# On transforming a spatial graph into a plane graph

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This article is a revised detailed version of the research announcement Ref. 9) introducing a complexity of a spatial graph, which is useful to transform a spatial graph (without degree one vertices) into a plane graph. We also introduce related topological invariants for every spatial graph, called the *warping degree*, and $\gamma$-*warping degree* and $(\gamma, \Gamma)$-*warping degree*. We also generalize the usual unknotting number of a knot to every spatial graph and introduce related topological invariants for every spatial graph, called the $\gamma$-*unknotting number*, $\Gamma$-*unknotting number* and $(\gamma, \Gamma)$-*unknotting number*. These invariants are used to define "semi-topological" invariants for a spatial graph with degree one vertices, meaningful even for a knotted arc.

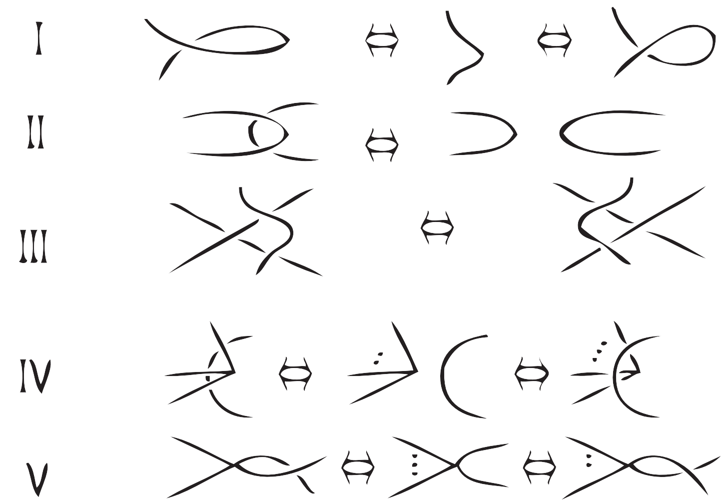


Fig. 1. Generalized Reidemeister moves

## §1. Introduction

In a research of proteins, molecules, or polymers, it is important to understand geometrically and topologically spatial graphs possibly with degree one vertices including knotted arcs. In this article, we introduce some numerical invariants generalizing the warping degree and the unknotting number of knots and links to a spatial graph (without degree 0 vertices) by defining first the invariants to a spatial graph without degree one vertices and then modifying the invariants into some semi-topological invariants of a spatial graph with degree one vertices.

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We begin with some basic terminologies on spatial graphs. Let $\Gamma$ be a finite graph which does not have any vertices of degrees 0 and 1. A *spatial graph* of $\Gamma$ is the image $G$ of a topological embedding $\Gamma \to \mathbf{R}^3$ such that there is an orientation-preserving homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ sending $G$ to a polygonal graph in $\mathbf{R}^3$. We consider a spatial graph $G$ by ignoring the degree two vertices which are useless in our topological argument. Let $\Gamma_i (i = 1, 2, \ldots, r)$ be an ordered set of the connected components of $\Gamma$, and $G_i = G(\Gamma_i)$ the corresponding spatial subgraph of $G = G(\Gamma)$. The spatial graph $G$ is called a *link* if $\Gamma$ is the disjoint union of finitely many loops, and it is *trivial* if it is the boundary of mutually disjoint disks. A *knot* is a link with one component. For a general reference of knots, links and spatial graphs, we refer Ref. 7). Let $T_i$ be a maximal tree of $G_i$. By definition, $T_i = \emptyset$ if $G_i$ is a knot, and $T_i$ is one vertex if $G_i$ has just one vertex of degree $\geqq 3$. The union $T = \cup_{i=1}^{r} T_i$ is called a *basis* of $G$, and the pair $(G, T)$ a *based* spatial graph. A spatial graph $G$ is *equivalent* to a spatial graph $G'$ if there is an orientation-preserving homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ such that $h(G) = G'$. Let $[G]$ be the class of spatial graphs $G'$ which are equivalent to $G$. It is well-known that two spatial graphs $G$ and $G'$ are equivalent if and only if any diagram $D_G$ of $G$ is deformed into any diagram $D_{G'}$ of $G'$ by a finite sequence of the *generalized Reidemeister moves* (see Fig. 1), where only the moves I-III are needed for links which we call the *Reidemeister moves*. Let $[D_G]$ be the class of diagrams obtained from a diagram $D_G$ of $G$ by the generalized Reidemeister moves, which is identified with the class $[G]$. In §2, we explain the concepts of

