$ is equivalent to a block sum of $A_1^{\varepsilon(m)}(\omega)$,

$$\begin{pmatrix} 0 & D_{23}(\omega) \\ D_{32}(\omega) & 0 \end{pmatrix}, \begin{pmatrix} 0 & D_{45}(\omega) \\ D_{54}(\omega) & 0 \end{pmatrix}$$

and a zero matrix. This implies that $\tau_{x\pm 0}(A_\natural^{\varepsilon(m)}(t))=\tau_{x\pm 0}(A_1^{\varepsilon(m)}(t))$ for all x. By Lemmas 4.1 and 5.5 note that $\mathrm{sign}(A_*+A'_*)=\tau_{x\pm 0}(A_*^{\varepsilon(m)}(t))=0$ for all x. Then we have $\tau_{x\pm 0}(A^{\varepsilon(m)}(t))=\sum_{a\in[-1,1]}\tau_{x\pm 0}(A_a^{\varepsilon(m)}(t))$ for all x, so that by Lemmas 4.1 and 5.5

$$\tau_{x+\varepsilon(m)0}(A^{\varepsilon(m)}(t))=\sum_{\varepsilon(m)a\leq\varepsilon(m)x}\mathrm{sign}(A_a+A'_a)$$

and

$$\tau_{x-\varepsilon(m)0}(A^{\varepsilon(m)}(t))=\sum_{\varepsilon(m)a<\varepsilon(m)x}\mathrm{sign}(A_a+A'_a)\,.$$

Next, we show that $\mathrm{sign}(A_\natural+A'_\natural)=\mathrm{sign}(A_1+A'_1)$. When $\varepsilon(m)=1$, this is clear, because by the identity $\mathrm{Int}_V(x, y)=L^+(x, y)+\varepsilon(m)L^+(y, x)$ and Lemma 2.4, $A_\natural+A'_\natural$ is a block sum of $A_1+A'_1$,

$$\begin{pmatrix} 0 & C_{23}+C'_{32} \\ C_{32}+C'_{23} & 0 \end{pmatrix}, \begin{pmatrix} 0 & C_{45}+C'_{54} \\ C_{54}+C'_{45} & C_{55}+C'_{55} \end{pmatrix}$$

and a zero matrix. When $\varepsilon(m)=-1$, Lemmas 4.1 and 5.5 imply that

$$\tau_{x\pm 0}(A_\natural^{-1}(t))=\mathrm{sign}(A_\natural+A'_\natural)$$

and $\tau_{x\pm 0}(A_1^{-1}(t))=\mathrm{sign}(A_1+A'_1)$ for all x. Since

$$\tau_{x\pm 0}(A_\natural^{-1}(t))=\tau_{x\pm 0}(A_1^{-1}(t))\,,$$

we have $\mathrm{sign}(A_\natural+A'_\natural)=\mathrm{sign}(A_1+A'_1)$. Hence

$$\mathrm{sign}(A+A')=\sum_{a\in[-1,1]}\mathrm{sign}(A_a+A'_a)\,, \qquad \text{for} \quad \mathrm{sign}(A_*+A'_*)=0\,.$$

The identity $\sigma_a^{\varepsilon(m)}(A)=\mathrm{sign}(A_a+A'_a)$ for all a is now easily established. By combining it with Lemma 6.1, we complete the proof of the Main Theorem.

Appendix A: The intersection and linking numbers of singular chains in a topological manifold

A topological manifold is understood to be a paracompact Hausdorff manifold. We consider singular p- and q-chains c_1 and c_2 in an oriented topological n-manifold X such that

$$p+q=n \quad \text{and} \quad |\partial c_1|\cap|c_2|=|\partial c_2|\cap|c_1|=\emptyset,$$

where $|c|$ denotes the underlying space of a chain c, which is a compact subset of X. Then it is easy to find a neighborhood system $(N_c^1, N_o^1; N_c^2, N_o^2)$ of $(|c_1|, |\partial c_1|; |c_2|, |\partial c_2|)$ such that $N_o^i\subset N_c^i$ and N_o^i are open in X and N_c^i are closed in X and $N_o^1\cap N_c^2=N_o^2\cap N_c^1=\emptyset$. Since $N_c^1\times N_o^2$ and $N_o^1\times N_c^2$ are open in $N_c^1\times N_c^2$, we have the Künneth isomorphism

$$H_*(N_c^1, N_o^1)\otimes H_*(N_c^2, N_o^2)\cong H_*((N_c^1, N_o^1)\times(N_c^2, N_o^2)),$$

taking real coefficients. So the cross product

$$\{c_1\}\times\{c_2\}\in H_n((N_c^1, N_o^1)\times(N_c^2, N_o^2))$$

of $\{c_1\}\in H_p(N_c^1, N_o^1)$ and $\{c_2\}\in H_q(N_c^2, N_o^2)$ is well defined. Let

$$U\in H^n(X\times X, X\times X-\delta(X))$$

be the orientation class of X given by the orientation of X. Let

$$i\colon (N_c^1, N_o^1)\times(N_c^2, N_o^2)\subset(X\times X, X\times X-\delta(X))$$

be the inclusion.

Definition A.1. The *intersection number* $\mathrm{Int}_X(c_1, c_2)$ of c_1 and c_2 is defined by the identity

$$\mathrm{Int}_X(c_1, c_2)=\varepsilon(p)\varepsilon_{X\times X}[U\cap i_*(\{c_1\}\times\{c_2\})]$$

(cf. Dold [D, p. 197]).

