SUPPLEMENT TO A CHORD DIAGRAM OF A RIBBON SURFACE-LINK

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Abstract

A revised proof is given to an assertion on chord diagrams of a ribbon surface-link.

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1 Statement of result

In a previous paper [**K**], it is shown as the main result (Theorem 4.1) that any two cord diagrams of faithfully equivalent ribbon surface-links are deformed into each other by a finite number of the moves M_0, M_1, M_2 on the chord diagrams. Blake Winter recently suggested that a procedure of increasing chords in the argument of [**K**] is not clear for equivalences of one-fusion ribbon 2-knots, for T. Yasuda in [**Y**] has given the necessary example to increase a chord for a one-fusion ribbon 2-knot. The author would like to thank him for this suggestion.

Corollary 4.7 of [**K**] was an essential part to obtain the main result of [**K**] and is revised as Theorem in this paper. Terminologies, notations and references are completely borrowed from [**K**].

A connected chord graph $(o;\alpha)$ is called *in a standard shape* if it is in the shape of Fig. 23 of [**K**], namely the based loop system o consists of the loops o_i $(i = 0, 1, 2, \ldots, n)$ and the chord α_i spans o_0 and o_i for every i with $1 \leqq i \leqq n$ and the chord α_j with $j > n$ is a self-connecting chord attached to o_0. A disconnected chord graph $(o; \alpha)$ is said to be *in a standard shape* if every connected component of $(o; \alpha)$ is in a standard shape. Unless otherwise mentioned, a chord graph is in a standard shape.

For a chord diagram $C(o; \alpha)$, assume by the move M_0 that the loops in the based loop system o are mutually disjoint simple loops with counterclockwise orientation in the plane so that o bounds the system d of mutually disjoint oriented disks d_i $(i = 0, 1, 2, \ldots, n)$ in the plane. Then the pair (d, α) is called a *disk chord system*. Corollary 4.7 of [**K**] was stated as follows:.

Corollary 4.7. After a finite number of the moves M_0, M_1, M_2 on the chord diagrams $C(o; \alpha)$ and $C(o; \alpha')$, the chords α are homotopic to the chords α' in $\mathbf{R}^3$ by a homotopy relative to the based loops o.

In this paper, the following theorem with the exact assumption of Corollary 4.7 included here for convenience is shown:

Theorem. Let $f' : \mathbf{R}^3[-3, 3] \to \mathbf{R}^3[-3, 3]$ be an orientation-preserving homeomorphism preserving $\mathbf{R}^3[-3]$ and $\mathbf{R}^3[3]$, respectively. Let $(\bar{o}, \bar{\alpha})$ and $(\bar{o}, \bar{\alpha}')$ be chord graphs in $\mathbf{R}^3[1]$ with $f'(\bar{o}) = \bar{o}$ such that $f'(\bar{\alpha})$ is homotopic to $\bar{\alpha}'$ in $\mathbf{R}^3[-3, 3]$ by a homotopy relative to $o[-3, 3]$. Assume that the restriction $f'|_{\bar{o} \cup u[-3,3]}$ is the identity map for a simple arc system $\bar{u}$ in $\bar{o}$ containing the attaching points of the homotopically corresponding chord systems $\bar{\alpha}$, $\bar{\alpha}'$. Then every chord diagram $C(o; \alpha)$ is deformed into a chord diagram $C(o; \alpha')$ by a finite number of the moves M_0, M_1, M_2 on $C(o; \alpha)$, $C(o; \alpha')$ and homotopic deformations of the chord systems $\bar{\alpha}$, $\bar{\alpha}'$ in $\mathbf{R}^3[-3, 3]$ by homotopies relative to $o[-3, 3]$.

Theorem and Corollary 4.7 are the same assertion except that the homotopies and the moves M_0, M_1, M_2 are mixed in Theorem. If $f'(\bar{\alpha})$ is homotopic to $\bar{\alpha}'$ in $\mathbf{R}^3[-3, 3]$ by a homotopy relative to $o[-3, 3]$, then the chord system $\bar{f}'(\bar{\alpha})$ in $\mathbf{R}^3[1]$ obtained from $f'(\bar{\alpha})$ by the projection $\mathbf{R}^3[-3, 3] \to \mathbf{R}^3[1]$ is homotopic to $\bar{\alpha}'$ in $\mathbf{R}^3[1]$ by a homotopy relative to $\bar{o}$. Then by Lemma 4.6 their chord diagrams are deformable into each other by the moves M_0, M_1, M_2. Thus, Theorem and Corollary 4.7 are equivalent statements.