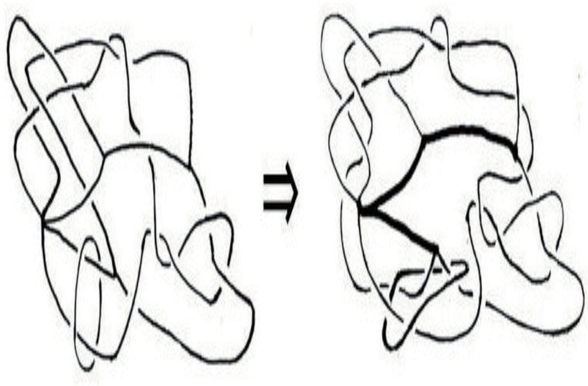

Fig. 2. Deforming a diagram into a based diagram

a monotone diagram, the warping degree and the complexity for a spatial graph without degree one vertices. In §3, some concepts of the unknotting number for a spatial graph without degree one vertices are explained. A relationship between the warping degree and the unknotting number together with related invariants is also investigated there. In §4, we explain how these invariants are applied to a spatial graph with degree one vertices.

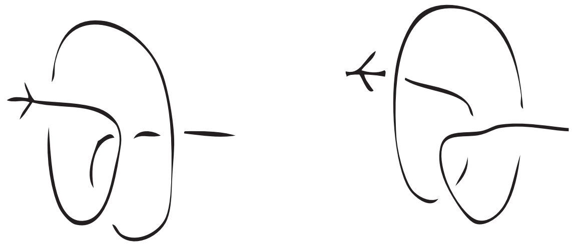

Fig. 3. Monotone edge diagrams

## §2. Monotone diagram, warping degree and complexity

Our spatial graph $G$ is obtained from a basis $T$ by adding *edges* (consisting of arcs or loops) $\alpha_k$ $(k = 1, 2, \cdots, m)$. Let $D$ be a diagram of $G$. Let $DT$ and $D\alpha_k$ be the subdiagrams of $D$ corresponding to the basis $T$ and the edge $\alpha_k$, respectively. The diagram $D$ is a *based diagram* (on a basis $T$) and denoted by $(D; T)$ if there are no crossing points of $D$ belonging to $DT$. We can deform every diagram into a based diagram by a finite sequence of the generalized Reidemeister moves (see Figure 2). An edge diagram $D\alpha_k$ is *monotone* if there is an orientation on $\alpha_k$ such that a point going along the oriented diagram $D\alpha_k$ from the origin vertex meets first the upper crossing point at every crossing point (see Figure 3), where we take a suitable non-crossing point as a starting point if $\alpha_k$ is a loop. A sequence of the edges $\alpha_k$ $(k = 1, 2, \cdots, m)$ for a based spatial graph $(G, T)$ is *regularly ordered* if any edge belonging to the connected based graph $(G_i, T_i)$ is ordered to be smaller than any edge belonging to the connected based graph $(G_j, T_j)$ for every $i < j$. A based diagram $(D; T)$ of a based spatial graph $(G, T)$ is *monotone* if there is a regularly ordered edge sequence $\alpha_k$ $(k = 1, 2, \cdots, m)$ of $(G, T)$ such that the edge diagram $D\alpha_k$ is monotone for all $k$ and the edge diagram $D\alpha_k$ is upper than the edge diagram $D\alpha_{k'}$ for every $k < k'$. The *warping degree* $d(D; T)$ of a based diagram $(D; T)$ is the least number of crossing changes on the edge diagrams $D\alpha_i (i = 1, 2, \cdots, m)$ needed to obtain a monotone diagram from $(D; T)$ (see Figure 4). For $T = \emptyset$, we denote $d(D; T)$ by $d(D)$. When the edges $\alpha_i$ $(i = 1, 2, \cdots, m)$ are previously oriented, we can also define the *oriented warping degree* $\vec{d}(D; T)$ (or $\vec{d}(D)$ for $T = \emptyset$) of $D$ by