By the naturality of the cross product, one can easily check that $\mathrm{Int}_X(c_1, c_2)$ is independent of any choice of a neighborhood system $(N_c^1, N_o^1; N_c^2, N_o^2)$. Clearly, we have $\mathrm{Int}_{X'}(c_1, c_2)=\mathrm{Int}_X(c_1, c_2)$ for any n-submanifold X' containing $|c_1|$ and $|c_2|$. Let $N'_1\subset N_1$ and $N'_2\subset N_2$ be subspaces of X such that $N'_1\cap N_2=N'_2\cap N_1=\emptyset$. Then Int_X induces a pairing

$$H_p(N_1, N_1') \times H_q(N_2, N_2') \longrightarrow R\,,$$

called the *intersection pairing*. Similary, $\mathrm{Int}_X(c_2, c_1)$ is defined.

Proposition A.2. $\mathrm{Int}_X(c_2, c_1) = \varepsilon(pq)\ \mathrm{Int}_X(c_1, c_2)$.

Proof. Let T be the self-map of $(X \times X, X \times X - \delta(X))$ interchanging the factors. By Spanier [S, pp. 235 and 305], we have

$$T_* i_*(\{c_1\} \times \{c_2\}) = \varepsilon(pq) i'_*(\{c_2\} \times \{c_1\})\,,$$

where $i'\colon (N_c^2, N_o^2) \times (N_c^1, N_o^1) \subset (X \times X,\ X \times X - \delta(X))$, and $T^* U = \varepsilon(n) U$. The desired identity follows.

The family $\{H_n(X, (X-K) \cup \partial X) \mid K$ is compact in $X\}$ forms an inverse system (directed by inclusion on K), whose limit is denoted by $H_n^c(X, \partial X)$. By [S, p. 301], the orientation of X determines a unique element of $H_n^c(X, \partial X)$, which we call the *fundamental class* of X and denote by $[X]$. We consider that ∂X is a disjoint union $\partial_1 X + \partial_2 X$, where $\partial_i X$ may be empty. The cohomology with compact support $H_c^p(X, \partial_1 X)$ is the limit of the direct system $\{H^p(X, (X-K) \cup \partial_1 X) \mid K$ is compact in $X\}$. The cap product $\cap [X]\colon H_c^p(X, \partial_1 X) \to H_q(X, \partial_2 X)$ is well defined by taking the limit of the usual cap product

$$\cap [X]_K\colon H^p(X, (X-K) \cup \partial_1 X) \longrightarrow H_q(X, \partial_2 X)\,,$$

where $[X]_K$ is the image of $[X]$ under the projection $H_n^c(X, \partial X) \to H_n(X, (X-K) \cup \partial X)$.

The Poincaré Duality Theorem: $\cap [X]\colon H_c^p(X, \partial_1 X) \cong H_q(X, \partial_2 X)$.

The proof of the case $\partial_1 X = \partial_2 X = \varnothing$ is given by, for example, Milnor/Stasheff [M/S]. The cases $\partial_1 X = \varnothing$ and $\partial_1 X = \partial X$ are then obtained by considering the following commutative and sign-commutative diagrams with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_c^p(X, \partial X) & \longrightarrow & H_c^p(DX) & \longrightarrow & H_c^p(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cap[X]} & & \cong\downarrow{\scriptstyle \cap[DX]} & & \downarrow{\scriptstyle \cap[X]} & & \\ 0 & \longrightarrow & H_q(X) & \longrightarrow & H_q(DX) & \longrightarrow & H_q(X, \partial X) & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc}
\longrightarrow H_c^{p-1}(\partial X) & \longrightarrow & H_c^p(X, \partial X) & \longrightarrow & H_c^p(X) & \longrightarrow & H_c^p(\partial X) & \longrightarrow \\
\cong \downarrow \cap[\partial X] & & \downarrow \cap[X] & & \downarrow \cap[X] & & \cong \downarrow \cap[\partial X] & \\
\longrightarrow H_q(\partial X) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X, \partial X) & \longrightarrow & H_{q-1}(\partial X) & \longrightarrow,
\end{array}$$

where DX denotes the double of X (cf. Komatu/Nakaoka/Sugawara [K/N/S, p. 656] for these techniques). The above general case is then obtained from the following sign-commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\longrightarrow H_c^{p-1}(\partial_1 X) & \longrightarrow & H_c^p(X, \partial_1 X) & \longrightarrow & H_c^p(X) & \longrightarrow & H_c^p(\partial_1 X) & \longrightarrow \\
\cong \downarrow \cap[\partial_1 X] & & \downarrow \cap[X] & & \cong \downarrow \cap[X] & & \cong \downarrow \cap[\partial_1 X] & \\
\longrightarrow H_q(\partial_1 X) & \longrightarrow & H_q(X, \partial_2 X) & \longrightarrow & H_q(X, \partial X) & \longrightarrow & H_{q-1}(\partial_1 X) & \longrightarrow.
\end{array}$$

There is another description of Poincaré duality by means of the slant product (cf. [S]). Let $X^i = X - \partial_i X$. Let $N_o(\partial_i X)$ be an open collar neighborhood of $\partial_i X$ in X if $\partial_i X \neq \emptyset$, or $\emptyset$ if $\partial_i X = \emptyset$. Let $X_c^i = X - N_o(\partial_i X)$. For a compact subset $K \subset X_c^2 - \partial_1 X$ we define a map

$$U/ : H_q(K \cup N_o(\partial_2 X), N_o(\partial_2 X)) \to H^p(X_c^2, X_c^2 - K)$$

by the identity

$$(U/)(x) = [U | (X_c^2, X_c^2 - K) \times (K \cup N_o(\partial_2 X), N_o(\partial_2 X))]/x .$$

Note that $X_c^2 - K$ is a cobounded neighborhood of $\partial_1 X$ in X_c^2. Passing to the direct limit on K, we have a map (also denoted by $U/$)

$$H_q(X^1, N_0(\partial_2 X)) \longrightarrow H_c^p(X_c^2, \partial_1 X) .$$

By using the natural isomorphisms

$$H_q(X^1, N_0(\partial_2 X)) \cong H_q(X, \partial_2 X) \quad \text{and} \quad H_c^p(X, \partial_1 X) \cong H_c^p(X_c^2, \partial_1 X) ,$$

we identify this map with a map $H_q(X, \partial_2 X) \to H_c^p(X, \partial_1 X)$.

