KNOTS IN THE STABLE 4-SPACE; AN OVERVIEW

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Let M be a smooth connected oriented $(n+2)$-manifold. We call an oriented n-sphere smoothly imbedded in $Int(M)$ a G-knot in M, and a knot in M if it is null-homotopic in M. A G-link (link, resp.) in M is a disjoint union of finitely many G-knots (knots, resp.) in M. A G-link L in M is called a B-link in M if L is null-homologous in M. Two G-links L_1, L_2 in M are equivalent if there is an orientation-preserving diffeomorphism of M sending L_1 to L_2 orientation-preservingly. The equivalence classes of G-links are called the types of G-links. The stable 4-space, which we denote by SR^4, is any smooth 4-manifold diffeomorphic to the universal covering space of $T^4 \# S^2 x S^2$, which is obtained from the 4-space, R^4, by stabilization, that is, by a connected sum with countably infinite copies of $S^2 x S^2$ (cf. § 1). The stable 4-space SR^4 is so huge that we can imbed all 3-manifolds smoothly and properly in it. The purpose of this paper is to give an overview of the author's recent results on several properties of links in SR^4. No proof is given here. Details will be published elsewhere in separated forms ([12]). In § 1 we discuss the stable 4-space SR^4 itself. We characterize open 4-manifolds which become SR^4 after stabilization. It follows that any exotic 4-space (cf. Gompf[5]) becomes SR^4 after stabilization. In § 2 we characterize the group of a link in SR^4 and discuss related topics. In § 3 the torsion pairing invariant of a B-link in SR^4 is introduced. In § 4 we observe that there are many knots in

SR^4 with no minimal Seifert manifolds and discuss the supporting degree. In § 5 we observe that there are many links in SR^4 which are not I-equivalent. In § 6 the Triviality Theorem for a G-link in SR^4 is stated and the prime decomposition and the unknotting number of a knot in SR^4 are discussed.

§1 The stable 4-space

Let W be a non-compact connected oriented smooth n-manifold $(n > 1)$. Let $\{B_i\}_{i=1}^{\infty}$ be a family of disjoint n-balls B_i in $Int(W)$. This family is said to be discrete, if for each compact set C in W $C \cap B_i = \emptyset$ except a finite number of i. Then note that $W - \bigcup_{i=1}^{\infty} Int(B_i)$ is a smooth manifold with boundary $\bigcup_{i=1}^{\infty} \partial B_i \cup \partial W$. W is connected at infinity, if for each compact set C in W there is a compact set C' in W such that $C' \supset C$ and $W-C'$ is connected. W is 1-connected at infinity, if W is connected at infinity and for each compact set C in W with $W-C$ connected there is a compact set C' in W such that $C' \supset C$ and $W-C'$ is connected and the natural homomorphism $\pi_1(W-C') \to \pi_1(W-C)$ is trivial.

Lemma 1.1. If W is connected at infinity, then for any two discrete families $\{B_i\}_{i=1}^{\infty}$, $\{B_i'\}_{i=1}^{\infty}$ there is an orientation-preserving diffeomorphism of W sending B_i to B_i' for all i.

Let $n = 4$ and assume that W is connected at infinity. Taking a connected sum of W and countably infinite copies of $S^2 \times S^2$ along a discrete family of 4-balls in W, we obtain a smooth 4-manifold whose diffeomorphism type is independent of a choice of a discrete family of 4-balls in W by Lemma 1.1.

Definition. This 4-manifold is the stabilization of W and denoted by SW.

Theorem 1.2. SW is diffeomorphic to the stable 4-space SR^4 if and only if W is a 1-connected spin open 4-manifold which is 1-connected at infinity.

For example, the stabilization of a punctured K3-surface is diffeomorphic to SR^4.

Corollary 1.2.1. Any W stated in Theorem 1.2 is smoothly imbedded in SR^4 with complement diffeomorphic to SR^4_+ of the upper half 4-space R^4_+.

Corollary 1.2.2. For any compact smooth 4-submanifold W of SR^4 with ∂W a 3-sphere, $SR^4\text{-Int}(W)$ is diffeomorphic to SR^4 with an open 4-ball removed.

Corollary 1.2.3. For any exotic 4-space $\tilde{R}^4$, $S\tilde{R}^4$ is diffeomorphic to SR^4 and $\tilde{R}^4$ is smoothly imbedded in SR^4 with complement diffeomorphic to SR^4_+.