considering only the crossing changes on the edge or loop diagrams $D\alpha_i$ $(i = 1, 2, \cdots, m)$ along the specified orientations. Similar notions on links have been discussed by W. B. R. Lickorish and K. C. Millett,[10] S. Fujimura,[4] T. S. Fung,[5] M. Okuda[13] and M. Ozawa[14] considering the ascending number of an oriented link. A. Shimizu[16),17] also established an equality between the warping degrees and the crossing number of a link diagram. In particular, A. Shimizu characterized the alternating knot diagrams by establishing the inequality $\vec{d}(D) + \vec{d}(-D) \leqq c(D) - 1$ for every knot diagram $D$ with crossing number $c(D) > 0$ where the equality holds if and only if $D$ is an alternating diagram. The *warping degree* $d(G)$ of $G$ is the minimum of the warping degrees $d(D; T)$ for all based diagrams $(D; T) \in [D_G]$. The *complexity* of a based diagram $(D, T)$ is the pair $cd(D; T) = (c(D; T), d(D; T))$ together with the dictionary order. This notion was introduced in Ref. 8) for an oriented ordered link diagram. A. Shimizu also observed that the dictionary order on $cd(D;T)$ is equivalent to the numerical order on $c(D;T)^2 + d(D;T)$ by using the inequality $d(D;T) \leqq c(D;T)$. The *complexity* $\gamma(G) = (c_\gamma(G), d_\gamma(G))$ of $G$ is the minimum (in the dictionary order) of the complexities $cd(D;T)$ for all based diagrams $(D;T) \in [D_G]$, where the topological invariants $c_\gamma(G)$ and $d_\gamma(G)$ are called the *$\gamma$-crossing number* and the *$\gamma$-warping degree* of $G$, respectively. The *crossing number* of $G$ is given by $c(G) = \min_{D\in[D_G]} c(D)$ which has the inequality $c(G) \leqq c_\gamma(G)$. The following properties (1) and (2) on $G$ give a reason why we call $\gamma(G)$ the complexity of $G$:

(1) *If $d_\gamma(G) > 0$, then there is a crossing change on any based diagram $(D,T)$ of $G$ with $cd(D;T) = \gamma(G)$ to obtain a spatial graph $G'$ with $\gamma(G') < \gamma(G)$* (see Fig. 4). We have *$d_\gamma(G) = 0$ if and only if the spatial graph $G$ is equivalent to $G'$ with a monotone diagram $(D';T')$ with $c(D';T') = c_\gamma(G)$.*

(2) *If $c_\gamma(G) > 0$, then there is a spatial graph $G'$ with $c_\gamma(G') < c_\gamma(G)$, so that* $\gamma(G') < \gamma(G)$, by any splice on any based diagram $(D,T)$ *of $G$ with* $cd(D;T) = \gamma(G)$ (see Fig. 4). *The crossing number $c_\gamma(G) = 0$ if and only if $c(G) = 0$, i.e., $G$ is equivalent to a graph in a plane.*

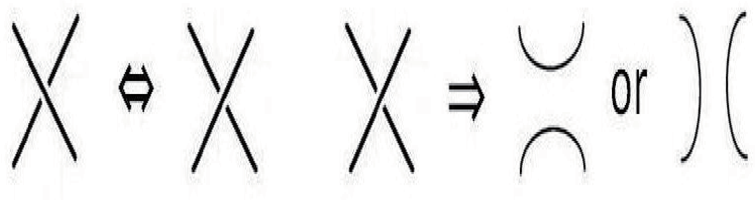


Fig. 4. A crossing change in the left-hand side and a splice in the right-hand side

