On 4-dimensional universe for every 3-dimensional manifold *

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Abstract

A boundary-less connected oriented 4-manifold is called a universe for every 3-manifold if every closed connected oriented 3-manifold is embedded in it, and a punctured universe if every punctured 3-manifold is embedded in it, which is known to be an open 4-manifold. We introduce types 1, 2 and full universes as refined notions of a universe and a punctured universe and investigate some relationships among them. After introducing some topological invariants for every (possibly non-compact) oriented 4-manifold which we call the topological indexes, we show infinity and independence on some topological indexes of every universe.

Keywords: Universe, Punctured universe , Topological index, 3-manifold, Punctured 3-manifold, Type 1 embedding, Type 2 embedding, Signature theorem

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1. Introduction

Throughout this paper, by a *closed 3-manifold* we mean a closed connected oriented 3-manifold and by a *punctured 3-manifold* a punctured manifold of a closed connected oriented 3-manifold. Then we know that for every compact oriented 4-manifold, there is a closed 3-manifold whose punctured 3-manifold is not embeddable in it (see [4][1]) and hence any oriented 4-manifold with every punctured 3-manifold embedded must be non-compact. This motivates us to put the following definition:

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[1]The non-orientable version is also known in [11], but we do not discuss it here. Also, by an embedding we will mean a smooth or piecewise-linear embedding.

Definition. A *universe* is an open connected oriented 4-manifold U with every closed 3-manifold M embedded. A *punctured universe* is an open connected oriented 4-manifold U with every punctured 3-manifold M^0 embedded.

Then we ask a question: *What topological shapes a universe and a punctured universe have ?*

In this question, we introduce the following topological indexes

$$\hat{\beta}_d(Y)(d=1,2),\ \delta(Y),\ \delta_i(Y)\,(i=0,1,2),\ \rho(Y),\ \rho_i(Y)\,(i=0,1,2)$$

of every (possibly, non-compact) oriented 4-manifold Y, which are obtained from homological arguments and are topological invariants of Y with values taken in $\{0,1,2,\dots,+\infty\}$. We apply these invariants to a punctured universe, a universe and their refined universes, namely types 1, 2 and full universes to obtain our main result (Theorem 3.3) which is stated as follows:

For a punctures universe U, we show that one of the topological indexes $\hat{\beta}_2(U)$, $\delta_0(U)$, $\rho_0(U)$ is $+\infty$. Further, in every case, there is a punctured spin universe U with the other topological indexes taken 0.

For a type 1 universe U, we show that one of the topological indexes $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ is $+\infty$. We have always $\hat{\beta}_1(U) \geqq 1$, but in the case of $\rho_1(U)=+\infty$, we can add the condition that $\hat{\beta}_1(U)=+\infty$. Further, in every case, there is a type 1 spin universe U with the other topological indexes on $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ taken 0.

For a type 2 universe U, we show that one of the topological indexes $\hat{\beta}_2(U)$, $\delta_2(U)$ is $+\infty$. Further, in every case, there is a type 2 spin universe U with the other topological index taken 0.

For a universe U, we show that one of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$, $\rho(U)$ is $+\infty$. In the case of $\rho(U)=+\infty$, we can add the condition that $\hat{\beta}_1(U)=+\infty$. Further, in every case, there is a spin universe U with the other topological indexes on $\hat{\beta}_2(U)$, $\delta(U)$ and $\rho(U)$ taken 0.

For a full universe U, we show that one of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$ is $+\infty$. We have always $\hat{\beta}_1(U) \geqq 1$. Further, in every case, there is a full spin universe U with the other topological index on $\hat{\beta}_2(U)$ and $\delta(U)$ taken 0.

In Section 2, we introduce types 1, 2 and full universes as refined notions of a universe and a punctured universe. We explain some relationships among them in Theorem 1.1. In Section 3, the topological indexes of every oriented 4-manifold are defined and our main result (Theorem 3.3) is stated. The existence part of universes in our main result (Theorem 3.3) is shown in this section with some examples. In

Section 4, we establish a non-compact 4-manifold version of the signature theorem for an infinite cyclic covering of a compact oriented manifold given in [3], which is needed to prove the infinity of some topological indexes stated in Theorem 3.3. In Section 5, we introduce a notion of a loose embedding needed as a tool connecting an embedding argument with an argument of an infinite cyclic covering. In Section 6, we complete the proof of Theorem 3.3.

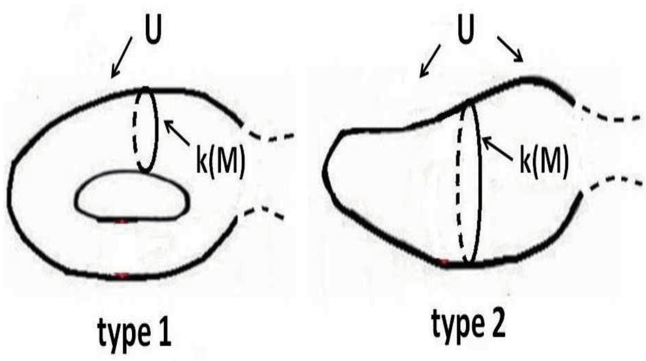

Figure 1: Universes of types 1 and 2

2. Types 1, 2 and full universes as refined notions of a universe and a punctured universe

Let $\mathbb{M}$ be the set of closed 3-manifolds M, and $\mathbb{M}^0$ the set of punctured 3-manifolds M^0. It is useful to denote the members of $\mathbb{M}$ and $\mathbb{M}^0$ by $M_i\,(i = 1, 2, 3, \dots)$ and $M_i^0\,(i = 1, 2, 3, \dots)$, respectively. For a connected open oriented 4-manifold U, we note that there are two types of embeddings $k : M \to U$. An embedding $k : M \to U$ is *of type 1* if $U\backslash k(M)$ is connected, and *of type 2* if $U\backslash k(M)$ is disconnected (see Fig. 1). If there is a type 1 embedding $k : M \to U$, then there is an element $x \in H_1(U; Z)$ with the intersection number $\mathrm{Int}_U(x, k(M)) = +1$, so that the intersection form $\mathrm{Int}_U : H_1(U; Z) \times H_3(U; Z) \to Z$ induces an epimorphism

$$I_d : H_d(U; Z) \to Z$$

for $d = 1, 3$ such that the composite $I_3 k_* : H_3(M; Z) \to H_3(U; Z) \to Z$ is an isomorphism and the composite $I_1 k_* : H_1(M; Z) \to H_1(U; Z) \to Z$ is the 0-map. This suggests that we must consider more refined universes as follows:

Definition. The universe U is a *type 1 universe* if every closed 3-manifold is type 1 embeddable in U, a *type 2 universe* if every M is type 2 embedded in U, and a *full universe* if U is a type 1 universe and a type 2 universe.

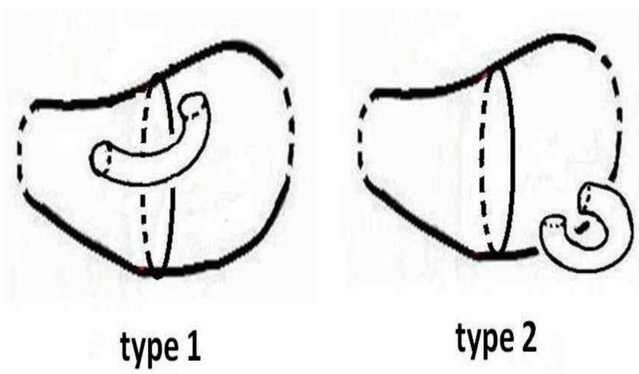

Figure 2: Creating a full universe from a type 2 universe

Our central problem is to characterize the topological shapes of punctured, types 1, 2 and full universes. We note that a full universe is obtained from a type 2 universe by taking a connected sum with $S^1 \times S^3$ (see Fig. 2). We first establish the following theorem.

Theorem 2.1. The following assertions hold.

$$
(1)\quad \text{Full universe} \;\begin{array}{c} \text{Type 1 universe} \\ \nearrow \qquad\qquad \searrow \\ \\ \searrow \qquad\qquad \nearrow \\ \text{Type 2 universe} \end{array}\; \text{Universe} \to \text{Punctured universe.}
$$

(2) Type 1 universe $\not\to$ Full universe.

(3) Type 2 universe $\not\to$ Full universe.

(4) Universe $\not\to$ Type 1 universe.

(5) Universe $\not\to$ Type 2 universe.

(6) Punctured universe $\not\to$ Universe.

Proof. (1) is obvious by definition. To see (3) and (4), we note that the stable 4-space $SR^4 = R^4 \#_{i=1}^{+\infty} S^2 \times S_i^2$ considered in [5] is a type 2 spin universe because every closed 3-manifold M bounds a simply connected spin 4-manifold whose double is the connected sum of some copies of $S^2 \times S^2$. Since $H_1(SR^4; Z) = 0$, we see that any closed 3-manifold cannot be type 1 embedded in SR^4, showing (3) and (4). To see (2) and (5), we consider a type 1 spin universe

$$U_{SP} = R^4 \#_{i=1}^{+\infty} M_i \times S^1$$

which we call the S^1-*product universe*. We use a notion of a *linking form*, namely a non-singular symmetric bilinear form $\ell : G \times G \to Q/Z$ on a finite abelian group G. The linking form ℓ is *split* if ℓ is hyperbolic, i.e., G is a direct sum $H' \oplus H''$ with $\ell(H', H') = \ell(H'', H'') = 0$ or ℓ is the orthogonal sum of a linking form $\ell_H : H \times H \to Q/Z$ and its inverse $-\ell_H : H \times H \to Q/Z$. Then we have the following lemma:

Lemma 2.2. If a closed 3-manifold M with $H_1(M; Z)$ a finite abelian group is type 2 embeddable in the product universe U_{SP}, then the linking form

$$\ell : H_1(M; Z) \times H_1(M; Z) \to Q/Z$$

is split.