Proposition A.3. *The inverse of the Poincaré duality map* $\cap [X]$ *is* $\varepsilon(pn) U/$.

Proof. We prove it for the case $\partial_1 X = \partial_2 X = \emptyset$. The general case then follows from an argument similar to the above proof of the Poincaré Duality Theorem. Let $y \in H_q(X)$. Take a compact subset $K \subset X$ and $y_K \in H_q(K)$ so that $i_*^K(y_K) = y$, where i^K: $K \subset X$. Let p_i be the projection $X \times X \to X$ to the i^{th} factor and $p_1^K = p_1 | X \times K$ and $p_2^K = p_2 | K \times X$ and $\bar{p}_1^K = p_1 | K \times X$ with image K. Let $U_K = U | (X, X-K) \times K$ and $U'_K = U | K \times (X, X-K)$. Let $T' = T | X \times K$

with image $K\times X$ and $T''=T|(X, X-K)\times K$ with image $K\times(X, X-K)$ and $i\colon K\times(X, X-K)\subset(X\times X, X\times X-\delta(X))$. Then by [S, 6.1.6],

$$(\cap[X])(U/)(y)=(U_K/y_K)\cap[X]_K=p^K_{1*}[U_K\cap([X]_K\times y_K)]$$
$$=p^K_{2*}T'_*[U_K\cap([X]_K\times y_K)]=\varepsilon(pn)p^K_{2*}[U'_K\cap(y_K\times[X]_K)]$$
$$=\varepsilon(pn)p_{2*}[U\cap i_*(y_K\times[X]_K)],$$

since $T''^*U'_K=\varepsilon(n)U_k$ (for $T^*U=\varepsilon(n)U$) and $T''_*([X]_K\times y_K)=\varepsilon(nq)y_K\times[X]_K$ and $\varepsilon(nq+n)=\varepsilon(pn)$. By [S, 6.3.11], it equals

$$\varepsilon(pn)p_{1*}[U\cap i_*(y_K\times[X]_K)]=\varepsilon(pn)i^K_*\bar{p}^K_{1*}[U'_K\cap(y_K\times[X]_K)]$$
$$=\varepsilon(pn)i^K_*[(U'_K/[X]_K)\cap y_K]=\varepsilon(pn)y,$$

since $U'_K/[X]_K=1$ by [S, p. 301]. This completes the proof.

Proposition A.4. *For* $\{c_1\}\in H_p(X, \partial_1 X)$ *and* $\{c_2\}\in H_q(X, \partial_2 X)$, *we have* $\mathrm{Int}_X(c_1, c_2)=\varepsilon_X[(u_1\cup u_2)\cap[X]]$ *for* $u_1\in H^q_c(X, \partial_2 X)$ *and* $u_2\in H^p_c(X, \partial_1 X)$ *with* $u_i\cap[X]=\{c_i\}$.

Proof.

$$\mathrm{Int}_X(c_1, c_2)=\varepsilon(p)\varepsilon_{X\times X}[U\cap i_*(\{c_1\}\times\{c_2\})]$$
$$=\varepsilon(p)\varepsilon_X[(U/)(\{c_2\})\cap\{c_1\}] \qquad \text{(by [S, 6.1.6])}$$
$$=\varepsilon(pq)\varepsilon_X(u_2\cap\{c_1\})=\varepsilon(pq)\varepsilon_X[(u_2\cup u_1)\cap[X]]$$
$$=\varepsilon_X[(u_1\cup u_2)\cap[X]],$$

since $(U/)(\{c_2\})=\varepsilon(pn)u_2$ by Proposition A.3 and $\varepsilon(p+pn)=\varepsilon(pq)$. This completes the proof.

Next, we consider boundary p- and s-cycles z_1 and z_2 in X such that

$$p+s+1=n \quad\text{and}\quad |z_1|\cap|z_2|=\varnothing.$$

Definition A.5. The *linking number* $\mathrm{Link}_X(z_1, z_2)$ of z_1 and z_2 is defined by the identity $\mathrm{Link}_X(z_1, z_2)=\mathrm{Int}_X(c_1, z_2)$ for any $(p+1)$-chain c_1 with $\partial c_1=z_1$.

We see easily that $\mathrm{Int}_X(c_1, z_2)$ is independent of a choice of c_1 and $\mathrm{Link}_{X'}(z_1, z_2)=\mathrm{Link}_X(z_1, z_2)$ for any n-submanifold X' in which z_1 and z_2 are boundary cycles. For disjoint subspaces $X_1, X_2\subset X$, let $K_p(X_1)$ and $K_s(X_2)$ be the kernels of the natural maps $H_p(X_1)\to H_p(X)$ and $H_s(X_2)\to H_s(X)$, respectively. Then Link_X induces a pairing

$$K_p(X_1) \times K_s(X_2) \longrightarrow R,$$

called the *linking pairing*. Similarly, $\mathrm{Link}_X(z_2, z_1)$ is defined.

Proposition A.6. $\mathrm{Link}_X(z_2, z_1) = \varepsilon(ps+1)\mathrm{Link}_X(z_1, z_2)$.