Proof of Theorem. The proof will be done by assuming that $f'(\bar{\alpha}) = \bar{\alpha}'$. The following observation is used in our argument:

Observation 1. If a disk chord system $(\bar{d}', \bar{\alpha}')$ in $\mathbf{R}^3[1]$ is obtained from a disk chord system $(\bar{d}, \bar{\alpha})$ in $\mathbf{R}^3[1]$ by a homotopic deformation in $\mathbf{R}^4$ deforming the pair $(\bar{d}, \bar{d} \cap \bar{\alpha})$ into the pair $(\bar{d}', \bar{d}' \cap \bar{\alpha}')$ isotopically, then the chord system $\bar{\alpha}'$ is homotopic to the chord system $\bar{\alpha}$ in $\mathbf{R}^3[-3,3]$ by a homotopy relative to $o[-3,3]$ by regarding as $\partial d = \partial d' = o$.

The proof of Observation 1 is obtained by taking a homotopic deformation fixing the part $(\bar{d}, \bar{d} \cap \bar{\alpha})$. The following observation is a key observation for the proof of Theorem:

Observation 2. After a finite number of the moves M_0, M_1, M_2 on chord diagrams $C(o;\alpha)$, $C(o;\alpha')$, homotopic deformations of the chord systems $\bar{\alpha}$, $\bar{\alpha}'$ by homotopies relative to $o[-3,3]$ and isotopic deformations of f', the disk chord systems $(\bar{d};\bar{\alpha})$ and $(\bar{d};\bar{\alpha}')$ have the properties

$$f'(\bar{\alpha}) = \bar{\alpha}' \quad \text{and} \quad f'(\bar{\alpha} \cap \bar{d}_i) = \bar{\alpha}' \cap \bar{d}_i \quad (i = 0, 1, 2, \ldots, n).$$

Let B be an oriented 3-ball in $\mathbf{R}^3[1]$ containing the disk chord system $(\bar{d};\bar{\alpha})$. Let $(d^{f'};\alpha^{f'})$ be the disk chord system in the 3-ball $f'(B)$ obtained from $(\bar{d};\bar{\alpha})$ by the homeomorphism f', and $(\bar{d}^{f'};\bar{\alpha}^{f'})$ a disk chord system in $\mathbf{R}^3[1]$ obtained from $(d^{f'};\alpha^{f'})$ by deforming $f'(B)$ into $\mathbf{R}^3[1]$ in $\mathbf{R}^3[-3,3]$. Since $f'|_{\bar{o} \cup u[-3,3]} = 1$ (before deforming f'), the chord graph $(\bar{o};\bar{\alpha})$ is equivalent to the chord graph $(\bar{o}^{f'};\bar{\alpha}^{f'})$ in $\mathbf{R}^3[1]$, where $\bar{o}^{f'} = \partial \bar{d}^{f'}$. Hence a chord diagram $C(\bar{o};\bar{\alpha})$ is deformed into a chord diagram $C(\bar{o}^{f'};\bar{\alpha}^{f'})$ in by the move M_0 (see [**8, 9, 14**]). Since Observation 1 can be applied between $(\bar{d};\bar{\alpha}')$ and $(\bar{d}^{f'};\bar{\alpha}^{f'})$, Theorem is obtained.

Observation 2 is shown by an inductive argument on the component number r of the chord graph $(\bar{o};\bar{\alpha})$ in $\mathbf{R}^3[1]$, which will be done from now.