Before proving Lemma 2.2, the proof of Theorem 2.1 will be completed by using Lemma 2.2. In fact, the lens space $L(p, q)$ with $p \neq 0, \pm 1$ is not type 2 embeddable in U_{SP} by Lemma 2.2, showing (2) and (5). To see (6), for $I = [0, 1]$ we consider a punctured spin universe

$$U_{IP} = R^4 \#_{i=1}^{+\infty} \mathrm{int}(M_i^0 \times I),$$

which we call the *I-product punctured universe.* Suppose that there is an embedding $k : M \to U_{IP}$ for a closed 3-manifold $M \in \mathbb{M}$. We note that every element of $H_1(U_{IP}; Z)$ is represented by the sum of 1-cycles in $\mathrm{int}(M_i^0 \times I)$ for a finite number of i which can be moved to be disjoint from $k(M)$. This means that the intersection number $\mathrm{Int}(M, H_1(U_{IP}; Z)) = 0$, showing that the embedding k is not of type 1 and hence k must be of type 2. Regarding $I \subset S^1$, we can consider $U_{IP} \subset U_{SP}$. Then the composite embedding $M \xrightarrow{k} U_{IP} \subset U_{SP}$ is still of type 2, because the boundary $\partial(M_i^0 \times I)$ is connected. Thus, if $H_1(M; Z)$ is a finite abelian group, then the linking form $\ell : H_1(M; Z) \times H_1(M; Z) \to Q/Z$ splits by Lemma 2.2. Thus, the lens space $L(p, q)$ with $p \neq 0, \pm 1$ is not embeddable in U_{IP}, implying that U_{IP} is not any universe, showing (6). This completes the proof of Theorem 2.1 except the proof of Lemma 2.2. □

The proof of Lemma 2.2 is given as follows:

Proof of Lemma 2.2. By an S^1-*semi-product* 4-manifold, we mean a 4-manifold which is the connected sum of S^1-products $M_i \times S^1$ $(i = 1, 2, \ldots, m)$ for some m. Assume that M is type 2 embedded in U_{SP}. Then M is type 2 embedded in an S^1-semi-product 4-manifold. We show the following assertion:

(2.2.1) If $H_1(M; Z)$ is a finite abelian group, then M is type 2 embedded in an S^1-semi-product 4-manifold X consisting of the connected summands $M_i \times S^1$ ($i = 1, 2, \ldots, m$) such that there is a point $p_i \in S^1$ with $M_i \times p_i \cap M = \emptyset$ for every i.

Proof of (2.2.1). We see that M is embedded in an S^1-semi-product 4-manifold $X_U = \#_{j=1}^{s} M_j \times S^1$. The n-fold cyclic covering $M_1 \times S^1 \to M_1 \times S^1$ associated with the n-fold cyclic covering $S^1 \to S^1$ induces an n-fold cyclic covering $X^{(1)} \to X_U$ such that $X^{(1)}$ is an S^1-semi-product 4-manifold consisting of the connected summand $M_1 \times S^1$ and the trivial lifts of the other connected summands $M_i \times S^1$ ($j = 2, \ldots, s$). Since $H_1(M; Z)$ is finite, the manifold M is also trivially lifted to $X^{(1)}$. We note that M is type 2 embedded in X_U if and only if $\mathrm{Int}_{X_U}(M, H_1(X_U; Z)) = 0$. Since $H_1(X^{(1)}; Z)$ is generated by loops which are lifts of loops in X_U, we see that any trivial lift M' of M to $X^{(1)}$ has $\mathrm{Int}_{X^{(1)}}(M', H_1(X^{(1)}; Z)) = 0$ and hence is type 2 embedded. Taking n sufficiently large, we have $M_1 \times p_1 \cap M = \emptyset$ for a point $p_1 \in S^1$. Applying the same arguments inductively to $M_i \times S^1$ ($i = 2, 3, \ldots, s$), we obtain the conclusion of (2.2.1). □

By (2.2.1), for $I = [0, 1]$ we may consider that M is type 2 embedded in the connected sum

$$Y = M_1 \times I \# M_2 \times I \# \ldots \# M_m \times I,$$

so that M splits Y into two compact 4-manifolds A and B whose boundaries ∂A and ∂B have the form

$$\partial A = M \cup \partial_A Y, \quad \partial B = (-M) \cup \partial_B Y,$$

where

$$\begin{aligned} \partial_A Y &= M_1 \times \partial I \cup M_2 \times \partial I \cup \cdots \cup M_s \times \partial I, \\ \partial_B Y &= M_{s+1} \times \partial I \cup M_{s+2} \times \partial I \cup \cdots \cup M_m \times \partial I. \end{aligned}$$

We show the following assertion:

(2.2.2) The following natural sequence

$$(\#) \quad 0 \to \mathrm{tor} H_2(A, M \cup \partial_A Y; Z) \stackrel{\partial_*}{\to} \mathrm{tor} H_1(M \cup \partial_A Y; Z) \stackrel{i_*}{\to} \mathrm{tor} H_1(A; Z) \to 0$$

on the homology torsion parts is a split exact sequence.

By (2.2.2), the linking form

$$\ell^+ : \mathrm{tor} H_1(M \cup \partial_A Y; Z) \times \mathrm{tor} H_1(M \cup \partial_A Y; Z) \to Q/Z$$

is split. This is because $\mathrm{im}\partial_*$ is a direct summand of $\mathrm{tor}H_1(M \cup \partial_A Y; Z)$ and $(\mathrm{im}\partial_*)^\perp = \mathrm{im}\partial_*$ with respect to ℓ^+. Since the linking form

$$\ell_0 : \mathrm{tor}H_1(\partial_A Y; Z) \times \mathrm{tor}H_1(\partial_A Y; Z) \to Q/Z$$

is split and the linking form ℓ^+ is an orthogonal sum of the linking forms ℓ and ℓ_0, we see from [9] that the linking form $\ell : H_1(M; Z) \times H_1(M; Z) \to Q/Z$ is split. This completes the proof of Lemma 2.2 except the proof of (2.2.2).

Proof of (2.2.2). Let

$$\begin{aligned}\partial_A^0 Y &= M_1 \times 0 \cup M_2 \times 0 \cup \cdots \cup M_s \times 0,\\ \partial_B^0 Y &= M_{s+1} \times 0 \cup M_{s+2} \times 0 \cup \cdots \cup M_m \times 0.\end{aligned}$$

Further, let $\partial^0 Y = \partial_A^0 Y \cup \partial_B^0 Y$. Since $H_2(M; Z) = H_2(Y, \partial^0 Y; Z) = 0$, the Mayer-Vietoris exact sequence

$$H_2(M; Z) \to H_2(A, \partial_A^0 Y; Z) \oplus H_2(B, \partial_B^0 Y; Z) \to H_2(Y, \partial^0 Y; Z)$$

implies that

$$H_2(A, \partial_A^0 Y; Z) = H_2(B, \partial_B^0 Y; Z) = 0.$$

Since $H_1(M \cup \partial_A^0 Y, \partial_A^0 Y; Z) = H_1(M; Z)$ is finite and $H_2(A, \partial_A^0 Y; Z) = 0$, we see from the exact sequence

$$H_2(A, \partial_A^0 Y; Z) \to H_2(A, M \cup \partial_A^0 Y; Z) \to H_1(M \cup \partial_A^0 Y, \partial_A^0 Y; Z)$$

that $H_2(A, M \cup \partial_A^0 Y; Z)$ is finite. Because j_* passes through the finite abelian group $H_2(A, M \cup \partial_A^0 Y; Z)$, we see that the image of the homomorphism $j_* : H_2(A; Z) \to H_2(A, M \cup \partial_A Y; Z)$ is finite. Thus, the semi-exact sequence

$$\mathrm{tor}H_2(A, M \cup \partial_A Y; Z) \xrightarrow{\partial_*} \mathrm{tor}H_1(M \cup \partial_A Y; Z) \xrightarrow{i_*} \mathrm{tor}H_1(A; Z)$$

is exact. We construct a monomorphism

$$\pi : \mathrm{tor}H_1(A; Z) \to \mathrm{tor}H_1(M \cup \partial_A Y; Z)$$

with the identity

$$i_*\pi = 1 : \mathrm{tor}H_1(A; Z) \xrightarrow{\pi} \mathrm{tor}H_1(M \cup \partial_A Y; Z) \xrightarrow{i_*} \mathrm{tor}H_1(A; Z).$$

Then we see that the sequence (#) is a split exact sequence, because i_* is onto and ∂_* is injective, for $\partial_* : \mathrm{tor}H_2(A, M \cup \partial_A Y; Z) \to \mathrm{tor}H_1(M \cup \partial_A Y; Z)$ is Poincaré dual to the epimorphism $i_* : \mathrm{tor}H_1(M \cup \partial_A Y; Z) \to \mathrm{tor}H_1(A; Z)$.