Proof. Let c_1 and c_2 be $(p+1)$- and $(s+1)$-chains in X whose boundaries are z_1 and z_2, respectively. Let $(N_c^1, \emptyset; N_c^2, N_o^2)$ and $(N_c'^1, N_o'^1; N_c'^2, \emptyset)$ be neighborhood systems of $(|z_1|, \emptyset; |c_2|, |z_2|)$ and $(|c_1|, |z_1|; |z_2|, \emptyset)$ used to defined the intersection numbers. Let $\tau: C_\#(X) \otimes C_\#(X) \to C_\#(X \times X)$ be the Eilenberg/Zilber chain equivalence (cf. [S, p. 232]). Using that $\partial(c_1 \otimes c_2) = (\partial c_1) \otimes c_2 + \varepsilon(p+1) c_1 \otimes \partial c_2$ (cf. [S, p. 228]), we have

$$\{\tau(z_1 \otimes c_2)\} + \varepsilon(p+1)\{\tau(c_1 \otimes z_2)\} = 0$$

in $H_b(X \times X, X \times X - \delta(X))$, noting that $\tau(z_1 \otimes c_2)$ and $\tau(c_1 \otimes z_2)$ are cycles in $(X \times X, X \times X - \delta(X))$. Since τ induces chain equivalences

$$C_\#(N_c^1) \otimes C_\#(N_c^2, N_o^2) \longrightarrow C_\#(N_c^1 \times (N_c^2, N_o^2))$$

and

$$C_\#(N_c'^1, N_o'^1) \otimes C_\#(N_c'^2) \longrightarrow C_\#(N_c'^1, N_o'^1) \times N_c'^2),$$

it follows that

$$i_*(\{z_1\} \times \{c_2\}) = \{\tau(z_1 \otimes c_2)\} \quad \text{and} \quad i'_*(\{c_1\} \times \{z_2\}) = \{\tau(c_1 \otimes z_2)\}$$

in $H_n(X \times X, X \times X - \delta(X))$, where

$$i: N_c^1 \times (N_c^2, N_o^2) \subset (X \times X, X \times X - \delta(X))$$

and

$$i': (N_c'^1, N_o'^1) \times N_c'^2 \subset (X \times X, X \times X - \delta(X)).$$

Thus,

$$i_*(\{z_1\} \times \{c_2\}) = \varepsilon(p) i'_*(\{c_1\} \times \{z_2\})$$

and

$$\begin{aligned} \mathrm{Int}_X(z_1, c_2) &= \varepsilon(p)\varepsilon_{X \times X}[U \cap i_*(\{z_1\} \times \{c_2\})] \\ &= \varepsilon_{X \times X}[U \cap i'_*(\{c_1\} \times \{z_2\})] = \varepsilon(p+1)\,\mathrm{Int}_X(c_1, z_2). \end{aligned}$$

By Proposition A.2, this implies that

$$\mathrm{Link}_X(z_2, z_1) = \varepsilon(ps+1)\mathrm{Link}_X(z_1, z_2).$$

This completes the proof.

Appendix B: Proof of the Duality Theorem

Lemma B.1. *Assume that $\gamma \in H^1(M; Z)$ has a leaf V. Then there is an element μ' in $T_{n-1}(\tilde{M}, \partial\tilde{M})$ with $(t-1)\mu'=0$ such that for any $\tilde{T}^q(\tilde{M}, \partial_1\tilde{M})$,*

$$\cap\,\mu' : \tilde{T}^q(\tilde{M}, \partial_1\tilde{M}) \longrightarrow T_{n-q-1}(\tilde{M}, \partial_2\tilde{M})$$

is an R-isomorphism.

Proof. We consider that $\tilde{M}$ is the union of M_i''s, as it is stated in § 1. Let $\tilde{M}_r^+ = M_u' \cup M_{r+1}' \cup \cdots$ and $\tilde{M}_s^- = M_{-s}' \cup M_{-s-1}' \cup \cdots$ be the submanifolds of $\tilde{M}$. By considering the Mayer/Vietoris sequence of $(\tilde{M}; \tilde{M}_r^+ \cup \partial_1\tilde{M}, \tilde{M}_s^- \cup \partial_1\tilde{M})$ and then taking the limits $r, s \to +\infty$, we obtain the following exact sequence

$$\longrightarrow H_c^q(\tilde{M}, \partial_1\tilde{M}) \longrightarrow \varinjlim\{H^q(\tilde{M}, \tilde{M}_r^+ \cup \partial_1\tilde{M})\} \oplus \varinjlim\{H^q(\tilde{M}, \tilde{M}_s^- \cup \partial_1\tilde{M})\}$$
$$\xrightarrow{\{j_r^{+*}\}+\{j_s^{-*}\}} H^q(\tilde{M}, \partial_1\tilde{M}) \xrightarrow{\ \delta\ } H_c^{q+1}(\tilde{M}, \partial_1\tilde{M}) \longrightarrow,$$

where j_r^+ and j_s^- denote the natural inclusions. We use the Poincaré duality

$$\cap[\tilde{M}] : H_c^{q+1}(\tilde{M}, \partial_1\tilde{M}) \cong H_{n-q-1}(\tilde{M}, \partial_2\tilde{M}),$$

stated in Appendix A. By considering the case that $\partial_1\tilde{M}=\varnothing$ and $q=0$, we let $\mu'=\delta(1)\cap[\tilde{M}]\in H_{n-1}(\tilde{M}, \partial\tilde{M})$. Since $t1=1$ and $t[\tilde{M}]=[\tilde{M}]$, we see that $(t-1)\mu'=0$. For any q, the composite

$$H^q(\tilde{M}, \partial_1\tilde{M}) \xrightarrow{\ \delta\ } H_c^{q+1}(\tilde{M}, \partial_1\tilde{M}) \xrightarrow{\cap[\tilde{M}]} H_{n-q-1}(\tilde{M}, \partial_2\tilde{M})$$

is given by $\cap\,\mu'$. In fact, for $u \in H^q(\tilde{M}, \partial_1\tilde{M})$,

$$\delta(u)\cap[\tilde{M}] = \delta(u \cup 1)\cap[\tilde{M}] = (u \cup \delta(1))\cap[\tilde{M}]$$
$$= u \cap (\delta(1)\cap[\tilde{M}]) = u \cap \mu'.$$