Remark 1.2.4. The existence of an exotic 4-space $\tilde{R}^4$ with $\tilde{R}^4 \#^m_{i=1} S^2 x S^2_i$ diffeomorphic to $R^4 \#^m_{i=1} S^2 x S^2_i$ for some finite m is equivalent to the existence of an exotic 4-sphere whose punctured manifold is an exotic 4-space. The author is grateful to R. Gompf for correcting an error of the first draft as above and informing him that the existence of an exotic 4-sphere whose punctured manifold is diffeomorphic to R^4 is equivalent to the failure of the 4-dimensional smooth Schönflies conjecture, which is still conceivable, for as J. Tao observed, any punctured homotopy n-sphere with $n > 4$ is diffeomorphic to R^n.

Recall that a map is proper if the preimage of any compact set is compact.

Theorem 1.3. Any (possibly, non-compact or disconnected) smooth oriented 3-manifold is smoothly and properly imbeddable in SR^4.

The idea of the proof is to construct a 4-manifold which contains a given 3-manifold and has the conditions of Theorem 1.2. Since SR^4 is a proper smooth submanifold of R^5, we have the following Corollary, generalizing a result of Hirsch [7] to the non-compact case:

Corollary 1.3.1. Any smooth oriented 3-manifold is smoothly and properly imbeddable in R^5 with a trivial normal bundle.

Remark 1.3.2. For each compact connected oriented topological 4-manifold there are infinitely many compact connected oriented 3-manifolds which are not topologically imbeddable in it (cf. [10]).

§2 The groups of links

Let L be a G-link in a smooth connected oriented (n+2)-manifold M. A smooth compact oriented (n+1)-submanifold V of M is called a Seifert manifold for L if $\partial V = L$ and V has no components of closed manifolds. Let $\langle t \rangle$ be the infinite cyclic group generated by a letter t.

Lemma 2.1. The following conditions on a G-link L in M are equivalent:

(1) L is a B-link,

(2) L has a Seifert manifold,

(3) There is an epimorphism $\gamma: \pi_1(M-L) \to \langle t \rangle$ sending each meridian of L to t.

We call γ in (3) a B-*epimorphism*. A link in M is *trivial* if it has a Seifert manifold consisting of disjoint (n+1)-balls. We consider a G-link L in SR^4. Let $E(L) = SR^4-L$. We call $G(L) = \pi_1(E(L))$ the *group* of L.

Definition. We say that a B-link L in SR^4 is G-*ribbon* if it has a Seifert manifold diffeomorphic to $S^3\#_{i=1}^{m} S^1 \times S_i^2$ with open 3-balls removed.

The normal closure of elements $x_1, x_2, \ldots, x_s$ in a group G is denoted by $(x_1, x_2, \ldots, x_s)^G$.

Theorem 2.2. The group $G(L)$ of an r-component link L in SR^4 is a finitely presented group with $H_1(G(L)) \cong \oplus_r Z$ and $(m_1, m_2, \ldots, m_r)^{G(L)} = G(L)$ for meridians $m_1, m_2, \ldots, m_r$ of L. Further, for any finitely presented group G and elements $m_1, m_2, \ldots, m_r$ with $H_1(G) \cong \oplus_r Z$ and $(m_1, m_2, \ldots, m_r)^G = G$, there is an r-component G-ribbon link in SR^4 whose group is G and whose meridians are $m_1, m_2, \ldots, m_r$.

Theorem 2.2'. The group $G(L)$ of an r-component B-link L in SR^4 is a finitely presented group with unique B-epimorphism and $(m_1, m_2, \ldots, m_r)^{G(L)} = G(L)$ for meridians $m_1, m_2, \ldots, m_r$ of L. Further, for any finitely presented group G and an epimorphism $\gamma: G \to \langle t \rangle$ and elements $m_1, m_2, \ldots, m_r$ such that $(m_1, m_2, \ldots, m_r)^G = G$ and $\gamma(m_i) = t$, $i = 1, 2, \ldots, r$, there is an r-component G-ribbon B-link in SR^4 whose group is G and whose B-epimorphism is γ and whose meridians are $m_1, m_2, \ldots, m_r$.