To construct a monomorphism π, we note that the Mayer-Vietoris exact sequence

$$0 = H_2(Y, \partial^0 Y; Z) \to H_1(M; Z) \to H_1(A, \partial^0_A Y; Z) \oplus H_1(B, \partial^0_B Y; Z) \\ \to H_1(Y, \partial^0 Y; Z) = Z^{m-1}$$

induces a natural isomorphism

$$j^A_* + j^B_* : H_1(M; Z) \cong \mathrm{tor} H_1(A, \partial^0_A Y; Z) \oplus \mathrm{tor} H_1(B, \partial^0_B Y; Z).$$

Then we can construct a monomorphism

$$\pi^A : \mathrm{tor} H_1(A, \partial^0_A Y; Z) \to H_1(M; Z)$$

so that

$$j^A_* \pi^A = 1 : \mathrm{tor} H_1(A, \partial^0_A Y; Z) \xrightarrow{\pi^A} H_1(M; Z) \xrightarrow{j^A_*} \mathrm{tor} H_1(A, \partial^0_A Y; Z).$$

Since j^A_* passes through the natural homomorphisms $j^M_* : H_1(M; Z) \to \mathrm{tor} H_1(A; Z)$ and $j^0_* : \mathrm{tor} H_1(A; Z) \to \mathrm{tor} H_1(A, \partial^0_A Y; Z)$, we have

$$j^0_*(j^M_* \pi^A) = j^A_* \pi^A = 1.$$

Using that $H_2(A, \partial^0_A Y; Z) = 0$ and j^0_* is onto, we see that the sequence

$$0 \to \mathrm{tor} H_1(\partial^0_A Y; Z) \xrightarrow{i^0_*} \mathrm{tor} H_1(A; Z) \xrightarrow{j^0_*} \mathrm{tor} H_1(A, \partial^0_A Y; Z) \to 0$$

obtained from the homology sequence of the pair $(A, \partial^0_A Y)$ is a split exact sequence, by which we can define a homomorphism

$$\pi^0 : \mathrm{tor} H_1(A; Z) \to \mathrm{tor} H_1(\partial^0_A Y; Z)$$

with the identity $i^0_* \pi^0 = 1 - j^M_* \pi^A j^0_*$. We define the homomorphism

$$\pi' = \pi^A j^0_* + \pi^0 : \mathrm{tor} H_1(A; Z) \to H_1(M; Z) \oplus \mathrm{tor} H_1(\partial^0_A Y; Z).$$

This homomorphism

$$j^M_* + i^0_* : H_1(M; Z) \oplus \mathrm{tor} H_1(\partial^0_A Y; Z) \to \mathrm{tor} H_1(A; Z)$$

has the identity

$$(j^M_* + i^0_*)\pi' = 1 : \mathrm{tor} H_1(A; Z) \to \mathrm{tor} H_1(A; Z).$$

In fact, we have

$$\begin{aligned} (j^M_* + i^0_*)\pi'(x) &= j^M_* \pi^A j^0_*(x) + i^0_* \pi^0(x) \\ &= j^M_* \pi^A j^0_*(x) + x - j^M_* \pi^A j^0_*(x) = x \end{aligned}$$

for all $x \in \mathrm{tor}H_1(A;Z)$. The direct sum $H_1(M;Z) \oplus \mathrm{tor}H_1(\partial_A^0 Y;Z)$ is identified with the homology $\mathrm{tor}H_1(M \cup \partial_A^0 Y;Z)$. Then we can extend the homomorphisms $j_*^M + i_*^0$ and π' to the natural homomorphism $i_* : \mathrm{tor}H_1(M \cup \partial_A Y;Z) \to \mathrm{tor}H_1(A;Z)$ and a homomorphism $\pi : \mathrm{tor}H_1(A;Z) \to H_1(M \cup \partial_A Y;Z)$ with $i_*\pi = 1$ where the value of π on the direct summand $\mathrm{tor}H_1(\partial_A Y \backslash \partial_A^0 Y;Z)$ of $\mathrm{tor}H_1(M \cup \partial_A Y;Z)$ is taken 0. Thus, we have a desired monomorphism π, showing (2.2.2). □

This completes the proof of Lemma 2.2.

3. Homology of a universe and a punctured universe

Let Y be an orientable possibly non-compact 4-manifold. For the intersection form

$$\mathrm{Int} : H_d(Y;Z) \times H_{4-d}(Y;Z) \to Z,$$

we define the *dth null homology* of Y to be the subgroup

$$O_d(Y;Z) = \{x \in H_d(Y;Z) |\, \mathrm{Int}(x, H_{4-d}(Y;Z)) = 0\}$$

of the dth homology group $H_d(Y;Z)$ and the *dth non-degenerate homology* of Y to be the quotient group

$$\hat{H}_d(Y;Z) = H_d(Y;Z)/O_d(Y;Z).$$

We have the following lemma:

Lemma 3.1. $\hat{H}_d(Y;Z)$ is a free abelian group.

Proof. We first note that the induced intersection form

$$\mathrm{Int} : \hat{H}_d(Y;Z) \times \hat{H}_{4-d}(Y;Z) \to Z$$

is non-degenerate and $\hat{H}_*(Y;Z)$ is a torsion-free abelian group. Thus, if Y is compact, then $\hat{H}_*(Y;Z)$ is a free abelian group. Assume that Y is non-compact. Let

$$Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \dots$$

be an ascending sequence of compact 4-submanifolds Y_n of Y with $\cup_{n=1}^{+\infty} Y_n = Y$. We find elements $x_i^Q \in H_d(Y_1;Q)$ $(i = 1, 2, \dots, m_1)$ representing a Q-basis for $\hat{H}_d(Y_1;Z) \otimes Q$ and elements $y_i^Q \in H_{4-d}(Y_1;Q)$ representing a Q-basis for $\hat{H}_{4-d}(Y_n;Z) \otimes Q$ $(i = 1, 2, \dots, m_1)$ with the Q-intersection numbers $\mathrm{Int}_Q(x_i^Q, y_j^Q) = \delta_{i,j}$ for all i, j. Then the elements x_i^Q $(i = 1, 2, \dots, m_1)$ and y_i^Q $(i = 1, 2, \dots, m_1)$ are regarded as linearly independent elements of $H_d(Y_2;Q)$ and $H_{4-d}(Y_2;Q)$, respectively. Taking the orthogonal complements of the Q-subspaces generated by these elements with respect to the Q-intersection form $\mathrm{Int}_Q : H_d(Y_2;Q) \times H_{4-d}(Y_2;Q) \to Q$, we can add new members $x_i^Q \in H_d(Y_2;Q)$ $(i = m_1+1, m_1+2, \dots, m_2)$ and $y_i^Q \in H_{4-d}(Y_2;Q)$ $(i = m_1+1, m_1+2, \dots, m_2)$ with $\mathrm{Int}_Q(x_i^Q, y_j^Q) = \delta_{i,j}$ for all i,j to form Q-bases for $\hat{H}_d(Y_2;Z) \otimes Q$ and $\hat{H}_{4-d}(Y_2;Z)$. By continuing this process, we have elements $x_i \in H_d(Y;Z)$ $(i = 1,2,3,\dots)$ forming a Q-basis for $\hat{H}_d(Y;Z) \otimes Q$ and $y_i \in H_{4-d}(Y;Z)$ $(i = 1,2,3,\dots)$ forming a Q-basis for $\hat{H}_{4-d}(Y;Z) \otimes Q$ with $\mathrm{Int}(x_i, y_i) \neq 0$ and $\mathrm{Int}(x_i, y_j) = 0$ for all i,j with $i \neq j$. Let Z^* be the free abelian subgroup of $\hat{H}_{4-d}(Y;Z)$ such that y_i $(i = 1,2,3,\dots)$ form a basis, and $\hom^f(Z^*, Z)$ the free subgroup of $\hom(Z^*, Z)$ consisting of homomorphisms $f : Z^* \to Z$ taking the value 0 except a finite number of y_i $(i = 1,2,3,\dots)$. Then, since the intersection form $\mathrm{Int} : \hat{H}_d(Y;Z) \times \hat{H}_{4-d}(Y;Z) \to Z$ induces a monomorphism $\hat{H}_d(Y;Z) \to \hom^f(Z^*, Z)$, we see that $\hat{H}_d(Y;Z)$ is a free abelian group. □

The Z-rank $\hat{\beta}_d(Y)$ of $\hat{H}_d(Y;Z)$ is our first topological index of Y. The proof of Lemma 3.1 also implies the following corollary.