Let D^q be the image of $\{j_r^{+*}\}+\{j_s^{-*}\}$ in $H^q(\tilde{M}, \partial_1\tilde{M})$. By [K1, Lemma 1.5], $D^q \subset B^q(\tilde{M}, \partial_1\tilde{M})$, so that the natural map $\tilde{T}^q(\tilde{M}, \partial_1\tilde{M}) \to H^q(\tilde{M}, \partial_1\tilde{M})/D^q$ is injective. Since δ induces an injection $H^q(\tilde{M}, \partial_1\tilde{M})/D^q \to H_c^{q+1}(\tilde{M}, \partial_1\tilde{M})$, we see that

$$\cap\,\mu' : H^q(\tilde{M}, \partial_1\tilde{M})/D^q \to H_{n-q-1}(\tilde{M}, \partial_2\tilde{M})$$

is injective. Thus,

$$\cap\,\mu' : \tilde{T}^q(\tilde{M}, \partial_1\tilde{M}) \longrightarrow H_{n-q-1}(\tilde{M}, \partial_2\tilde{M})$$

is injective. The identity $(tu)\cap\mu'=t^{-1}(u\cap\mu')$ implies

$$\tilde{T}^q(\tilde{M}, \partial_1\tilde{M})\cap\mu'\subset T_{n-q-1}(\tilde{M}, \partial_2\tilde{M}).$$

So we obtain a monomorphism

$$\cap\mu' : \tilde{T}^q(\tilde{M}, \partial_1\tilde{M})\longrightarrow T_{n-q-1}(\tilde{M}, \partial_2\tilde{M}).$$

Similarly, we have a monomorphism

$$\cap\mu' : \tilde{T}^{n-q-1}(\tilde{M}, \partial_2\tilde{M})\longrightarrow T_q(\tilde{M}, \partial_1\tilde{M}),$$

which shows that

$$\dim_R T_{n-q-1}(\tilde{M}, \partial_2\tilde{M})\leq\dim_R T_q(\tilde{M}, \partial_1\tilde{M}).$$

This implies that

$$\cap\mu' : \tilde{T}^q(\tilde{M}, \partial_1\tilde{M})\longrightarrow T_{n-q-1}(\tilde{M}, \partial_2\tilde{M})$$

is an isomorphism. This completes the proof.

Lemma B.2. *There is one and only one element μ in $T_{n-1}(\tilde{M}, \partial\tilde{M})$ with the properties* (i) *and* (ii).

Proof. If γ has a leaf V, then we noted in § 1 that $\bar{I}_*([V])$ has (i) and (ii). Let $\mu=\bar{I}_*([V])$. To see the uniqueness of μ, note that $T_0(\tilde{M})_1$ is $R\langle t\rangle$-isomorphic to a direct sum of some copies of $R\langle t\rangle/(t-1)$ (cf. [K2, Lemma 1.1]). By Lemma B.1, $T_{n-1}(\tilde{M}, \partial\tilde{M})_1$ is also so. By the Wang exact sequence (cf. [Mi]), the natural map $T_{n-1}(\tilde{M}, \partial\tilde{M})_1\to H_{n-1}(M, \partial M)$ is injective, showing the uniqueness of μ. If γ has no leaf, then we consider $M_P=M\times CP^2$ and $\gamma_P\in H^1(M_P; Z)$ corresponding to γ, which has a leaf V_P by [K/S]. By the isomorphism $T_{(n+4)-1}(\tilde{M}_P, \partial\tilde{M}_P)\cong T_{n-1}(\tilde{M}, \partial\tilde{M})\otimes H_4(CP^2)$, we have one and only one $\mu\in T_{n-1}(\tilde{M}, \partial\tilde{M})$ such that $\bar{I}_*([V_P])=\mu\times[\mathrm{CP}^2]$. This completes the proof.

Lemma B.3. *When M is connected and $\gamma\neq 0$, we have* (D1).

Proof. First, assume that γ has a leaf V. By Lemma B.1 and [K2, Lemma 1.1], $T_{n-1}(\tilde{M}, \partial\tilde{M})_1\cong T_0(\tilde{M})_1\cong R$. Since μ' and μ are non-zero, there is a non-zero r in R with $\mu=r\mu'$. By the proof of Lemma B.1, we have a duality

$$\cap\mu : \tilde{T}^q(\tilde{M}, \partial_1\tilde{M})\cong T_{n-q-1}(\tilde{M}, \partial_2\tilde{M})$$

and a monomorphism

$$\cap\,\mu:\ H^q(\tilde{M},\partial_1\tilde{M})/D^q \longrightarrow H_{n-q-1}(\tilde{M},\partial_2\tilde{M})\,.$$

Since $\cap\,\mu$ is a t-anti-map, we see that $B^q(\tilde{M},\partial_1\tilde{M})/D^q$ is $R\langle t\rangle$-free. Let

$$l:\ B^q(\tilde{M},\partial_1\tilde{M})/D^q \longrightarrow B^q(\tilde{M},\partial_1\tilde{M})$$

be a right inverse $R\langle t\rangle$-homomorphism of the natural map

$$B^q(\tilde{M},\partial_1\tilde{M}) \longrightarrow B^q(\tilde{M},\partial_1\tilde{M})/D^q\,.$$