## §3. Unknotting number

A spatial graph $G$ is *unknotted* if $d(G) = 0$ (see Figure 5). This notion is related to some notions by T. Endo and T. Otsuki,[3)] R. Shinjo[18)] and M. Ozawa and Y. Tsutsumi,[14)] which the referee of this article suggested kindly. A spatial graph $G$ is *$\gamma$-unknotted* if $d_\gamma(G) = 0$. Let $\gamma(\Gamma) = (c_\gamma(\Gamma), d_\gamma(\Gamma))$ be the minimum of $\gamma(G)$ for all spatial graphs $G$ of a graph $\Gamma$. Then we have $d_\gamma(\Gamma)) = 0$. We called $\gamma(\Gamma)$ and $c_\gamma(\Gamma)$ the *complexity* and *crossing number* of the graph $\Gamma$, respectively. A spatial graph $G$ is *$\Gamma$-unknotted* if $G$ is a spatial graph of $\Gamma$ with $\gamma(G) = \gamma(\Gamma)$. By definitions, we have the following relations:

$$\text{“}\Gamma\text{-unknotted”} \Rightarrow \text{“}\gamma\text{-unknotted”} \Rightarrow \text{“unknotted”}.$$

A link $L$ is unknotted if and only if $L$ is a trivial link, and a spatial graph $G$ of a plane graph $\Gamma$ is $\Gamma$-unknotted if and only if $G$ is equivalent to a graph in a

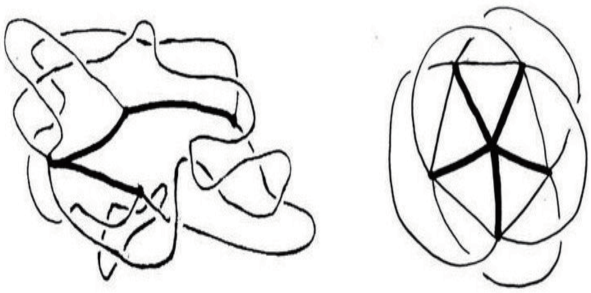

Fig. 5. Unknotted spatial graphs

plane. A *constituent link* of a spatial graph $G$ is a link contained in $G$. Every loop component in an unknotted graph $G$ is seen to be a trivial knot which can be splitted from the other connected components of $G$. The spatial graph in Figure 6 is an unknotted but non-$\Gamma$-unknotted graph since it is a plane graph with a constituent Hopf link. In spite of the Conway-Gordon Theorem in Ref. 2) stating that every spatial 6-complete graph $K_6$ contains a non-trivial constituent link and every spatial 7-complete graph $K_7$ contains a non-trivial constituent knot, we have the following lemma for unknotted graphs:

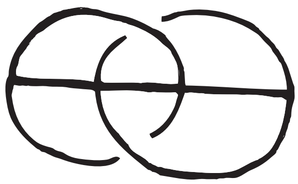

Fig. 6. An unknotted plane graph with a Hopf constituent link

**Lemma 3.1.** Let $\Gamma$ be a graph without loop components.
(1) For every graph $\Gamma$, there are only finitely many unknotted graphs $G$ of $\Gamma$ up to equivalences.
(2) Every unknotted graph $G = G(\Gamma)$ is a split union of unknotted connected graph components $G_i (i = 1, 2, \ldots, r)$ of $G$ with $G_i$ equivalent to a trivial bouquet of circles after a basis of $G_i$ is shrunk to a vertex for every $i$.
(3) By a sequence of edge reductions, illustrated in Figure 7, every unknotted graph $G$ is deformed into a basis of $G$.

*Proof of Lemma 3.1.* We note that there are only finitely many choices of bases $T$ of $\Gamma$. For every basis $T$ of $\Gamma$, there are finitely many choices of regularly ordered edge sequences of $(\Gamma, T)$. For every basis $T$ put on the $xy$-plane $\mathbf{R}^2$ in $\mathbf{R}^3$, there is an embedding of every regularly ordered edge sequence of $(\Gamma, T)$ into the upper half 3-space $\mathbf{R}^3_+$ such that the induced based spatial graph $(G, T)$ meets $\mathbf{R}^2$ only with $T$ and has a monotone diagram with the basis $T$. Every unknotted based graph $(G, T)$