Corollary 3.2. For an ascending sequence $Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \dots$ of compact 4-submanifolds Y_n for a non-compact oriented 4-manifold Y with $\cup_{n=1}^{+\infty} Y_n = Y$, we have

$$\hat{\beta}_d(Y_n) \leqq \hat{\beta}_d(Y_{n+1})\,(n = 1,2,3,\dots) \quad \text{and} \quad \lim_{n \to +\infty} \hat{\beta}_d(Y_n) = \hat{\beta}_d(Y).$$

For an abelian group G, let $G^{(2)} = \{x \in G \,|\, 2x = 0\}$, which is a direct sum of some copies of Z_2. For $M^0 \in \mathbb{M}^0$, let $\delta(M^0 \subset Y)$ be the minimal Z-rank of the image $\mathrm{im}[k_*^0 : H_2(M^0;Z) \to H_2(Y;Z)]$, and $\rho(M^0 \subset Y)$ the minimal Z_2-rank of $\mathrm{im}[k_*^0 : H_2(M^0;Z) \to H_2(Y;Z)]^{(2)}$, for all embeddings $k^0 : M^0 \to Y$. By taking the value 0 for the non-embeddable case, we define the following topological invariants of Y:

$$\begin{aligned} \delta_0(Y) &= \sup\{\delta(M^0 \subset Y) \,|\, M^0 \in \mathbb{M}^0\}, \\ \rho_0(Y) &= \sup\{\rho(M^0 \subset Y) \,|\, M^0 \in \mathbb{M}^0\}. \end{aligned}$$

Let $\delta(M \subset Y)$ be the minimal Z-rank of the image $\mathrm{im}[k_* : H_2(M;Z) \to H_2(Y;Z)]$, and $\rho(M \subset Y)$ the minimal Z_2-rank of $\mathrm{im}[k_* : H_2(M;Z) \to H_2(Y;Z)]^{(2)}$, for all embeddings $k : M \to Y$. By taking the value 0 for the non-embeddable case, we define the following invariants of Y:

$$\begin{aligned} \delta(Y) &= \sup\{\delta(M \subset Y) \,|\, M \in \mathbb{M}\}, \\ \rho(Y) &= \sup\{\rho(M \subset Y) \,|\, M \in \mathbb{M}\}. \end{aligned}$$

Restricting all embeddings $k : M \to Y$ to all embeddings $k : M \to Y$ of type i for $i = 1,2$, we obtain the topological indexes $\delta_i(Y)$ and $\rho_i(Y)$ $(i = 1,2)$ of Y in place of $\delta(Y)$ and $\rho(Y)$.

Our main result concerns a behavior on the topological indexes for a punctured universe and a universe, and their refined universes, namely types 1, 2 and full universes, which are stated as follows:

Theorem 3.3.
(1) If U is a punctured universe, then one of the following cases (1.1)-(1.3) holds.

(1.1) $\hat{\beta}_2(U) = +\infty$.
(1.2) $\delta_0(U) = +\infty$.
(1.3) $\rho_0(U) = +\infty$.

Further, in every case, there is a punctured spin universe U with the other topological indexes taken 0.

(2) If U is a type 1 universe, then one of the following cases (3.1)-(3.3) holds.

(3.1) $\hat{\beta}_2(U) = +\infty$ and $\hat{\beta}_1(U) \geqq 1$.
(3.2) $\delta_1(U) = +\infty$ and $\hat{\beta}_1(U) \geqq 1$.
(3.3) $\rho_1(U) = +\infty$ and $\hat{\beta}_1(U) = +\infty$.

Further, in every case, there is a type 1 spin universe U with the other topological indexes on $\hat{\beta}_2(U)$, $\delta_1(U)$ and $\rho_1(U)$ taken 0.

(3) If U is a type 2 universe, then one of the following cases (3.1) and (3.2) holds.

(3.1) $\hat{\beta}_2(U) = +\infty$.
(3.2) $\delta_2(U) = +\infty$.

Further, in every case, there is a type 2 spin universe U with the other topological index taken 0.

(4) If U is a universe, then one of the following cases (4.1)-(4.3) holds.

(4.1) $\hat{\beta}_2(U) = +\infty$.
(4.2) $\delta(U) = +\infty$.
(4.3) $\rho(U) = +\infty$ and $\hat{\beta}_1(U) = +\infty$.

Further, in every case, there is a spin universe U with the other topological indexes on $\hat{\beta}_2(U)$, $\delta(U)$ and $\rho(U)$ taken 0.

(5) If U is a full universe, then one of the following cases (5.1) and (5.2) holds.

(5.1) $\hat{\beta}_2(U) = +\infty$ and $\hat{\beta}_1(U) \geqq 1$.
(5.2) $\delta(U) = +\infty$ and $\hat{\beta}_1(U) \geqq 1$.

Further, in every case, there is a full spin universe U with the other topological index on $\hat{\beta}_2(U)$ and $\delta(U)$ taken 0.

In the following Examples 3.4-3.6, we give some examples on a punctured universe, a universe, and their refined universes, namely types 1, 2 and full universes, which are sufficient to see the existence assertions on (1)-(5) of Theorem 3.3.

Example 3.4. The stable 4-space $SR^4 = R^4 \#_{i=1}^{+\infty} S^2 \times S^2_i$ has the following property:

(3.4.1) For every $M \in \mathbb{M}$, there is a type 2 embedding $k : M \to SR^4$ inducing the trivial homomorphism $k_* = 0 : H_2(M; Z) \to H_2(SR^4; Z)$.

Thus, $U = SR^4$ is a punctured and type 2 spin universe with $\hat{\beta}_2(U) = +\infty$, $\hat{\beta}_1(U) = 0$, $\delta_0(U) = \delta_2(U) = 0$ and $\rho_0(U) = \rho_2(U) = 0$. Further, $U_S = S^1 \times S^3 \# SR^4$ is a punctured, type 1, type 2, full spin universe with

$$\begin{aligned}
&\hat{\beta}_2(U_S) = +\infty,\ \hat{\beta}_1(U_S) = 1,\\
&\delta_0(U_S) = \delta_1(U_S) = \delta_2(U_S) = \delta(U_S) = 0,\\
&\rho_0(U_S) = \rho_1(U_S) = \rho_2(U_S) = \rho(U_S) = 0.
\end{aligned}$$

Proof of (3.4.1). Let W be a simply connected spin 4-manifold with $\partial W = M$ whose double DW is homeomorphic to the connected sum X of some copies of $S^2 \times S^2$. Since the natural homomorphism $i_* : H_2(M; Z) \to H_2(W; Z)$ is injective, we can represent a basis of the image of i_* by mutually disjoint 2-spheres S_i $(i = 1, 2, \dots, m)$ in W which we can find in the factors $S^2 \times p$'s of the connected summands $S^2 \times S^2$'s of an $S^2 \times S^2$-decomposition of DW, if necessary, by taking connected sums with some copies of $S^2 \times S^2$. By the surgeries of DW on S_i $(i = 1, 2, \dots, m)$, we obtain the connected sum X' of some copies of $S^2 \times S^2$ such that the inclusion $M \to X'$ induces the zero map $H_2(M; Z) \to H_2(X'; Z)$. Since the stable 4-space SR^4 is constructed from a punctured manifold of X', we have (3.4.1). □

Example 3.5. For $I = [0, 1]$, let W_i be a spin 4-manifold obtained from $M_i \times I$ by attaching 2-handles on $M_i \times 1$ along a basis for $H_1(M_i \times 1; Z)/(\text{torsions})$ to obtain that $H_1(W_i; Z)$ is a torsion abelian group. Then the natural homomorphism $H_2(M_i \times I; Z) \to H_2(W_i; Z)$ is an isomorphism, so that $H_2(W_i; Z)$ is a free abelian group. Let $W_i (i = 1, 2, \dots)$ be the 4-manifolds corresponding to the 3-manifolds $M_i \in \mathbb{M}$ $(i = 1, 2, \dots)$. We construct the open 4-manifolds

$$U_T = R^4 \#_{i=1}^{+\infty} \operatorname{int} W_i \quad \text{and} \quad U_{ST} = S^1 \times S^3 \# U_T.$$

The open 4-manifold U_T is a punctured and type 2 spin universe with

$$\begin{aligned}
&\hat{\beta}_2(U_T) = \hat{\beta}_1(U_T) = 0,\\
&\delta_0(U_T) = \delta_2(U_T) = +\infty,\\
&\rho_0(U_T) = \rho_2(U_T) = 0.
\end{aligned}$$

The open 4-manifold U_{ST} is a punctured, type 1, type 2 and full spin universe with

$$\hat{\beta}_2(U_{ST}) = 0,\ \hat{\beta}_1(U_{ST}) = 1,$$
$$\delta_0(U_{ST}) = \delta_1(U_{ST}) = \delta_2(U_{ST}) = \delta(U_{ST}) = +\infty,$$
$$\rho_0(U_{ST}) = \rho_1(U_{ST}) = \rho_2(U_{ST}) = \rho(U_{ST}) = 0.$$

Example 3.6. Let $Z_{/2} = Z[\frac{1}{2}]$ be a subring of Q. The *4-dimensional solid torus with three meridian disks* is a spin 4-manifold $D(T^3)$ with boundary the 3-dimensional torus T^3 which is obtained from the 4-disk D^4 by attaching the three 0-framed 2-handles along the Borromean rings L_B (see [8, 10]). For $s \geqq 2$, let $D(sT^3)$ be the disk sum of s copies of $D(T^3)$. Then the boundary $\partial D(sT^3)$ is the connected sum $\# sT^3$of s copies of T^3. For $s = 0$, we understand $D(sT^3) = S^4$ and $\# sT^3 = \emptyset$. Let

$$\mathbf{\Sigma} = S^1 \times S^3 \# D(sT^3) \quad \text{and} \quad \hat{\mathbf{\Sigma}} = S^4 \# D(sT^3) = D(sT^3).$$