Let $(\bar{o}^{(i)};\bar{\alpha}^{(i)})$ $(i = 1, 2, \ldots, r)$ be the connected components of $(\bar{o};\bar{\alpha})$. Let $\bar{d}^{(i)}$ be the disk sub-system of $\bar{d}$ with $\partial \bar{d}^{(i)} = \bar{o}^{(i)}$. To show this observation, the homeomorphism f' is regarded as a diffeomorphism on some smooth structures on $\mathbf{R}^3[-3,3]$ with $d[-3,3]$ as a smooth submanifold (see [**2**, p. 128]). Then the intersections $d[-3,3] \cap (f')^{-1}(\bar{d})$ and $d[-3,3] \cap f'(\bar{d})$ are considered as mutually disjoint simple loops and arcs including $\bar{o}$ by a transversality argument. Deform these simple loops into sets of mutually disjoint intersection annuli, denoted by a and a', respectively. Let a and a' have orientations induced from $(f')^{-1}(\bar{d})$ and $f'(\bar{d})$, respectively. Every annulus in a or a' is assumed to have a disk which is orientation-preservingly embedded into $\bar{d}$ by the projection $\mathbf{R}^3[-3,3] \to \mathbf{R}^3[1]$. Let e and e' be such disk systems in a and a' with the projection images $\bar{e}$ and $\bar{e}'$ in $\bar{d}$, respectively. By a choice of $\bar{e}$ and $\bar{e}'$, the disks in $f'(\bar{e})$ and $\bar{e}'$ are made disjoint in $\bar{d}$. Let

$$\begin{aligned} a^{(i)} &= a \cap d^{(i)}[-3,3], & a'^{(i)} &= a' \cap d^{(i)}[-3,3], \\ e^{(i)} &= e \cap d^{(i)}[-3,3], & e'^{(i)} &= e' \cap d^{(i)}[-3,3], \\ \bar{e}^{(i)} &= \bar{e} \cap \bar{d}^{(i)}, & \bar{e}'^{(i)} &= \bar{e}' \cap \bar{d}^{(i)}. \end{aligned}$$

Consider the connected component $(d^{(1)};\alpha^{(1)})$. Every point p' of the finite set $\Delta'^{(1)} = f'(\bar{\alpha}^{(1)} \cap (\bar{d}^{(1)} \setminus \bar{o}^{(1)}))$ is moved into an $e'^{(1)}$-part in the first meeting annuli in $a'^{(1)}$ by a slide of p' along $f'(\bar{d}^{(1)})$ avoiding the simple arcs in $d^{(1)}[-3,3] \cap f'(\bar{d}^{(1)})$ and the finite set $o[-3,3]) \cap f'(\bar{d}^{(1)})$. Then the set in the disk system $e'^{(1)}$ obtained from $\Delta'^{(1)}$ by these deformations is moved into $\bar{e}'^{(1)}$ vertically. Every point p of the finite set $\Delta^{(1)} = \bar{\alpha}^{(1)} \cap ((f')^{-1}(\bar{d}^{(1)}) \setminus \bar{\alpha}^{(1)} \cap \bar{d}^{(1)}$ is moved into an $e^{(1)}$-part in the first meeting annuli in $a^{(1)}$ by a slide of p along $(f')^{-1}(\bar{d}^{(1)})$ avoiding the simple arcs in $d^{(1)}[-3,3] \cap (f')^{-1}(\bar{d}^{(1)})$ and the finite set $o[-3,3]) \cap (f')^{-1}(\bar{d}^{(1)})$. Then the set in the disk system $e^{(1)}$ obtained from $\Delta^{(1)}$ by these deformations is moved into $\bar{e}^{(1)}$ vertically.

These deformations deform the chord systems $\bar{\alpha}$, $f'(\bar{\alpha}) = \bar{\alpha}'$ into chord systems $\tilde{\alpha}$, $f'(\tilde{\alpha}) = \tilde{\alpha}'$ in $\mathbf{R}^3[-3,3]$ (which may not be in $\mathbf{R}^3[1]$) on the based loop system $\bar{o}$. Then the homeomorphism f' is isotopically deformed so that

$$\tilde{\alpha}^{(1)} \cap \bar{d}^{(1)} = \tilde{\alpha}^{(1)} \cap (f')^{-1}(\bar{d}^{(1)}) \subset A^{(1)}$$

by letting $A^{(1)} = \bar{e}^{(1)} \cup (f')^{-1}(\bar{e}'^{(1)})$. It is noted from construction that