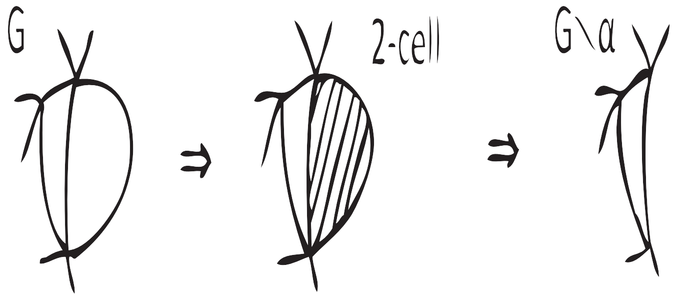


Fig. 7. An edge reduction

is realized in this way up to equivalences, and unique up to equivalences for a given regularly ordered edge sequence of $(\Gamma, T)$. By this explanation, we have (1), (2) and (3), where for (3) we can delete the edges in the order of the regularly ordered edge sequence by using edge reductions, completing the proof.

As a corollary of (2), we see that *every edge of an unknotted based graph* $(G, T)$ *is contained in a trivial constituent knot.* Let $O$ be the set of unknotted graphs of $\Gamma$, and $\mathcal{D}_O$ the set of diagrams of the spatial graphs in $O$. Let $O_\Gamma$ be the set of $\Gamma$-unknotted spatial graphs, and $\mathcal{D}_{O_\Gamma}$ the set of diagrams of the spatial graphs in $O_\Gamma$. For a spatial graph $G$, let $O^G_\gamma$ be the set of unknotted graphs with monotone diagrams obtained by $d_\gamma(G)$ times crossing changes from all the based diagrams $(D;T) \in [D_G]$ with $cd(D;T) = \gamma(G)$. The sets $O^G_\gamma$ and $O_\Gamma$ may be disjoint in general. For example, for the spatial graph $G$ in Figure 6, the set $O^G_\gamma$ consists of only $G$ with $\gamma(G) = (2, 0)$ by noting that $G$ has a Hopf link as a constituent link. Since the abstract graph $\Gamma$ of $G$ is a planar graph, the set $O_\Gamma$ consists of only a graph $G_0$ in a plane with $\gamma(G_0) = (0, 0)$. Thus, $O^G_\gamma \cap O_\Gamma = \emptyset$. This example motivates us to define the number

$$d^\Gamma_\gamma(G) = d_\gamma(G) + \rho(O^G_\gamma, O_\Gamma),$$

called the $(\gamma, \Gamma)$-*warping degree* of $G$, where $\rho(,)$ denotes the $x$-distance (i.e., Gordian distance) function on the spatial graphs of $\Gamma$. The *unknotting number* $u(G)$ and the $\Gamma$-*unknotting number* $u^\Gamma(G)$ of $G$ are respectively defined by the identities:

$$u(G) = \rho(G, O) \text{ and } u^\Gamma(G) = \rho(G, O_\Gamma).$$

Let $[D_G]_\gamma = \{(D;T) \in [D_G] \mid c(D;T) = c_\gamma(G)\}$. By regarding $\rho$ as the diagrammatical Gordian distance function, the $\gamma$-*unknotting number* $u_\gamma(G)$ and $(\gamma, \Gamma)$-*unknotting number* $u^\Gamma_\gamma(G)$ are defined as follows:

$$u_\gamma(G) = \rho([D_G]_\gamma, \mathcal{D}_O) \text{ and } u^\Gamma_\gamma(G) = \rho([D_G]_\gamma, \mathcal{D}_{O_\Gamma}).$$

We have the following theorem:

**Theorem 3.2.** The topological invariants $u(G)$, $u^\Gamma(G)$, $u^\Gamma_\gamma(G)$, $d(G)$, $u_\gamma(G)$, $d_\gamma(G)$ and $d^\Gamma_\gamma(G)$ satisfy the following inequalities and are distinct topological invariants:

$$\begin{array}{ccccccc} u(G) & \leqq & u^\Gamma(G) & \leqq & u^\Gamma_\gamma(G) & \leqq & d^\Gamma_\gamma(G) \\ & & & & \vee\!\| & & \vee\!\| \\ & & & & u_\gamma(G) & \leqq & d_\gamma(G) \\ & & & & \vee\!\| & & \vee\!\| \\ & & & & u(G) & \leqq & d(G) \end{array}$$