A *Samsara 4-manifold* on $M \in \mathbb{M}$ is a compact oriented spin 4-manifold Σ with $\partial \Sigma = \# sT^3$ and with $Z_{/2}$-homology of $\mathbf{\Sigma}$ for some $s \geqq 0$ such that there is a type 1 embedding $k : M \to \Sigma$ inducing the trivial homomorphism

$$k_* = 0 : H_2(M; Z_{/2}) \to H_2(\Sigma; Z_{/2}).$$

We also call $\mathbf{\Sigma}$ the *standard Samsara 4-manifold* on S^3. In [8], we showed that there is a Samsara 4-manifold Σ_i on every $M_i \in \mathbb{M}$ $(i = 1, 2, 3, \dots)$. Let R^4_+ be the upper-half 4-space with boundary the 3-space R^3. Let

$$\Sigma R^4_+ = R^4_+ \natural_{i=1}^{+\infty} \Sigma_i$$

be the 4-manifold obtained from R^4_+ by making the connected sums with the closed Σ_i's and the disk sums with the bounded Σ_i's. We call the open 4-manifold $U_{SM} = \mathrm{int}(\Sigma R^4_+)$ a *Samsara universe*, which is a punctured and type 1 spin universe with

$$\hat{\beta}_2(U_{SM}) = 0,\ \hat{\beta}_1(U_{SM}) = +\infty,$$
$$\delta_0(U_{SM}) = \delta_1(U_{SM}) = 0,$$
$$\rho_0(U_{SM}) = \rho_1(U_{SM}) = +\infty.$$

Let $\mathbf{\Sigma} R^4_+$ be the 4-manifold obtained from R^4_+ by making the connected sums with countably many copies of $S^1 \times S^3$ and the disk sums with countably many copies of $D(T^3)$, and

$$\mathbf{\Sigma} R^4 = \mathrm{int}(\mathbf{\Sigma} R^4_+).$$

Every Samsara universe U_{SM} has the same $Z_{/2}$-homology as $\mathbf{\Sigma} R^4$. By Theorem 3.3, we can see that any Samsara universe U_{SM} is not any type 2 universe. A *reduced Samsara 4-manifold* on $M^0 \in \mathbb{M}^0$ is a compact oriented spin 4-manifold $\hat{\Sigma}$ with $\partial \hat{\Sigma} = \# sT^3$

and with $Z_{/2}$-homology of $\hat{\mathbf{\Sigma}}$ for some $s \geqq 0$ such that there is a punctured embedding $k^0 : M^0 \to \hat{\Sigma}$ inducing the trivial homomorphism

$$k^0_* = 0 : H_2(M^0; Z_{/2}) \to H_2(\hat{\Sigma}; Z_{/2}).$$

We also call $\hat{\mathbf{\Sigma}}$ the *standard reduced Samsara 4-manifold* on the punctured 3-sphere $(S^3)^0$. We obtain a reduced Samsara 4-manifold $\hat{\Sigma}_i$ on M^0_i from a Samsara 4-manifold Σ_i on M_i by a surgery of Σ_i killing a generator of $\hat{H}_1(\Sigma_i; Z) = Z$, and conversely we obtain a Samsara 4-manifold Σ_i on M_i from a reduced Samsara 4-manifold $\hat{\Sigma}_i$ on M^0_i by the surgery of $\hat{\Sigma}_i$ along the 2-knot $S^2_i = \partial M^0_i$ (see [8]).

Let

$$\hat{\Sigma} R^4_+ = R^4_+ \natural_{i=1}^{+\infty} \hat{\Sigma}_i$$

be the 4-manifold obtained from R^4_+ by making the connected sums with the closed Σ_i's and the disk sums with the bounded Σ_i's. We call the open 4-manifold

$$U_{RS} = \mathrm{int}(\hat{\Sigma} R^4_+)$$

a *reduced Samsara universe*, which is a punctured spin universe with the following topological indexes

$$\begin{aligned}
&\hat{\beta}_2(U_{RS}) = \hat{\beta}_1(U_{RS}) = 0,\\
&\delta_0(U_{RS}) = 0,\\
&\rho_0(U_{RS}) = +\infty.
\end{aligned}$$

Let $\hat{\mathbf{\Sigma}} R^4_+$ be the 4-manifold obtained from R^4_+ by making the disk sums with countably many copies of $D(T^3)$, and

$$\hat{\mathbf{\Sigma}} R^4 = \mathrm{int}(\hat{\mathbf{\Sigma}} R^4_+).$$

Every reduced Samsara universe U_{RS} has the same $Z_{/2}$-homology as $\hat{\mathbf{\Sigma}} R^4$. By [8, (3.1.4.1)], we can show that if a closed 3-manifold M with $H_1(M; Z)$ a finite abelian group is embedded in U_{RS}, then the linking form $\ell_p : H_1(M; Z)_p \times H_1(M; Z)_p \to Q/Z$ restricted to the p-primary component $H_1(M; Z)_p$ of $H_1(M; Z)$ for every odd prime p is hyperbolic. Thus, U_{RS} is not any universe. Further, from [8, 3.1(4)], we can see that $\mathbf{\Sigma} R^4$ and $\hat{\mathbf{\Sigma}} R^4$ are not any punctured universe.

4. A non-compact version of the signature theorem for an infinite cyclic covering

We need a non-compact 4-manifold version of the signature theorem in [3] to prove Theorem 3.3 which is explained in this section.

Let Y be a non-compact oriented 4-manifold with boundary a closed 3-manifold B. Assume that $\hat{\beta}_2(Y) < +\infty$. We say that a homomorphism $\gamma : H_1(Y; Z) \to Z$

is *end-trivial* if there is a compact submanifold Y' of Y such that the restriction $\gamma|_{\mathrm{cl}(Y\backslash Y')} : H_1(Y\backslash Y'; Z) \to Z$ is the zero map. For any end-trivial homomorphism $\gamma : H_1(Y; Z) \to Z$, we take the infinite cyclic covering $(\tilde{Y}, \tilde{B})$ of (Y, B) associated with γ. Then $H_2(\tilde{Y}; Q)$ is a (possibly, infinitely generated) Γ-module for the principal ideal domain $\Gamma = Q[t, t^{-1}]$ of Laurent polynomials with rational coefficients. Consider the Γ-intersection form

$$\mathrm{Int}_\Gamma : H_2(\tilde{Y}; Q) \times H_2(\tilde{Y}; Q) \to \Gamma$$

defined by $\mathrm{Int}_\Gamma(x, y) = \sum_{m=-\infty}^{+\infty} \mathrm{Int}(x, t^{-m}y)t^m$ for $x, y \in H_2(\tilde{Y}; Q)$. Then we have the identities:

$$\mathrm{Int}_\Gamma\left(\overline{f(t)}x, y\right) = \mathrm{Int}_\Gamma(x, f(t)y) = f(t)\mathrm{Int}_\Gamma(x, y), \quad \mathrm{Int}_\Gamma(y, x) = \overline{\mathrm{Int}_\Gamma(x, y)},$$

where $\overline{}$ denotes the involution of Γ sending t to t^{-1}. Let

$$O_2(\tilde{Y}; Q)_\Gamma = \{x \in H_2(\tilde{Y}; Q) | \mathrm{Int}_\Gamma(x, H_2(\tilde{Y}; Q)) = 0\}$$

and

$$\hat{H}_2(\tilde{Y}; Q)_\Gamma = H_2(\tilde{Y}; Q)/O_2(\tilde{Y}; Q)_\Gamma,$$

which is a torsion-free Γ-module. We show the following lemma:

Lemma 4.1. If $\hat{\beta}_2(Y) < +\infty$, then $\hat{H}_2(\tilde{Y}; Q)_\Gamma$ is a free Γ-module of finite rank.

Proof. We split Y by a compact 4-submanifold $Y' \supset B$ of Y and $Y'' = \mathrm{cl}(Y\backslash Y')$ such that$\hat{\beta}_2(Y') = \hat{\beta}_2(Y)$ and Y'' is trivially lifted to $\tilde{Y}$. Then we note that $\hat{H}_2(\tilde{Y}''; Q)_\Gamma = 0$. For $B_0 = Y' \cap Y''$, since $H_2(\tilde{Y}'; Q)$ and $H_1(\tilde{B}_0; Q)$ are finitely generated Γ-modules, the Mayer-Vietoris sequence

$$H_2(\tilde{Y}'; Q) \oplus H_2(\tilde{Y}''; Q) \to H_2(\tilde{Y}; Q) \to H_1(\tilde{B}_0; Q)$$

shows that $\hat{H}_2(\tilde{Y}; Q)_\Gamma$ is a finitely generated, torsion-free Γ-module, so that it is a free Γ-module of finite rank. □

Let $A(t)$ be a Γ-Hermitian matrix representing the Γ-intersection form Int_Γ on $\hat{H}_2(\tilde{Y}; Q)_\Gamma$. For $x \in (-1, 1)$ let $\omega_x = x + \sqrt{1 - x^2}i$, which is a complex number of norm one. For $a \in (-1, 1)$ we define the signature invariant of $\tilde{Y}$ by

$$\tau_{a\pm 0}(\tilde{Y}) = \lim_{x \to a\pm 0} \mathrm{sign}A(\omega_x).$$

The signature invariants $\sigma_a(\tilde{B})$ $(a \in [-1, 1])$ of $\tilde{B}$ are also defined in [1, 2] by the quadratic form

$$b : \mathrm{tor}_\Gamma H_1(\tilde{B}; Q) \times \mathrm{tor}_\Gamma H_1(\tilde{B}; Q) \to Q$$

on the Γ-torsion part $\mathrm{tor}_\Gamma H_1(\tilde{B};Q)$ of $H_1(\tilde{B};Q)$. For $a \in [-1,1]$, let

$$\sigma_{[a,1]}(\tilde{B}) = \sum_{a \leqq x \leqq 1} \sigma_x(\tilde{B}),$$
$$\sigma_{(a,1]}(\tilde{B}) = \sum_{a < x \leqq 1} \sigma_x(\tilde{B}).$$

We show the following theorem which is a non-compact version of the signature theorem given in [3].