Theorem 2.2". The group $G(L)$ of an r-component G-link L in SR^4 is a finitely presented group with $(m_1, m_2, \ldots, m_r)^{G(L)} = G(L)$ for meridians $m_1, m_2, \ldots, m_r$ of L. Further, for any finitely presented group G and elements $m_1, m_2, \ldots, m_r$ with $(m_1, m_2, \ldots, m_r)^G = G$, there is an r-component G-link in SR^4 whose group is G and whose meridians are $m_1, m_2, \ldots, m_r$ (cf. González-Acuña[6]).

A G-ribbon link in SR^4 is obtained from a G-link L^* in SR^4 with a trivial normal bundle by the following operation (F):

(F) Perform a surgery on the boundary of a collar $L^* \times [0,1]$ of L^* in SR^4 along disjointly imbedded 1-handles, not meeting $L^* \times (0,1)$.

Definition. A link in SR^4 is Q-ribbon if it is obtained by the operation (F) from a G-link L^*, equivalent to the union of a trivial link in R^4 and some $p_j \times S_j^2$'s under an identification $SR^4 = R^4 \#_{i=1}^{\infty} S^2 \times S_i^2$.

Theorem 2.3. The class of the groups of Q-ribbon links in SR^4 equals that of closed oriented surfaces smoothly imbedded in R^4.

Corollary 2.3.1. For each $r > 0$ there are infinitely many r-component G-ribbon links in SR^4 whose groups are not isomorphic to the groups of Q-ribbon links in SR^4.

A compact smooth 4-submanifold W of SR^4 is called a support of a G-link L in SR^4 if ∂W is a 3-sphere and $L \subset \mathrm{Int}(W)$. The closed 4-

manifold W^+, obtained from W by attaching a 4-ball is called a <u>closed support</u> of a G-link L in SR^4. By Lemma 1.1 and Corollary 1.2.2, the G-link L in SR^4 is obtained from L in $\mathrm{Int}(W)$ by stabilization along a discrete family of 4-balls disjoint from L. Then we call that <u>the G-link L in SR^4 is obtained from a G-link L in</u> W, $\mathrm{Int}(W)$ or W^+ <u>by stabilization</u>. If L is a link (B-link, resp.) in SR^4, then L is also a link (B-link, resp.) in any support W of L. For a link L^1 in S^3 with components k_i, $i = 1,2,\dots,r$, we consider the product $S^1 x (S^3, L^1)$. Choose disjoint 3-balls B_i, $i = 1,2,\dots,r$, in S^3 so that $B_i \cap k_i$ is an unknotted arc in B_i. Perform surgeries on $S^1 x S^3$ replacing $S^1 x B_i$ by $D^2 x \partial B_i$. Then we obtain from $S^1 x (S^3, L^1)$ an r-component link, $S(L^1)$, in $S^4 \#_{i=1}^{r-1} S^2 x S_i^2$.

<u>Definition</u>. A link L in SR^4 is the <u>surgery-spun link</u> of a link L^1 in S^3, if it is obtained from the link $S(L^1)$ stated above by stabilization.

The type of L in SR^4 is uniquely determined by the type of L^1 in S^3. The group $G(L)$ is meridian-preservingly isomorphic to the group $\pi_1(S^3 - L^1)$. Further, L is Q-ribbon and each component of L is contained in a 4-ball in SR^4 as the spun 2-knot of the corresponding component of L^1.

<u>Lemma</u> 2.4. For a G-link L in SR^4 the following are equivalent:

(1) Any map $f: S^2 \to SR^4$ is homotopic to a map $f': S^2 \to SR^4$ with $f'S^2 \cap L = \emptyset$,

(2) L is a link in SR^4 with $H_2(G(L)) = 0$.

<u>Definition</u>. A link L in SR^4, given by Lemma 2.4 is called a

flexible link.

Corollary 2.4.1. For the surgery-spun link L in SR^4 of a link L^1 in S^3 the following are equivalent:

(1) L is flexible,

(2) L is contained in a 4-ball in SR^4,

(3) The link L^1 is completely splittable.

By Theorem 2.3 and, for example, Litherland's result[17], we have the following:

Corollary 2.4.2. For each $r > 0$ there are infinitely many r-component Q-ribbon links in SR^4 that are not flexible links.

By Theorem 2.2, Kervaire's result[13] and Yajima's results[22],[23], we have the following:

Corollary 2.4.3. The class of the groups of flexible links in SR^4 equals that of links in S^{n+2} for $n > 2$ and is contained in that of Q-ribbon links in SR^4.