Further, for every graph $\Gamma$ and any integer $n \geqq 0$, there is a spatial graph $G$ of $\Gamma$ such that

$$u(G) = u^\Gamma(G) = u^\Gamma_\gamma(G) = d(G) = u_\gamma(G) = d_\gamma(G) = d^\Gamma_\gamma(G) = n.$$

Fig. 8. Kinoshita's $\theta$-curve

*Proof.* The inequalities are easily obtained by the definitions. We show that these invariants are mutually distinct. The spatial graph $G$ in Figure 6 is directly seen to have $u(G) = d(G) = u_\gamma(G) = d_\gamma(G) = 0$ and $u^\Gamma(G) = u^\Gamma_\gamma(G) = d^\Gamma_\gamma(G) = 1$. We take $\Gamma = S^1$ to consider a knot $K$. For the knot $K = 5_2$ which is a twist knot, we have $u(K) = u^\Gamma_\gamma(K) = d(K) = 1$ because T. S. Fung[5)] and M. Ozawa[14)] showed that a knot $K$ with $d(K) = 1$ is characterized by a twist knot. We also see $d_\gamma(K) = d^\Gamma_\gamma(K) = 2$ by A. Shimizu[16)] since $\rho(O^K_\gamma, O_\Gamma) = 0$. For the knot $K = 6_2$, we have $u(K) = u^\Gamma_\gamma(K) = 1$, but $d(K) = d^\Gamma_\gamma(K) = 2$ since $K$ is not any twist knot. For $K = 10_8$, we have $u(K) = u^\Gamma(K) = 2 < u_\gamma(K) = u^\Gamma_\gamma(K) = 3$ by a result of S. A. Bleiler[1)] and Y. Nakanishi.[12)] The *Kinoshita $\theta$-curve* $G$ in Figure 8 has $c_\gamma(G) = 7$ because every tangle diagram obtained from $G$ by removing an open ball neighborhood of every edge has at least 7 crossings by the classification of algebraic tangles with crossing numbers $\leqq 6$ given by H. Moriuchi Ref. 11). Then we see that $u(G) = u^\Gamma_\gamma(G) = 1$. If $G$ is a $\theta$-curve with $d(G) = 1$, then we can show that the three constituent knots of $G$ consist of two trivial knots and one non-trivial twist knot. Thus, we have $d(G) = d^\Gamma_\gamma(G) = 2$. Taking a $\Gamma$-unknotted graph $G_0$, we have $d^\Gamma_\gamma(G_0) = 0$ (implying all the invariants vanish). These results are sufficient to know that the invariants $u$, $u^\Gamma$, $u^\Gamma_\gamma$, $d$, $u_\gamma$, $d_\gamma$, $d^\Gamma_\gamma$ are mutually distinct. To obtain the latter half, we use an argument of an infinite cyclic covering of a spatial graph $G$. By a *basic choice* of $G$, we mean a choice $\xi$ of a maximal tree $T$ and orientations of the remaining edges $\alpha_i$ $(i = 1, 2, \dots, s)$ in $G$. The number $s$ is called the *rank* of $G$. By the choice of $\xi$, we have a unique epimorphism from the group $\pi = \pi_1(\mathbf{R}^3 \backslash G)$