Theorem 4.2 (A non-compact version of the signature theorem).

$$\tau_{a-0}(\tilde{Y}) - \mathrm{sign} Y = \sigma_{[a,1]}(\tilde{B}),$$
$$\tau_{a+0}(\tilde{Y}) - \mathrm{sign} Y = \sigma_{(a,1]}(\tilde{B}).$$

Proof of Theorem 4.2. As it is discussed in Lemma 4.1, we split Y by a compact 4-submanifold $Y' \supset B$ and $Y'' = \mathrm{cl}(Y \backslash Y')$ such that $\hat{\beta}_2(Y') = \hat{\beta}_2(Y)$ (see Corollary 3.2) and Y'' is trivially lifted to $\tilde{Y}$. We use a variant argument of the proof of the Novikov addition theorem for infinite cyclic coverings (see [3]). We consider the homology over the quotient field $Q(\Gamma)$ of Γ. For $B_0 = Y' \cap Y''$, let $K_1(\tilde{B}_0; Q(\Gamma))$ be the kernel of the natural homomorphism

$$H_1(\tilde{B}_0; Q(\Gamma)) \to H_1(\tilde{Y}'; Q(\Gamma)) \oplus H_1(\tilde{Y}''; Q(\Gamma))$$

in the Mayer-Vietoris sequence of $(Y; Y', Y''; B_0)$. Let x_i $(i = 1, 2, \ldots, m)$ be a $Q(\Gamma)$-basis of the $Q(\Gamma)$-vector space $K_1(\tilde{B}_0; Q(\Gamma))$. This basis is extended to a $Q(\Gamma)$-basis x_i $(i = 1, 2, \ldots, m, m+1, \ldots, n)$ for $H_1(\tilde{B}_0; Q(\Gamma))$. A $Q(\Gamma)$-basis y_i $(i = 1, 2, \ldots, n)$ for $H_2(\tilde{B}_0; Q(\Gamma))$ is taken so that the $Q(\Gamma)$-intersection number $\mathrm{Int}_{Q(\Gamma)}(x_i, y_j) = \delta_{ij}$ in $\tilde{B}_0$ (see [1]). Let z_i $(i = 1, 2, \ldots, m)$ be "suspension elements" of x_i $(i = 1, 2, \ldots, m)$ in $H_2(\tilde{Y}; Q(\Gamma))$ (which are constructed from the $Q(\Gamma)$-basis x_i $(i = 1, 2, \ldots, m)$ of $K_1(\tilde{B}_0; Q(\Gamma))$ by using 2-chains in Y' and Y'' whose boundary cycles representing x_i). We regard y_i $(i = 1, 2, \ldots, m)$ as elements of $H_2(\tilde{Y}; Q(\Gamma))$ under the natural homomorphism $H_2(\tilde{B}_0; Q(\Gamma)) \to H_2(\tilde{Y}; Q(\Gamma))$. Then we have $\mathrm{Int}_{Q(\Gamma)}(z_i, y_j) = \delta_{ij}$ and $\mathrm{Int}_{Q(\Gamma)}(y_i, y_j) = 0$ in $\tilde{Y}$. Let $y'_{i'}$ $(i' = 1, 2, \ldots, n')$ be elements of $H_2(\tilde{Y}'; Q(\Gamma))$ such that y_i $(i = 1, 2, \ldots, m)$ and $y'_{i'}$ $(i' = 1, 2, \ldots, n')$ form a $Q(\Gamma)$-basis for $H_2(\tilde{Y}'; Q(\Gamma))$ and $y'_{i'}$ $(i' = 1, 2, \ldots, n')$ are orthogonal to the elements y_i, z_i $(i = 1, 2, \ldots, m)$ with respect to the $Q(\Gamma)$-intersection form $\mathrm{Int}_{Q(\Gamma)}$ in $\tilde{Y}$. Similarly, let $y''_{i''}$ $(i'' = 1, 2, \ldots, n'')$ be elements of $H_2(\tilde{Y}''; Q(\Gamma))$ such that y_i $(i = 1, 2, \ldots, m)$ and $y''_{i''}$ $(i'' = 1, 2, \ldots, n'')$ form a $Q(\Gamma)$-basis for $H_2(\tilde{Y}''; Q(\Gamma))$ and $y''_{i''}$ $(i'' = 1, 2, \ldots, n'')$ are orthogonal to the elements y_i, z_i $(i = 1, 2, \ldots, m)$ with respect to the $Q(\Gamma)$-intersection form $\mathrm{Int}_{Q(\Gamma)}$ in $\tilde{Y}$. Since $\hat{H}_2(\tilde{Y}''; Q)_\Gamma = 0$, we see that

$$\tau_{a\pm 0}(\tilde{Y}) = \tau_{a\pm 0}(\tilde{Y}').$$

By the version with $t = 1$ of this argument, we also have $\mathrm{sign}Y = \mathrm{sign}Y'$. Thus, by the compact version of the signature theorem in [3], we have

$$\tau_{a-0}(\tilde{Y}) - \mathrm{sign}Y = \tau_{a-0}(\tilde{Y}') - \mathrm{sign}Y' = \sigma_{[a,1]}(\tilde{B} \cup \tilde{B}_0) = \sigma_{[a,1]}(\tilde{B}),$$
$$\tau_{a+0}(\tilde{Y}) - \mathrm{sign}Y = \tau_{a-0}(\tilde{Y}') - \mathrm{sign}Y' = \sigma_{(a,1]}(\tilde{B} \cup \tilde{B}_0) = \sigma_{(a,1]}(\tilde{B}),$$

because $\sigma_x(\tilde{B}_0) = 0$ for all $x \in [-1, 1]$. □

Let $\kappa_1(\tilde{B})$ denote the Q-dimension of the kernel of the homomorphism $t - 1 : H_1(\tilde{B}; Q) \to H_1(\tilde{B}; Q)$. Then we have the following corollary:

Corollary 4.3. For every $a \in (-1, 1)$,

$$|\sigma_{(a,1]}(\tilde{B})| - \kappa_1(\tilde{B}) \leqq |\mathrm{sign}Y| + \hat{\beta}_2(Y) \leqq 2\hat{\beta}_2(Y).$$

Proof. In the proof of Theorem 4.2, we have

$$\sigma_{(a,1]}(\tilde{B}) + \mathrm{sing}Y = \tau_{a+0}(\tilde{Y}) = \tau_{a+0}(\tilde{Y}').$$

On the other hand, in [4, Theorem 1.6], it is shown that

$$|\tau_{a+0}(\tilde{Y}'))| - \kappa_1(\partial\tilde{Y}') \leqq \hat{\beta}_2(Y').$$

Since $\hat{\beta}_2(Y') = \hat{\beta}_2(Y)$ and $\partial\tilde{Y}' = \tilde{B} \cup \tilde{B}_0$ with $\sigma_{(a,1]}(\tilde{B}_0) = \kappa_1(\tilde{B}_0) = 0$, we have the desired inequalities. □

5. Loose embedding

Let M' be a compact connected oriented 3-manifold M', and U a possibly non-compact connected oriented 4-manifold. We say that an embedding $k' : M' \to U$ is *loose* if the kernel $K(M') = \ker(k'_* : H_2(M'; Z) \to H_2(U; Q)) \neq 0$. It is known that if the boundary $\partial M'$ of M' is $\emptyset$ or connected, then every indivisible $x \in K(M')$ is represented by a closed connected oriented surface F in M' which we call a *null-surface* of the loose embedding k' (see [6]). Then we have $sk'_*[F] = 0$ in $H_2(U; Z)$ for a positive integer s, which is assumed to be taken to be the smallest positive integer. We consider a loose embedding $k^0 : M^0 \to U$ for $M^0 \in \mathbb{M}^0$ which is regarded as the inclusion map $k^0 : M^0 \subset U$, and F as a null-surface of k^0. We use the following lemma:

Lemma 5.1. Assume that for a tubular neighborhood N_F of F in U, there is a compact connected oriented 3-manifold V in $\mathrm{cl}(U \backslash N_F)$ such that $[\partial V] = s[F]$ in $H_2(N_F; Z)$.