If $x\in B^q(\tilde{M},\partial_1\tilde{M})/D^q$ is non-zero, then $I_{2*}(I_1^*(l(x))\cap[V])=l(x)\cap\bar{I}_*([V])=l(x)\cap\mu=x\cap\mu\neq 0$, so that $I_1^*(l(x))\neq 0$. This implies that the composite

$$B^q(\tilde{M},\partial_1\tilde{M})/D^q \xrightarrow{\ l\ } B^q(\tilde{M},\partial_1\tilde{M}) \xrightarrow{\ I_1^*\ } H^q(V,\partial_1 V)$$

is injective. Noting that $H^q(V,\partial_1 V)$ has a finite R-dimension, we have that $B^q(\tilde{M},\partial_1\tilde{M})/D^q=0$. Then the duality

$$\cap\,\mu:\ \tilde{T}^q(\tilde{M},\partial_1\tilde{M})\cong T_{n-q-1}(\tilde{M},\partial_2\tilde{M})$$

induces a duality

$$\cap\,\mu:\ T^q(\tilde{M},\partial_1\tilde{M})\cong T_{n-q-1}(\tilde{M},\partial_2\tilde{M})\,.$$

If γ has no leaf, then we consider M_P, γ_P and V_P as in Lemma B.2. Let $\partial_i\tilde{M}_P=\partial_i\tilde{M}\times CP^2$. Note that

$$T^q(\tilde{M}_P,\partial_1\tilde{M}_P)=[T^q(\tilde{M},\partial_1\tilde{M})\otimes H^0(CP^2)]\oplus[T^{q-2}(\tilde{M},\partial_2\tilde{M})\otimes H^2(CP^2)]\oplus[T^{q-4}(\tilde{M},\partial_1\tilde{M})\otimes H^4(CP^2)]$$

and

$$T_{(n+4)-q-1}(\tilde{M}_P,\partial_2\tilde{M}_P)=[T_{n-q-1}(\tilde{M},\partial_2\tilde{M})\otimes H_4(CP^2)]\oplus[T_{n-q+1}(\tilde{M},\partial_2\tilde{M})\otimes H_2(CP^2)]\oplus[T_{n-q+3}(\tilde{M},\partial_2\tilde{M})\otimes H_0(CP^2)]\,.$$

For $\mu_P=\bar{I}_*([V_P])=\mu\times[CP^2]$, the duality

$$\cap\,\mu_P:\ T^q(\tilde{M}_P,\partial_1\tilde{M}_P)\cong T_{(n+4)-q-1}(\tilde{M}_P,\partial_2\tilde{M}_P)$$

induces, for example, a duality

$$\cap\,\mu\times[CP^2]:\ T^q(\tilde{M},\partial_1\tilde{M})\otimes H^0(CP^2)\cong T_{n-q-1}(\tilde{M},\partial_2\tilde{M})\otimes H_4(CP^2)\,,$$

which is equivalent to

$$\cap\,\mu:\ T^q(\tilde{M},\partial_1\tilde{M})\cong T_{n-q-1}(\tilde{M},\partial_2\tilde{M})\,.$$

This completes the proof.

Let $M_1, \cdots, M_c$ be the components of M such that $\gamma|M_j \neq 0$, $j=1, \cdots, c$. For the lift $\tilde{M}_j$ of M_j to $\tilde{M}$, let $\mu_j \in T_{n-1}(\tilde{M}_j, \partial\tilde{M}_j)$ have (i) and (ii). By Lemma B.2, we have $\mu=\mu_1+\cdots+\mu_c$.

Proof of (D1). Let $\partial_i\tilde{M}_j=(\partial_i\tilde{M})\cap\tilde{M}_j$. Then the duality

$$\cap\mu\colon T^q(\tilde{M}, \partial_1\tilde{M})\cong T_{n-q-1}(\tilde{M}, \partial_2\tilde{M})$$

is obtained as a direct sum of the dualities

$$\cap\mu_j\colon T^q(\tilde{M}_j, \partial_1\tilde{M}_j)\cong T_{n-q-1}(\tilde{M}_j, \partial_2\tilde{M}_j), \qquad j=1, \cdots, c,$$

of Lemma B.3. This completes the proof.

Proof of (D2). By the natural map $H^{n-1}(\tilde{M}, \partial\tilde{M})\to T^{n-1}(\tilde{M}, \partial\tilde{M})$, the cup product pairing

$$\cup\colon H^q(\tilde{M}, \partial_1\tilde{M})\times H^{n-q-1}(\tilde{M}, \partial_2\tilde{M})\longrightarrow H^{n-q-1}(\tilde{M}, \partial\tilde{M})$$

induces a pairing (also denoted by $\cup$)

$$H^q(\tilde{M}, \partial_1\tilde{M})\times H^{n-q-1}(\tilde{M}, \partial_2\tilde{M})\longrightarrow T^{n-1}(\tilde{M}, \partial\tilde{M}).$$

For this pairing, we have that if $u\in B^q(\tilde{M}, \partial_1\tilde{M})$ or $v\in B^{n-q-1}(\tilde{M}, \partial_2\tilde{M})$, then $u\cup v=0$, because

$$B^q(\tilde{M}, \partial_1\tilde{M})\cap\mu=B^{n-q-1}(\tilde{M}, \partial_2\tilde{M})\cap\mu=0,$$

and $u\cup v=0$ if and only if

$$(u\cup v)\cap\mu=u\cap(v\cap\mu)=(-1)^{q(n-q-1)}v\cap(u\cap\mu)$$

is 0 by the duality

$$\cap\mu\colon T^{n-1}(\tilde{M}, \partial\tilde{M})\cong T_0(\tilde{M}).$$

So there is an induced pairing (also denoted by $\cup$)

$$T^q(\tilde{M}, \partial_1\tilde{M})\times T^{n-q-1}(\tilde{M}, \partial_2\tilde{M})\longrightarrow T^{n-1}(\tilde{M}, \partial\tilde{M}).$$