onto $Z$ sending the meridian of every $\alpha_i$ to 1, whose abelianized kernel $\pi'/[\pi', \pi']$ denoted by $M(G, \xi)$ forms a finitely generated $\Lambda$-module for the Laurent polynomial ring $\Lambda = Z[t, t^{-1}]$. Let $m(G, \xi)$ be the minimal number of $\Lambda$-generators of $M(G, \xi)$. Let $K$ be an oriented knot with unkonotting number $u(K) = 1$ and a non-trivial Alexander polynomial $A(t)$. Let $G_n$ be a connected sum $G_0 \# nK$ of a $\Gamma$-unknotted graph $G_0$ with a basic choice $\xi_0$ and $n$ copies of $K$ along an edge of $G_0$. Using that the fundamental group $\pi = \pi_1(\mathbf{R}^3 \backslash G_0)$ is a free group with a basis consisting of the meridians $m_i$ $(i = 1, 2, \ldots, s)$ by Lemma 3.1 (2), we see that the $\Lambda$-module $M(G_n, \xi_n)$ with the basic choice $\xi_n$ inherited from $\xi_0$ is the direct sum $\Lambda^{s-1} \oplus (\Lambda/(A(t))^n$, so that we have $m(G_n, \xi_n) = s - 1 + n$ for the rank $s$ of $G_0$. The following lemma is proved by the method given in Ref. 6) where a stronger result has been shown for links.

**Lemma 3.3.** Let $G$ be a spatial graph of a graph $\Gamma$ with a basic choice $\xi$, and $G'$ a spatial graph obtained from $G$ by $k$ crossing changes on the oriented edges. Let $\xi'$ be the basic choice in $G'$ inherited from $\xi$. Then we have $k \geqq |m(G, \xi) - m(G', \xi')|$.

Assume that an unknotted graph $G'$ is obtained from $G_n$ by $k$ crossing changes on the oriented edges. Let $\xi'$ be the basic choice in $G'$ inherited from $\xi_n$. Although a basic choice $\xi''$ for the unknotted graph $G'$ does not always coincide with $\xi'$, we see from Lemma 3.1 (2) that the fundamental group $\pi = \pi_1(\mathbf{R}^3 \backslash G')$ is a free group of rank $s$, so that we can show $m(G', \xi') = s - 1$. Thus, we have $k \geqq n$ and $u(G_n) \geqq n$ by Lemma 3.3. Since $G_0$ is obtained from $G_n$ by $n$ crossing changes on the oriented edges, we have $d^{\Gamma}_{\gamma}(G) \leqq n$, so that $u(G_n) = d^{\Gamma}_{\gamma}(G) = n$. By the inequalities already established, the proof of Theorem 3.2 is completed.

## §4. A spatial graph with degree one vertices

In this section, we consider a spatial graph $G$ of a finite graph $\Gamma$ with the set $V$ of the degree one vertices $v_1, v_2, \ldots, v_m (m \geqq 1)$. We study the knottedness of a spatial graph $G$ with degree one vertices by considering the knotting problem on spatial graphs without degree one verticies associated with $G$. For this purpose, we need a concept of a normal graph. A spatial graph $G$ with degree one vertices is *normal* if we have the following properties (1) and (2):

(1) There is a set $X = \{x_1, x_2, \ldots, x_m\}$ of mutually distinct $m$ points in $G \backslash V$ such that the line segments $|v_i x_i| (i = 1, 2, \ldots, m)$ are mutually disjoint and intersect $G$ only in the set $V \cup X$. (We call the set $X$ a *coupling with* $V$.)

(2) There are only finitely many equivalence classes of the spatial graphs

$$G_X = G \cup_{i=1}^{m} |v_i x_i|$$

for all couplings $X$ with $V$.

We note that every polygonal spatial graph $G$ with degree one vertices which is not in a plane is normal and that if $G$ is a normal graph in a plane $\subset \mathbf{R}^3$, then the spatial graph $G_X$ is always a $\Gamma$-unknotted graph for every coupling $X$ with $V$. For every normal graph $G$, the *unknotting number* $u(G_X)$ of the spatial graph $G_X$ is defined as it is done in §3. An analysis on the dynamics of the invariant $u(G_X)$ for every coupling $X$ with $V$ will be useful in studying a knotted structure of the normal spatial graph $G$. The *unknotting number* $u(G)$ of a normal graph $G$ is defined to be

$$u(G) = \max\{u(G_X)|\, X \text{ is a coupling with } V\}.$$

Let $N(G)$ be the number of distinct equivalence classes on the spatial graphs $G_X$ for all couplings $X$ with $V$, and $n(G)$ the number of distinct equivalence classes of spatial graphs $G_X$ with $u(G_X) > 0$ for all couplings $X$ with $V$. The *knotting probability* of a normal graph $G$ is defined by the fraction