By arguments in § 3 we can show that there are infinitely many flexible links that are not G-ribbon (and hence Q-ribbon).

§ 3 The torsion pairing invariant

For a module H over the integral group ring Λ of $\langle t \rangle$, let TH be the Λ-torsion part of H and BH = H/TH. When H is finitely generated over Λ, let DH be the maximal finite Λ-submodule of H(cf. [11]). Let $E^1H = \mathrm{Ext}^1_\Lambda(H,\Lambda)$. For a B-link L in SR^4, let $\tilde{E}(L)$ be the covering space of $E(L)$ associated with the B-epimorphism. $H_q(\tilde{E}(L))$ is a finitely generated Λ-module when $q \neq 2$ and a direct sum of a finitely generated Λ-module and a free Λ-module of infinite rank when $q = 2$. The following is essentially a consequence of the Second Duality Theorem of [11]:

<u>Theorem</u> 3.1. For any B-link L in SR^4 there is a t-isometric symmetric non-singular pairing

$$\ell_L : D_L \times D_L \to Q/Z$$

such that

(1) (D_L, ℓ_L) is an invariant of the type of L, and

(2) There is a t-anti epimorphism $\theta : DH_1(\tilde{E}(L)) \to E^1(BH_2(\tilde{E}(L)))$ which is an invariant of the type of L and whose kernel is D_L.

If a knot K in SR^4 is obtained from a knot in S^4 by stabilization, then the pairing $\ell_K : D_K \times D_K \to Q/Z$ coincides with the Farber/Levine pairing of the original knot in S^4 (cf. [3],[16]). We call our pairing ℓ_L the <u>torsion pairing</u> of a B-link L in SR^4, the terms borrowed from [16].

<u>Proposition</u> 3.2. If a B-link L in SR^4 is G-ribbon, then $D_L = 0$.

<u>Corollary</u> 3.2.1. For each $r > 0$ there are infinitely many r-component flexible links that are not G-ribbon.

For a knot K in SR^4 we have $t-1 : H_1(\tilde{E}(K)) \cong H_1(\tilde{E}(K))$, so that $t-1 : D_K \cong D_K$. The following characterizes the torsion pairings of knots in SR^4:

Theorem 3.3. For any finite Λ-module D with $t-1: D \cong D$ and any t-isometric symmetric non-singular pairing $\ell: D \times D \to Q/Z$, there is a knot K in SR^4 such that (D_K, ℓ_K) is Λ-isomorphic to (D, ℓ) and $D_K = H_1(\tilde{E}(K))$.

It is unknown whether all pairs (D, ℓ) are realizable by knots in S^4 (cf. [16]).

§ 4 Knots with no minimal Seifert manifolds and the supporting degree

Let V be a Seifert manifold for a B-link L in a smooth connected oriented $(n+2)$-manifold M.

Definition. If the natural homomorphism $\pi_1(\mathrm{Int}(V_j)) \to \pi_1(M-L)$ is injective for each component V_j of V, then we say that the Seifert manifold V is minimal.

For $n = 1$ any B-link in M has a minimal Seifert surface. This is well-known by the loop theorem.

Proposition 4.1. In case $n > 2$, any B-link in M with a B-epimorphism whose kernel is finitely presented has a minimal Seifert manifold.

This is proved by a surgery argument(cf. Farrell[4,p. 325]). Note that a B-link L in SR^4 has a minimal Seifert manifold if and only if L has a minimal Seifert manifold in some closed support W^+ of L.

Proposition 4.2. Any surgery-spun link in SR^4 has a minimal Seifert manifold.

Definition. The supporting degree, sd(L), of a G-link L in SR^4 is the least number m such that L has a closed support diffeomorphic to $S^4 \#_{i=1}^{m} S^2 x S_i^2$.

Theorem 4.3. For each $m > 0$, there are infinitely many (up to equivalences) systems of knots K_i, $i = 0,1,\ldots,m$, in SR^4 such that

(0) K_0 is obtained from a fibered knot in S^4 by stabilization,

(1) Any two of $G(K_i)$, $i = 0,1,\ldots,m$, are meridian-preservingly isomorphic,

(2) No two of $H_2(\tilde{E}(K_i))$, $i = 0,1,\ldots,m$, are Λ-isomorphic,

(3) $sd(K_i) = i$, $i = 0,1,\ldots,m$, and

(4) For each $i \neq 0$, K_i has no minimal Seifert manifold.