$$A^{(1)} \subset \bar{d}^{(1)} \cap (f')^{-1}(\bar{d}^{(1)}).$$

Because the chord systems $\tilde{\alpha}$ and $\tilde{\alpha}'$ meet $d^{(1)}[-3,3]$ only in $A^{(1)}$, the chord systems $\tilde{\alpha}$ and $\tilde{\alpha}'$ can be pushed into $\mathbf{R}^3[1]$ by the projection $\mathbf{R}^3[-3,3] \to \mathbf{R}^3[1]$ and slight modifications of $\tilde{\alpha}$, $\tilde{\alpha}'$ and f' to obtain new chord systems on $\bar{o}$ in $\mathbf{R}^3[1]$, denoted also by $\bar{\alpha}$ and $\bar{\alpha}'$, respectively which have an additional property that

$$\bar{\alpha}^{(1)} \cap \bar{d}^{(1)} = \bar{\alpha}^{(1)} \cap (f')^{-1}(\bar{d}^{(1)}) \subset A^{(1)} \subset \bar{d}^{(1)} \cap (f')^{-1}(\bar{d}^{(1)}).$$

Let $\bar{d}^{*(1)}$ be a disk system in $A^{(1)}$, and $\bar{\beta}^{*(1)}$ a simple chord system for the based loop system $\bar{o}^{*(1)} = \partial \bar{d}^{*(1)}$ in the disk system $\bar{d}^{(1)}$ such that

(1) $\bar{\alpha} \cap \bar{d}^{(1)} = \bar{\alpha} \cap \bar{d}^{*(1)}$,
(2) $\bar{d}^{*(1)} \cap \bar{\beta}^{*(1)} = \bar{o}^{*(1)} \cap \bar{\beta}^{*(1)} = \partial \bar{\beta}^{*(1)}$,

(3) there is a strong deformation retraction $\bar{d}^{(1)} \to \bar{d}^{*(1)} \cup \bar{\beta}^{*(1)}$.

By the move M_1, a chord diagram $C(\bar{o};\bar{\alpha})$ is deformed into a chord diagram of a chord graph $(\bar{o}^{*(1)} \cup (\bar{o} \setminus \bar{o}^{(1)}); \bar{\alpha} \cup \bar{\beta}^{*(1)})$ in a non-standard shape, which is made in a standard shape by the chord slide move $M_{1,1}$ on the simple chord system $\bar{\beta}^{*(1)}$.

Let $d^{**(1)} = f'(\bar{d}^{*(1)})$ be a disk system in $f'(A^{(1)}) \subset \bar{d}^{(1)}$. Let $\bar{\beta}^{**(1)}$ be a simple chord system for the based loop system $\bar{o}^{**(1)} = \partial \bar{d}^{**(1)}$ in $\bar{d}^{(1)}$ such that

(1)$'$ $\bar{\alpha}' \cap \bar{d}^{(1)} = \bar{\alpha}' \cap \bar{d}^{**(1)})$,
(2)$'$ $\bar{d}^{**(1)} \cap \bar{\beta}^{**(1)} = \bar{o}^{**(1)} \cap \bar{\beta}^{**(1)} = \partial \bar{\beta}^{**(1)}$,
(3)$'$ there is a strong deformation retraction $\bar{d}^{(1)} \to \bar{d}^{**(1)} \cup \bar{\beta}^{**(1)}$.

By the move M_1, a chord diagram $C(\bar{o};\bar{\alpha}')$ is also deformed into a chord diagram of a chord graph $(\bar{o}^{**(1)} \cup (\bar{o} \setminus \bar{o}^{(1)}); \bar{\alpha}' \cup \bar{\beta}^{**(1)})$ in a non-standard shape, which is made in a standard shape with the identical based loop system by the chord slide move $M_{1,1}$ on the simple chord system $\bar{\beta}^{**(1)}$.

Let $\bar{f}'(\bar{\beta}^{*(1)})$ be the projection of $f'(\bar{\beta}^{*(1)})(\subset \mathbf{R}^3[-3,3])$ into $\mathbf{R}^3[1]$ which is a chord system on $\bar{o}^{**(1)}$ not meeting the interior of the disk system $\bar{d}^{**(1)}$ by construction.