Proof. Choose a compact connected 4-submanifold U' of U with $N_F \subset M^0 \times [-1,1] \subset U' \subset U$ and $M^0 \times 0 = M^0$ such that $K(M^0) = \ker(k^0_* : H_2(M^0;Z) \to H_2(U';Q))$ and $sk^0_*[F] = 0$ in $H_2(U';Z)$. Let $E' = cl(U' \backslash N_F)$. Then there is an indivisible element $z \in H_3(E', \partial N_F; Z) = H_3(U', N_F; Z)$ with $\partial_* z = s[F] \in H_2(N_F; Z)$ under $\partial_* : H_3(U', NF; Z) \to H_2(N_F; Z)$. Since $H_3(E', \partial N_F; Z) = H^1(E', \partial U'; Z)$, we have a compact oriented 3-manifold V' in $E' \subset \mathrm{cl}(U \backslash N_F)$ such that $z = [V'] \in H_3(E', \partial N_F; Z)$, and $\partial V' = s'' F'' \subset \partial N_F$ for a closed connected surface F'' and a factor $s'' > 0$ of s such that $[\partial V'] = s''[F''] = s[F] \in H_2(N_F; Z)$. Replace V' by a connected non-closed component V of V'. Then we still have $[\partial V] = s''[F''] = s[F] \in H_2(N_F; Z)$. □

Let $E_M = \mathrm{cl}(U \backslash M^0 \times [-1,1]) \subset E = \mathrm{cl}(U \backslash N_F)$. For a null-surface F of a loose embedding $k^0 : M^0 \subset U$, we define a homomorphism

$$\gamma : H_1(E_M; Z) \stackrel{i_*}{\to} H_1(E; Z) \stackrel{\mathrm{Int}_V}{\to} Z,$$

where i_* is a natural homomorphism and Int_V is defined by the identity $\mathrm{Int}_V(x) = \mathrm{Int}(x, V)$ for $x \in H_1(E; Z)$. We have the following lemma:

Lemma 5.2. i_* and Int_V are onto, so that γ is onto.

We call γ a *null-epimorphism* (associated with an null-surface F) of a loose embedding k^0.

Proof. Since $M^0 \backslash F$ is connected, every simple loop l in $U \backslash F$ meeting M^0 transversely is deformed in $U \backslash F$ into a simple loop l' in $U \backslash M^0$. Hence, i_* is onto. Then we have the Q-linking number $\mathrm{Link}_Q(F, m) = +1$ for a meridian m of F in ∂N_F and hence we see that m meets V with the intersection number s in E. Since V is connected, m is used to construct a simple loop m' in E meeting V transversely at just one point. Hence, Int_V is onto. □

We also need the following lemma:

Lemma 5.3. Every null-epimorphism $\gamma : H_1(E_M; Z) \to Z$ of a loose embedding $k^0 : M^0 \to U$ is end-trivial.

Proof. The infinite cyclic covering $\tilde{E}$ induced from the epimorphism Int_V is constructed from the infinite copies of $\mathrm{cl}(E \backslash V \times [0,1])$ by attaching them along the infinite copies of a bi-collar $V \times [0,1]$ of V in E. Thus, the restriction of Int_V to the non-compact part $\mathrm{cl}(E \backslash V \times [0,1])$ is the 0-map. Since the infinite cyclic covering $\tilde{E}_M \to E_M$ induced from γ is a restriction of the infinite cyclic covering $\tilde{E} \to E$, we see that γ is end-trivial. □

Let α be the reflection on the double $DM^0(=\partial E_M)$ of M^0 exchanging the two copies of M^0 orientation-reversely. A meridian m of F in $M^0 \times [-1,1]$ is deformed in $M^0 \times [-1,1]$ into a loop m' in $DM^0 = \partial E_M$ with $\alpha(m') = -m'$. Since $\mathrm{Int}_V([m]) = s$, the following lemma is directly obtained:

Lemma 5.4. We have $\dot\gamma(x_F) = s$ and $\alpha_*(x_F) = -x_F$ for the element $x_F = [m'] \in H_1(\partial E_M; Z)$ and the restriction $\dot\gamma : H_1(DM^0; Z) \to Z$ of γ.

Corollary 5.5. If s is odd, then the Z_2-reduction $\dot\gamma_2 : H_1(DM^0; Z) \to Z_2$ of $\dot\gamma$ is not α-invariant.

A homomorphism $\dot\gamma : H_1(DM^0; Z) \to Z$ satisfying the conclusion of Corollary 5.5 is called a Z_2-*asymmetric* homomorphism in [4, 8].

Proof. We can write x_F as $x' - \alpha_*(x')$ for the element $x' \in H_1(DM^0; Z)$ represented by a loop in M^0. Then

$$\dot\gamma(x_F) = \dot\gamma(x') - \dot\gamma\alpha_*(x') = s \equiv 1 \pmod 2,$$

which shows that γ_2 is not α-invariant. □

6. Completion of the proof of Theorem 3.3

Throughout this section, we make the proof of the remaining part of Theorem 3.3.

Completion of the proof of (1). For any positive integers n, c, we take n knots K_i $(1 \leqq i \leqq n)$ whose signatures $\sigma(K_i)$ $(1 \leqq i \leqq n)$ have the condition that

$$|\sigma(K_1)| > 2c \quad \text{and} \quad |\sigma(K_i)| > \sum_{j=1}^{i-1} |\sigma(K_j)| + 2c \quad (i = 2, 3, \ldots, n).$$

Let $M_i = \chi(K_i, 0)$ and $M = M_1 \# M_2 \# \ldots \# M_n$. We call M a *c-efficient 3-manifold of rank n.* The following calculation is made in [4, Lemma 1.3]:

(6.1.1) Every c-efficient 3-manifold M of any rank n has

$$|\sigma_{(-1,1]}(\widetilde{DM^0})| > 2c$$

for every Z_2-asymmetric homomorphism $\dot\gamma : H_1(DM^0; Z) \to Z$.

Suppose that a punctured universe U has

$$\hat\beta_2(U) = c < +\infty, \ \delta_0(U) = b < +\infty, \ \rho_0(U) = b' < +\infty.$$

Let M be a c-efficient 3-manifold of any rank $n > b+b'$. Suppose that M^0 is embedded in U. For the inclusion $k^0 : M^0 \subset U$, the kernel

$$K(M^0) = \ker[k^0_* : H_2(M^0; Z) \to H_2(U; Q)]$$

is a free abelian group of rank $d > b'$. Then there is a basis x_i $(i = 1, 2, \ldots, n)$ of $H_2(M^0; Z)$ such that x_i $(i = 1, 2, \ldots, d)$ is a basis of $K(M^0)$. Since $\rho_0(U) = b' < d$, we can find an indivisible element x in the basis x_i $(i = 1, 2, \ldots, d)$ such that the multiplied element rx for an odd integer r is represented by the boundary cycle of a 3-chain in U. Taking a closed connected oriented surface F in M^0 representing x, we have a null-epimorphism $\gamma : H_1(E_M : Z) \to Z$ (associated with an null-surface F) of the loose embedding k^0 whose restriction $\dot{\gamma} : H_1(DM^0 : Z) \to Z$ is a Z_2-asymmetric homomorphism. Then we obtain from (6.1.1) a contradiction that

$$2c < |\sigma_{(-1,1]}(\widetilde{DM^0})| \leqq 2c$$

because $\hat{\beta}_2(E_M) \leqq \hat{\beta}_2(U) = c$ and $\kappa_1(\widetilde{DM^0}) = 0$. Thus, at least one of $\hat{\beta}_2(U)$, $\delta_0(U)$, $\rho_0(U)$ must be $+\infty$. □

Completion of the proof of (2). Let U be a type 1 universe. We always have $\hat{\beta}_1(U) \geqq 1$. Since U is also a punctured universe, at least one of $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ must be $+\infty$ by (1). Suppose that a type 1 universe U has

$$b = \hat{\beta}_2(U) < +\infty,\ c = \delta_1(U) < +\infty,\ s = \hat{\beta}_1(U) < +\infty.$$

Then we show that there is a 3-manifold M which is not type 1 embeddable in U. Let $\hat{H}_1(U; Z) = Z^s$. Let U_u $(u = 1, 2, \ldots, 2^s - 1)$ be the connected double coverings of U induced from the epimorphisms $Z^s \to Z_2$. Let $\mathbb{M}_u$ be the subset of $\mathbb{M}$ consisting of M such that a type 1 embedding $k : M \to U$ is trivially lifted to $k_u : M \to U_u$. Since every type 1 embedding $M \to U$ lifts to U_u trivially for some u, we see that

$$\bigcup_{u=1}^{2^s-1} \mathbb{M}_u = \mathbb{M}.$$

Let U' be a compact 4-submanifold of U such that $U'' = \mathrm{cl}(U \backslash U')$ is trivially lifted to U_u for all u. Let U'_u and U''_u be the lifts of U' and U'' to U_u. Let

$$b' = \max\{\beta_2(U'_u)|\, u = 1, 2, \ldots, 2^{s-1}\}.$$

(6.2.1) $\mathrm{rank}(\mathrm{im}(k_u)_*) \leqq b + b'$ for any u.

Proof of (6.2.1). Let $K(M) = \ker(k_* : H_2(M; Z) \to H_2(U; Q))$. Let F_j $(j = 1, 2, \ldots, m)$ be a system of closed connected surfaces representing a basis for $K(M)$.