$$p(G) = \frac{n(G)}{N(G)}.$$

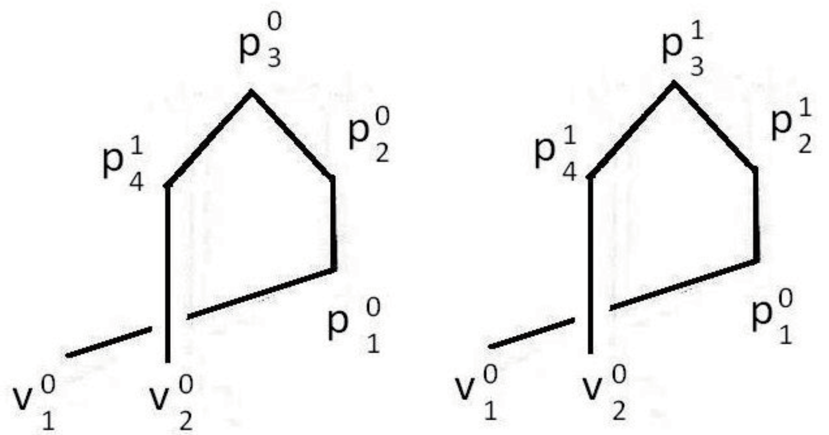


Fig. 9. Normal arcs

The graph $G$ is also called a $(p(G) \times 100)\%$-*knotted graph*. For example, we consider the polygonal arcs $G^{\varepsilon}$ $(\varepsilon = 0, 1)$ consisting of the ordered vertices $v_1^0$, $p_1^0$, $p_2^{\varepsilon}$, $p_3^{\varepsilon}$, $p_4^1$, $v_2^0$ where

$$v_1^0 = (0,0,0),\ p_1^0 = (3,1,0),\ p_2^{\varepsilon} = (3,2,\varepsilon),\ p_3^{\varepsilon} = (2,3,\varepsilon),\ p_4^1 = (1,2,1),\ v_2^0 = (1,0,0),$$

which are illustrated in Figure 9. It turns out that whereas the spatial graph $G_X^0$ is always unknotted, the spatial graph $G_X^1$ is classified into three equivalence classes consisting of an unknotted handcuff graph, an unknotted $\theta$-curve, and a knotted handcuff graph of unknotting number one caused from the observation that the line segment $|v_1^0 x_1|$ taking $x_1$ in an open line segment $(p_2^1, p_3^1)$ or $(p_2^1, q)$ with the midpoint $q$ of the line segment $|p_1^0 p_2^1|$ meets at an interior point of the triangle $\Delta v_2^0 x_2 p_4^1$ taking $x_2 = p_3^1$. This check will be relatively easily made because for every normal arc $G$, the spatial graph $G_X$ for every coupling $X$ with $V$ is a spatial $\theta$-curve or a spatial handcuff graph, so that $G_X$ is unknotted if and only if $G_X$ is equivalent to a graph in a plane. Thus, we have the unknotting numbers $u(G^0) = 0$ and $u(G^1) = 1$ and the knotting probabilities $p(G^0) = 0$ and $p(G^1) = \frac{1}{3}$. In other words, the arc $G^0$ is a 0%-knotted arc and the arc $G^1$ is a $\frac{100}{3}$%-knotted arc with $u(G^1) = 1$. Detailed studies on the knotting probability of a spatial arc will be made elsewhere.

In similar ways, the *warping degree* $d(G_X)$, the $\gamma$*-warping degree* $d_\gamma(G_X)$, the $(\gamma, \Gamma)$*-warping degree* $d_\gamma^\Gamma(G_X)$, the $\gamma$*-unknotting number* $u_\gamma(G_X)$, the $\Gamma$*-unknotting number* $u^\Gamma(G_X)$, and the $(\gamma, \Gamma)$*-unknotting number* $u_\gamma^\Gamma(G_X)$ and their related notions are defined for every normal spatial graph $G$.

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