Using the properties (1), (2), (1)$'$, (2)$'$, one can deform a chord diagram of a chord graph $(\bar{o}^{**(1)} \cup (\bar{o} \setminus \bar{o}^{(1)}); \bar{\alpha}' \cup \bar{f}'(\bar{\beta}^{*(1)})$ in a standard shape into a chord diagram of a chord graph $(\bar{o}^{**(1)} \cup (\bar{o} \setminus \bar{o}^{(1)}); \bar{\alpha}' \cup \bar{\beta}^{**(1)})$ in a standard shape by the moves M_1, M_2 and M_3 without changing the intersection data on $\bar{\alpha}'$ and $\bar{d}^{**(1)}$ (although the chord system $\bar{\alpha}'$ may be deformed homotopically). By this deformation, the homeomorphism f' is isotopically deformed to send the intersection data on $\bar{\alpha}$ and $\bar{d}^{*(1)}$ to the intersection data on $\bar{\alpha}$ and $\bar{d}^{**(1)}$ identically and the chord graph $(\bar{o}^{*(1)} \cup (\bar{o} \setminus \bar{o}^{(1)}); \bar{\alpha} \cup \bar{\beta}^{*(1)})$ to the chord graph $(\bar{o}^{**(1)} \cup (\bar{o} \setminus \bar{o}^{(1)}); \bar{\alpha}' \cup \bar{\beta}^{**(1)})$.

Next, apply the same argument to the connected component $(d^{(2)}; \alpha^{(2)})$ in the chord graph $(\bar{o}^{*(1)} \cup (\bar{o} \setminus \bar{o}^{(1)}); \bar{\alpha} \cup \bar{\beta}^{*(1)})$. Since we have

$$d^{*(1)}[-3,3] \cap (f')^{-1}(\bar{d}^{(2)}) = \emptyset \quad \text{and} \quad d^{**(1)}[-3,3] \cap f'(\bar{d}^{(2)}) = \emptyset$$

by choosing smaller disk systems $d^{*(1)}$ and $d^{**(1)}$, the homeomorphism f' is isotopically deformed to send the intersection data on $\bar{\alpha}$ and $\bar{d}^{*(1)} \cup \bar{d}^{*(2)}$ to the intersection data on $\bar{\alpha}'$ and $\bar{d}^{**(1)} \cup \bar{d}^{**(2)}$, and the chord graph $(\bar{o}^{*(1)} \cup \bar{o}^{*(2)} \cup (\bar{o} \setminus \bar{o}^{(1)} \cup \bar{o}^{(2)}); \bar{\alpha} \cup \bar{\beta}^{*(1)} \cup \bar{\beta}^{*(2)})$ to the chord graph $(\bar{o}^{*(1)} \cup \bar{o}^{*(2)} \cup (\bar{o} \setminus \bar{o}^{(1)} \cup \bar{o}^{(2)}); \bar{\alpha}' \cup \bar{\beta}^{**(1)} \cup \bar{\beta}^{**(2)})$.

By continuing this process, the homeomorphism f' is isotopically deformed to send the intersection data on $\bar{\alpha}$ and $\bar{d}^{*(1)} \cup \bar{d}^{*(2)} \cup \cdots \cup \bar{d}^{*(r)}$ to the intersection data on $\bar{\alpha}'$ and $\bar{d}^{**(1)} \cup \bar{d}^{**(2)} \cup \cdots \cup \bar{d}^{**(r)}$, and the chord graph

$$(\bar{o}^{*(1)} \cup \bar{o}^{*(2)} \cup \cdots \cup \bar{o}^{*(r)}; \bar{\alpha} \cup \bar{\beta}^{*(1)} \cup \bar{\beta}^{*(2)} \cup \cdots \cup \bar{\beta}^{*(r)})$$

to the chord graph

$$(\bar{o}^{*(1)} \cup \bar{o}^{*(2)} \cup \cdots \cup \bar{o}^{*(r)}; \bar{\alpha}' \cup \bar{\beta}^{**(1)} \cup \bar{\beta}^{**(2)} \cup \cdots \cup \bar{\beta}^{**(r)}).$$

Thus, Observation 2 is obtained and the proof of Theorem is completed. □

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References

[K] A. Kawauchi, A chord diagram of a ribbon surface-link, Journal of Knot Theory
and Its Ramifications 24 (2015), 1540002 (24 pages).

[Y] T. Yasuda, Ribbon 2-knots with distinct ribbon types, J. Knot Theory Ramifications, 18 (2009), 1509-1523.