Remark 4.3.1. Lee[15] gave, in our terminology, examples of non-flexible knots K in SR^4 with $sd(K) \neq 0$ (in fact, $sd(K) = 1$) by using the fact that any knot K' in SR^4 with $sd(K') = 0$ has $H_2(G(K')) = 0$. Theorem 4.3 tells us that there are many flexible knots K in SR^4 with $sd(K) \neq 0$. Tamura[21] also considered a similar problem on higher dimensional simple knots, such as genus one knots in $S^n x S^{n+1}$ $(n > 2)$.

The following proposition is useful to construct certain knots in SR^4 with no minimal Seifert manifolds (though it is not applicable to the proof of Theorem 4.3):

Proposition 4.4. If a knot K in SR^4 has a minimal Seifert manifold and the commutator subgroup $[G(K),G(K)]$ of $G(K)$ is finitely generated, then $[G(K),G(K)]$ is isomorphic to the fundamental group of a closed oriented 3-manifold.

The proof is similar to that of Neuwirth[20,Theorem 4.5.1].

Example 4.4.1. Let G be the fundamental group of the 0-surgery manifold of a non-trivial fibered knot in S^3. G is the group of a knot in $S^2 x S^2$ and hence in SR^4. Any knot in SR^4 with group G has no minimal Seifert manifold by Proposition 4.4, for $[G,G]$ is isomorphic to the fundamental group of a closed orientable surface of genus > 0, not to the fundamental group of a closed orientable 3-manifold.

Theorem 4.3 tells us that there are many knots K in SR^4 such that $[G(K),G(K)]$ is isomorphic to the fundamental group of a closed oriented 3-manifold but K has no minimal Seifert manifold. It is unknown whether all knots in S^4 have minimal Seifert manifolds. Theorem 4.3 is obtained as an application of the torsion pairing invariant.

§ 5 Cobordism

For any 2-component link in R^3 with linking number $\neq 0$, the components never bound disjoint compact oriented surfaces in R^4_+. Such a phenomenon does not happen for links in SR^4.

Proposition 5.1. For each link in SR^4, the components bound disjoint compact oriented smoothly imbedded 3-manifolds in $SR^4 x[0,+\infty)$.

Definition. Two G-links L, L' in SR^4 are I-equivalent, if there is a 3-manifold W, topologically imbedded in $SR^4 x[0,1]$, such that $W \cong Lx[0,1]$, $(\partial W) \cap SR^4 x0 = (-L)x0$, $(\partial W) \cap SR^4 x1 = L'x1$ and $(\partial W) \cap SR^4 x(0,1) = \emptyset$. Moreover, if W is smoothly imbedded in $SR^4 x[0,1]$, then G-links L, L' are cobordant.

A link in SR^4 is called a boundary link if the components have disjoint Seifert manifolds, and a weakly split link if the link has a disconnected Seifert manifold. The following is essentially due to Kervaire[14]:

<u>Proposition</u> 5.2. Every boundary link in SR^4 is cobordant to a trivial link. In particular, every knot in SR^4 is cobordant to a trivial knot.

It is unknown whether all links in S^4 (more generally, all flexible links in SR^4) are cobordant to trivial links (cf. Cochran[1]), but there are many non-flexible links in SR^4 which are not I-equivalent.

<u>Theorem</u> 5.3. For each $r > 1$ there are infinitely many (up to I-equivalences) r-component links L in SR^4 which are I-equivalent to none of flexible links, weakly split links and links with supporting degree $< r-1$.

<u>Theorem</u> 5.4. For each $r > 1$ there are infinitely many (up to I-equivalences) pairs of r-component non-flexible links L, L' in SR^4 such that the groups $G(L)$, $G(L')$ are meridian-preservingly isomorphic but the links L, L' are not I-equivalent.

We use the torsion pairing invariant to show Theorem 5.4.

§ 6 <u>Arithematic</u>

Let p be a point of a G-link L in SR^4. We call a half-open interval, α, smoothly and properly imbedded in SR^4 a <u>string relative to</u> (L,p) if $\partial\alpha = p$ and $(\alpha - p) \cap L = \emptyset$.

<u>Lemma</u> 6.1. For any two strings α, α' relative to (L,p), there is an orientation-preserving diffeomorphism of SR^4 sending α to α' and fixing L.