To show that the composite

$$T^q(\tilde{M}, \partial_1\tilde{M})\times T^{n-q-1}(\tilde{M}, \partial_2\tilde{M})\xrightarrow{\ \cup\ } T^{n-1}(\tilde{M}, \partial\tilde{M})\xrightarrow{\ \tilde{\mu}\ } R$$

is non-singular, suppose that $\tilde{\mu}(u\cup v)=0$ for all $v\in T^{n-q-1}(\tilde{M}, \partial_2\tilde{M})$. Using the identity

$$\tilde{\mu}(u\cup v)=\varepsilon_{\tilde{M}}((u\cup v)\cap\mu)=\varepsilon_{\tilde{M}}(u\cap(v\cap\mu)),$$

we see from the duality

$$\cap \mu: T^{n-q-1}(\tilde{M}, \partial_2\tilde{M}) \cong T_q(\tilde{M}, \partial_1\tilde{M})$$

that $\varepsilon_{\tilde{M}}(u \cap T_q(\tilde{M}, \partial_1\tilde{M})) = u(T_q(\tilde{M}, \partial_1\tilde{M})) = 0$, i.e., $u=0$. Similarly, if $\bar{\mu}(u \cup v) = 0$ for all $u \in T^q(\tilde{M}, \partial_1\tilde{M})$, then $v=0$. This completes the proof.

Appendix C: The Blanchfield duality for the Betti modules of infinite cyclic coverings of topological manifolds

We consider $\tilde{M}$ and $\partial\tilde{M} = \partial_1\tilde{M} + \partial_2\tilde{M}$ in § 1. We define a pairing

$$\widetilde{\mathrm{Int}}_{\tilde{M}}: H_p(\tilde{M}, \partial_1\tilde{M}) \times H_q(\tilde{M}, \partial_2\tilde{M}) \longrightarrow R\langle t\rangle$$

by the identity $\widetilde{\mathrm{Int}}_{\tilde{M}}(x, y) = \sum_{i\in Z} \mathrm{Int}_{\tilde{M}}(x, t^i y)t^{-i}$ for $x \in H_p(\tilde{M}, \partial_1\tilde{M})$ and $y \in H_q(\tilde{M}, \partial_2\tilde{M})$, $p+q=n$, where the sum is easily checked to be a finite sum. If we define $\widetilde{\mathrm{Int}}_{\tilde{M}}: H_q(\tilde{M}, \partial_2\tilde{M}) \times H_p(\tilde{M}, \partial_1\tilde{M}) \to R\langle t\rangle$ similarly, then we have that $\widetilde{\mathrm{Int}}_{\tilde{M}}(y, x) = \varepsilon(pq)\overline{\widetilde{\mathrm{Int}}_{\tilde{M}}(x, y)}$, where —— stands for the involution of $R\langle t\rangle$ sending t to t^{-1}. Note that $\widetilde{\mathrm{Int}}_{\tilde{M}}(f(t^{-1})x, y) = f(t)\widetilde{\mathrm{Int}}_{\tilde{M}}(x, y) = \widetilde{\mathrm{Int}}_{\tilde{M}}(x, f(t)y)$ for $f(t) \in R\langle t\rangle$, so that $\widetilde{\mathrm{Int}}_{\tilde{M}}$ induces a pairing (also denoted by $\widetilde{\mathrm{Int}}_{\tilde{M}}$)

$$B_p(\tilde{M}, \partial_1\tilde{M}) \times B_q(\tilde{M}, \partial_2\tilde{M}) \longrightarrow R\langle t\rangle\,.$$

The following was given by Blanchfield [B] when M is triangulated:

The Blanchfield Duality Theorem (for Betti Modules). *The pairing*

$$\widetilde{\mathrm{Int}}_{\tilde{M}}: B_p(\tilde{M}, \partial_1\tilde{M}) \times B_q(\tilde{M}, \partial_2\tilde{M}) \to R\langle t\rangle$$

is non-singular in the sense that the associated matrix of $\widetilde{\mathrm{Int}}_{\tilde{M}}$ *relative to any* $R\langle t\rangle$*-bases of* $B_p(\tilde{M}, \partial_1\tilde{M})$ *and* $B_q(\tilde{M}, \partial_2\tilde{M})$ *is invertible in* $R\langle t\rangle$.

To prove it, we need some preliminaries. We consider an infinite cyclic covering space pair $(\tilde{X}, \tilde{X}_1)$ of a compact topological pair (X, X_1). The singular chain complex $C_\#(\tilde{X}, \tilde{X}_1)$ forms a free chain complex over $R\langle t\rangle$. The cochain complex with compact support, $C_c^\#(\tilde{X}, \tilde{X}_1)$ is the subcomplex of the singular cochain complex $C^\#(\tilde{X}, \tilde{X}_1)$ consisting of all cochains f such that $fC_\#((\tilde{X}-K) \cup \tilde{X}_1) = 0$ for a compact $K \subset \tilde{X}$. Clearly, $C_c^\#(\tilde{X}, \tilde{X}_1)$ is a cochain complex over $R\langle t\rangle$. Let $C^\#_{R\langle t\rangle}(\tilde{X}, \tilde{X}_1)$ be the $R\langle t\rangle$-cochain complex $\mathrm{Hom}_{R\langle t\rangle}[C_\#(\tilde{X}, \tilde{X}_1), R\langle t\rangle]$. We define a map

$$\phi: C_c^\#(\tilde{X}, \tilde{X}_1) \longrightarrow C^\#_{R\langle t\rangle}(\tilde{X}, \tilde{X}_1)$$

by the identity $\phi(f)(x) = \sum_{i=Z} f(t^i x)t^{-i}$ for $f \in C_c^\#(\tilde{X}, \tilde{X}_1)$ and $x \in C_\#(\tilde{X}, \tilde{X}_1)$ where the sum is easily checked to be a finite sum. The following is directly proved:

Lemma C.1. *The map ϕ is a cochain $R\langle t\rangle$-homomorphism and natural with respect to maps between infinite cyclic covering space pairs, lifting maps between compact topological pairs.*

Note that the cohomology of $C_c^*(\tilde{X}, \tilde{X}_1)$ is $H_c^*(\tilde{X}, \tilde{X}_1)$. Let $H^*_{R\langle t\rangle}(\tilde{X}, \tilde{X}_1)$ be the cohomology of $C^*_{R\langle t\rangle}(\tilde{X}, \tilde{X}_1)$.