Let V_j be a compact oriented 3-manifold in U such that $\partial V_j = r_j k(F_j)$ for a positive integer r_j. Let $V_j'' = V_j \cap U''$ be a compact orientable 3-manifold. Then $\partial V_j'' = \dot{V}_j' \cup r_j F_j''$ where $F_j'' = k(F_j) \cap U''$ and $\dot{V}_j' = V_j \cap \partial U'$. Since V_j'' is trivially lifted to U_u, we see that the 2-cycle $k_u(F_j)$ is Q-homologous to the rational 2-cycle

$$\frac{1}{r_j}[r_j k_u(\mathrm{cl}(F_j \backslash F_j'')) + k_u(\dot{V}_j')]$$

in U_u' for all j. This means that $(k_u)_*(K(M))$ is in the image of the natural homomorphism $H_2(U_u'; Q) \to H_2(U_u; Q)$. Hence we have $\mathrm{rank}((k_u)_*(K(M))) \leqq b'$. Since $\mathrm{rank}(\mathrm{im}(k_*)) \leqq b$, we have $\mathrm{rank}(\mathrm{im}(k_u)_*) \leqq b + b'$. □

For any positive integers n, c, we take n knots K_i $(1 \leqq i \leqq n)$ whose local signatures $\sigma_{(a,1)}(K_i)$ $(1 \leqq i \leqq n)$ have the condition that there are numbers $a_i \in (-1,1)$ $(i = 1, 2, \ldots, n)$ such that

$$|\sigma_{(a_1,1]}(K_1)| > 2c, \;\; |\sigma_{(a_i,1]}(K_i)| > \sum_{j=1}^{i-1} \left|\sigma_{(a,1]}(K_j)\right| + 2c \quad (i = 2, 3, \ldots, n)$$

for every $a \in (-1,1)$ (see [7]). Let $M_i = \chi(K_i, 0)$ be the 0-surgery manifold along K_i, and $M = M_1 \# M_2 \# \ldots \# M_n$. We call M a strongly *c-efficient 3-manifold of rank n*. For this 3-manifold M, we say that a homomorphism $\dot{\gamma} : H_1(DM^0; Z) \to Z$ is *symmetric* if $\dot{\gamma}|_{\alpha(M_i^0)} = \pm \dot{\gamma}|_{M_i^0}$ for all i, where α is the reflection on the double DM^0. Otherwise, $\dot{\gamma}$ is said to be an *asymmetric homomorphism.* The following calculation is also seen from [4, Lemma 1.3]:

(6.2.2) For every strongly c-efficient 3-manifold M of any rank n and every asymmetric homomorphism $\dot{\gamma} : H_1(DM^0; Z) \to Z$, we have a number $a \in (-1,1)$ such that

$$|\sigma_{(a,1]}(\widetilde{DM^0}))| > 2c.$$

For example, if M is constructed from the knots K_i $(i = 1, 2, \ldots, n)$ with K_i the ic^+-fold connected sum of the trefoil knot for any fixed integer $c^+ > c$, then M is a strongly c-efficient 3-manifold of rank n. We show that every strongly c-efficient 3-manifold M of rank $> b + b'$ is not type 1 embedded in U. Suppose that M is type 1 embedded in U and lifts trivially in U_u. Let $U(M)$ and $U_u(M) = U(M) \cup tU(M)$ be the 4-manifolds obtained respectively from U and U_u by splitting along M, where t denotes the double covering involution. Let $\partial U(M) = M_0 \cup -M_1$ and $\partial U_u(M) = M_0 \cup -M_2$, where M_0, M_1, M_2 are the copies of M. Since the natural homomorphism $H_2(M; Z) \to H_2(U; Q)$ is not injective, there is a non-zero element $[C] \in H_2(M; Z)$ such that $C = \partial C$ for a 3-chain D in U_u and $C = \partial D_*$ for a 3-chain D_* in U which is the image of D under the covering projection $U_u \to U$. The 3-chains $D$and D_* define 3-chains D', D'' and D''' in $U(M)$ such that

$$\partial D' = C_1'' - (C_0 + C_0'),$$
$$\partial D'' = C_1' - C_0'',$$
$$\partial D''' = (C_1' + C_1'') - (C_0 + C_0' + C_0'')$$

for some 2-cycles C_u, C_u', C_u'' in M_u $(u = 0, 1)$ (see Fig. 3). Since $\hat{\beta}_2(U(M)) \leqq c$, the non-zero end-trivial homomorphism $\dot{\gamma} : H_1(DM^0; Z) \to Z$ defined by any 3-chain in $U(M)$ must be symmetric by Corollary 4.3 and (6.2.2) because every strongly c-efficient 3-manifold M has $\kappa_1(\widetilde{DM^0}) = 0$. Let

$$[C] = \sum_{i=1}^{m} a_i x_i,\ [C'] = \sum_{i=1}^{m} a_i' x_i,\ [C''] = \sum_{i=1}^{m} a_i'' x_i$$

in $H_1(M; Z)$ with x_i a generator of $H_1(M_i; Z) \cong Z$. By the symmetry conditions on D', D'' and D''', we have the following relations:

$$a_i'' = \varepsilon_i(a_i + a_i'),\ a_i' = \varepsilon_i' a_i'',\ a_i' + a_i'' = \varepsilon_i''(a_i + a_i' + a_i''),$$

where $\varepsilon_i, \varepsilon_i', \varepsilon_i'' = \pm 1$ for all i. Then we have

$$(1 + \varepsilon_i')a_i'' = \varepsilon_i''(\varepsilon_i + 1)a_i''.$$

If $\varepsilon_i \varepsilon_i' = -1$, then we have $a_i'' = a_i' = a_i = 0$ for all i. If $\varepsilon_i \varepsilon_i' = 1$, then we have $a_i = 0$ for all i. Hence we have $[C] = 0$, contradicting that $[C] \neq 0$. Hence M is not type 1 embeddable in U. □

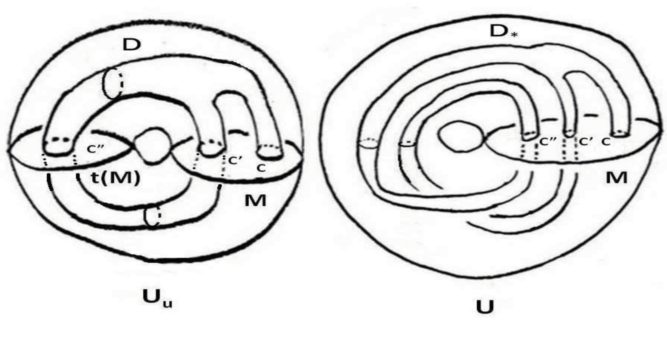

Figure 3: A situation of 3-chains

Completion of the proof of (3). Let U be a type 2 universe. Suppose that

$$\hat{\beta}_2(U) = c < +\infty, \quad \delta_2(U) = b < +\infty.$$

Let $M \in \mathbb{M}$ be a c-efficient 3-manifold of any rank $n > b$. Let $k : M \subset U$ be a type 2 embedding which is a loose embedding. let U' and U'' be the 4-manifolds obtained from U by splitting along M. For U' or U'', say U', we have a null-surface F in M and a positive (not necessarily odd) integer r such that the natural homomorphism $H_2(M; Z) \to H_2(U'; Z)$ sends $r[F]$ to 0. Taking the minimal positive integer r, we have a compact connected oriented 3-manifold V in U' with $\partial V = rF$. This 3-manifold V defines an end-trivial epimorphism $\gamma : H_1(U'; Z) \to Z$ whose restriction $\dot{\gamma} : H_1(M; Z) \to Z$ is equal to $r\dot{\gamma}_F$ for the epimorphism $\dot{\gamma}_F : H_1(M; Z) \to Z$ defined by F. Let $\tilde{M}$ and $\tilde{M}_F$ denote the infinite cyclic coverings of M induced from $\dot{\gamma}$ and $\dot{\gamma}_F$, respectively. Let $(1 \leqq) i_1 < i_2 < \cdots < i_s (\leqq n)$ be the enumeration of i such that the Z_2-reduction of $\dot{\gamma}_F$ restricted to the connected summand M_i of M is non-trivial. By a calculation made in [4, Lemma 1.3], we have

$$\sigma_{(-1,1]}(\tilde{M}_F) = \sum_{j=1}^{s} \sigma(K_{i_j}),$$

so that $|\sigma_{(-1,1]}(\tilde{M}_F)| > 2c$. By [4, Lemma 1.3], we also have

$$\sigma_{(-1,1]}(\tilde{M}_F) = \sigma_{(a,1]}(\tilde{M})$$

for some $a \in (-1, 1)$. Then, since $\hat{\beta}_2(U') \leqq \hat{\beta}_2(U) = c$ and $\kappa_1(\tilde{M}) = 0$, we obtain from Corollary 4.3 a contradiction that

$$2c < |\sigma_{(a,1]}(\tilde{M})| \leqq 2c.$$

Hence $\hat{\beta}_2(U)$ or $\delta_2(U)$ must be $+\infty$. □

Completion of the proof of (4). Let U be a universe. Assume that

$$\hat{\beta}_2(U) = c < +\infty \quad \text{and} \quad \delta(U) < +\infty.$$

By the proof of (3), for every infinite family of strongly c-efficient 3-manifolds of infinitely many ranks n any member must be type 1 embeddable to U. By the proof of (2), we have $\rho(U) = +\infty$ and $\hat{\beta}_1(U) = +\infty$. □

Completion of the proof of (5). Since a full universe U is a type 1 and type 2 universe, the desired result follows from (2) and (3). □

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