For G-knots K_i in SR^4 and strings α_i relative to (K_i, p_i), $i = 1,2$, we choose tubular neighborhoods $T(\alpha_i)$ so that $(T(\alpha_i), T(\alpha_i) \cap K_i)$ are diffeomorphic to $(D^3 x[0,+\infty), D^2 x1)$, where $D^2 \subset D^3$ is a standard disk pair. We attach $(SR^4 - Int(T(\alpha_i)), K_i - Int(T(\alpha_i) \cap K_i))$; $i = 1,2$, by an orientation-reversing diffeomorphism between the boundary pairs. The result is a pair (SR^4, K') with K' a G-knot in SR^4. The type of K' does not depend on any choice of strings by Lemma 6.1 and is determined only by the types of K_i.

Definition. The G-knot K' is the sum of K_i, $i = 1,2$, and denoted by $K_1 \# K_2$.

Definition. A G-knot K in SR^4 is prime if $K = K_1 \# K_2$ means that K_1 or K_2 is trivial.

Theorem 6.2. Any knot in SR^4 is the sum of finitely many prime knots in SR^4.

This result is a combination of Maeda's Theorem[18] and the following Triviality Theorem for G-links, essentially due to Matumoto[19]:

Theorem 6.3 (Triviality Theorem). A G-link L in SR^4 is a trivial link if and only if $G(L)$ is a free group with a meridian basis. In particular, a G-knot K in SR^4 is a trivial knot if and only if $G(K) \cong Z$.

To state Maeda's Theorem, we consider a pair (G,m) with the following conditions:(1) G is a finitely presented group,(2) $H_1(G)$ Z,(3) $(m)^G = G$.

We call this pair a pair of type K (cf. Theorem 2.2). For pairs (G_i, m_i), $i = 1,2$, of type K the pair $(G_1 * G_2 / m_1 = m_2, m_1)$ is also of type K and called the <u>sum</u> of (G_i,m_i), $i = 1,2$, and denoted by $(G_1,m_1)*(G_2,m_2)$. A pair (G,m) is <u>prime</u> if $(G,m) = (G_1,m_1)*(G_2,m_2)$ means G_1 or $G_2 \cong Z$. A knot K in SR^4 is prime when $(G(K),m)$ is prime for an meridian m of K, by the Triviality Theorem. Two pairs (G,m), (G',m') are <u>equivalent</u> if there is an isomorphism from G to G' sending m to m'.

<u>Maeda's Theorem</u>.*) Any pair (G,m) of type K is the sum of finitely many prime pairs. Moreover, the sum is unique (up to equivalences and the orders of sums).

It is still unknown whether the sum in Theorem 6.2 is unique. Next, to discuss the unknotting number of a knot in SR^4, we note that the boundary surface of a solid torus smoothly imbedded in SR^4 is unique up to ambient isotopies of SR^4 (cf. [8]). We call this surface a <u>trivial surface</u> in SR^4. Using that a knot has a Seifert manifold, we can transform any knot in SR^4 into a trivial surface by surgeries along a finite number of imbedded 1-handles (cf. [8]).

<u>Definition</u>([9]). The least number of these imbedded 1-handles is the <u>unknotting number</u> of a knot K in SR^4 and denoted by $u(K)$.

Let c be a null-homologous, smooth simple closed curve in SR^4-K. Perform a spin surgery $SR^4-T(c) \cup D^2 \times \partial D^3$ along a tubular neighborhood $T(c) = S^1 \times D^3$ of c. Since the result is also diffeomorphic to SR^4, we obtain a new knot K' in SR^4. We can show that any knot in SR^4 is transformed into a trivial knot by spin surgeries along tubular neighborhoods of a finite number of such curves c.

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*)The finiteness part was independently proved by Dunwoody/Fenn[2].

Definition. The least number of these curves c is the *weak unknotting number* of a knot K in SR^4 and denoted by $u_w(K)$.

Let $b(K)$ be the least number of meridian generators of $G(K)$ and $w(K)$, the least number of elements $x_1, x_2, \ldots, x_s$ of $G(K)$ such that $(x_1, x_2, \ldots, x_s)^{G(K)} = [G(K), G(K)]$ and $e(K)$, the least number of Λ-generators of $H_1(\tilde{E}(K))$. Then we have the following:

Theorem 6.4. $e(K) \leq w(K) = u_w(K) \leq u(K) \leq b(K)-1$.

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