Lemma C.2. *If (X, X_1) is a compact polyhedral pair, then*

$$\phi^*: H_c^*(\tilde{X}, \tilde{X}_1) \to H^*_{R\langle t\rangle}(\tilde{X}, \tilde{X}_1)$$

is an isomorphism.

Proof. Let (X', X_1') be a finite simplicial pair which is a triangulation of (X, X_1) and $(\tilde{X}', \tilde{X}_1')$, the lift of (X', X_1'). Let $C_f^*(\tilde{X}', \tilde{X}_1')$ be the finite simplicial cochain complex and $C^*_{R\langle t\rangle}(\tilde{X}', \tilde{X}_1') = \mathrm{Hom}_{R\langle t\rangle}[C_*(\tilde{X}', \tilde{X}_1'), R\langle t\rangle]$ for the simplicial chain complex $C_*(\tilde{X}', \tilde{X}_1')$, which is $R\langle t\rangle$-free of finite rank. Then the map $\phi': C_f^*(\tilde{X}', \tilde{X}_1') \to C^*_{R\langle t\rangle}(\tilde{X}', \tilde{X}_1')$ defined by

$$\phi'(f)(x) = \sum_{i\in Z} f(t^i x) t^{-i}$$

is easily seen to be bijective. So the induced map

$$\phi'^*: H_f^*(\tilde{X}', \tilde{X}'_1) \longrightarrow H^*_{R\langle t\rangle}(\tilde{X}', \tilde{X}'_1)$$

is an isomorphism. Since there are natural isomorphisms $H^*_{R\langle t\rangle}(\tilde{X}, \tilde{X}_1) \cong H^*_{R\langle t\rangle}(\tilde{X}', \tilde{X}_1')$ [Use the universal coefficient theorem over $R\langle t\rangle$] and $H_c^*(\tilde{X}, \tilde{X}_1) \cong H_f^*(\tilde{X}', \tilde{X}_1')$, we see that ϕ^* is an isomorphism and complete the proof.

By [K/S] every compact manifold pair is homotopy equivalent to a compact polyhedral pair. So we see from the naturality of ϕ and Lemma C.2 the following:

Corollary C.3. *For a compact manifold pair (X, X_1),*

$$\phi^*: H_c^*(\tilde{X}, \tilde{X}_1) \longrightarrow H^*_{R\langle t\rangle}(\tilde{X}, \tilde{X}_1)$$

is an isomorphism.

Proof of the Blanchfield Duality Theorem. Let $x \in H_p(\tilde{M}, \partial_1\tilde{M})$ and $y = \{c_y\} \in H_q(\tilde{M}, \partial_2\tilde{M})$, and $x_B \in B_p(\tilde{M}, \partial_1\tilde{M})$, $y_B \in B_q(\tilde{M}, \partial_2\tilde{M})$, the images of x, y. By Proposition A.4, $\mathrm{Int}_{\tilde{M}}(x, y) = \varepsilon_{\tilde{M}}(u \cap y) = f_u(c_y)$ for $u = \{f_u\} \in H_c^q(\tilde{M}, \partial_2\tilde{M})$ with $u \cap [\tilde{M}] = x$. So, $\tilde{\mathrm{I}}\mathrm{nt}_{\tilde{M}}(x, y) = \sum_{i=Z} f_u(t^i c_y) t^{-1} = \phi(f_u)(c_y)$. By the universal coefficient theorem over $R\langle t\rangle$, note that

$$H^*_{R\langle t\rangle}(\tilde{M}, \partial_2\tilde{M})/\mathrm{Tor}_{R\langle t\rangle}H^*_{R\langle t\rangle}(\tilde{M}, \partial_2\tilde{M}) = \mathrm{Hom}_{R\langle t\rangle}[H_*(\tilde{M}, \partial_2\tilde{M}), R\langle t\rangle]$$
$$= \mathrm{Hom}_{R\langle t\rangle}[B_*(\tilde{M}, \partial_2\tilde{M}), R\langle t\rangle].$$

Let β be the composite t-anti-isomorphism (cf. Corollary C.3)

$$B_p(\tilde{M}, \partial_1\tilde{M}) \xleftarrow[\cong]{\cap[\tilde{M}]} H^q_c(\tilde{M}, \partial_2\tilde{M})/\mathrm{Tor}_{R\langle t\rangle}H^q_c(\tilde{M}, \partial_2\tilde{M})$$
$$\xrightarrow[\cong]{\phi^*} \mathrm{Hom}_{R\langle t\rangle}[B_q(\tilde{M}, \partial_2\tilde{M}), R\langle t\rangle].$$

Then $\tilde{\mathrm{I}}\mathrm{nt}_{\tilde{M}}(x_B, y_B) = \phi(f_u)(c_y) = \beta(x_B)(y_B)$. This implies that $\tilde{\mathrm{I}}\mathrm{nt}_{\tilde{M}}$ is non-singular and completes the proof.

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Department of Mathematics
Osaka City University