Knotting probability of an arc diagram

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Abstract. The knotting probability of an arc diagram is defined as the quadruplet of four kinds of finner knotting probabilities which are invariant under a reasonable deformation containing an isomorphism on an arc diagram. In a separated paper, it is shown that every oriented spatial arc admits four kinds of unique arc diagrams up to isomorphisms determined from the spatial arc and the projection, so that the knotting probability of a spatial arc is defined. The definition of the knotting probability of an arc diagram uses the fact that every arc diagram induces a unique chord diagram representing a ribbon 2-knot. Then the knotting probability of an arc diagram is set to measure how many non-trivial ribbon genus 2 surface-knots occur from the chord diagram induced from the arc diagram. The conditions for an arc diagram with the knotting probability 0 and for an arc diagram with the knotting probability 1 are given together with some other properties and some examples.

1. Introduction

A *spatial arc* L is an oriented polygonal arc in the 3-space $\mathbf{R}^3$, which is considered as a model of a protein or a linear polymer in science. The following question on science is an interesting question that can be set as a mathematical question:

Question. How a linear scientific object such as a non-circular molecule (e.g. a non-circular DNA, protein, linear polymer, etc.) is considered as a knot object ?

An *arc diagram* D is a diagram of a spatial arc in the plane with only *crossing points* (i.e., transversely meeting double points with over-under information) and with the starting point s and the terminal point t as single points. An *inbound arc diagram* is an arc diagram such that the points s and t are in the same connected region of the plane divided by the arc diagram, where the region containing s and t is called the *front region*. Two arc diagrams D and D' in the plane P are *isomorphic* if there is an orientation-preserving self-homeomorphism $f : P \to P$ sending D to D' which preserves the crossing points of D and D'. The map f is called an *isomorphism* from D to D'. In an illustration of an arc diagram, it is convenient to illustrate an arc diagram with smooth edges in the class of isomorphic arc diagrams instead of a polygonal arc diagram.

2010 *Mathematics Subject Classification.* 57M25; 57Q45.

Key words and phrases. Spatial arc, Arc diagram, Ribbon surface-knot, Chord diagram, Knotting probability.

In this paper, the knotting probability $p(D)$ of an arc diagram D is defined to be the quadruplet of the knotting probabilities $p^{\mathrm{I}}(D)$, $p^{\mathrm{II}}(D)$, $p^{\mathrm{III}}(D)$, $p^{\mathrm{IV}}(D)$ of types I, II, III, IV which are determined from D uniquely up to isomorphisms. The notation $p(D) = 0$ is used if all of them are 0 and otherwise, $p(D) > 0$. Also, the notation $p(D) = 1$ is used if all of them are 1 and otherwise, $p(D) < 1$. The definition of the knotting probability of an arc diagram uses an argument on a chord diagram derived from a ribbon surface-knot in the 4-space $\mathbf{R}^4$. For simple arc diagrams, the knotting probability are easily calculable.

In the separated paper [**10**], it is shown that every oriented spatial arc admits a unique arc diagrams up to isomorphisms determined from the spatial arc and the projection direction, so that the knotting probability of a spatial arc is defined.

We mention here that a knotting probability of a circular knot is studied by Deguchi and Tsurusaki [**2**] (see also E. Uehara and T. Deguchi [**2**]) from the viewpoint of a random knotting, which is independent of our viewpoint. The case of a spatial arc is also studied by Millett, Dobay and Stasiak in [**12**] from the random knotting viewpoint with the same motivation as the present question[1]. We also note that a knotting probability of a spatial arc was defined in [**4, 6**] from a knotting structure of a spatial graph but with the demerit that it depends on the heights of the crossing points of a diagram of the spatial arc.

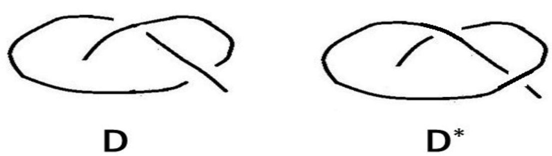

FIGURE 1. A 2-crossing arc diagram D and its mirror image D^*

In § 2, some resent results in [**5, 7, 8, 9**] on a chord diagram derived from a ribbon surface-knot are explained. In § 3, the knotting probability $p(D)$ of an arc diagram D is defined by an argument transforming an arc diagram into a chord diagram. From the argument, it will be seen that the knotting probability measures how a fixed spatial arc tends to be a non-trivial ribbon surface-knot of genus 2. In Theorem 3.3 which is a main result, it is shown that for an arc diagram D, the algebraic condition to be $p(D) > 0$ and the algebraic condition to be $p(D) = 1$ are given. Further, it is shown that the mirror image D^* of an inbound arc diagram D has the same knotting probability as D. In § 4, several examples are given. For example, the knotting probability $p(D)$ for the diagram D in Fig. 1 is calculated to be

$$p(D) = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}).$$

The knotting probability $p(D)$ is independent of a choice of the orientation of an arc diagram D. However, for the mirror image D^* of D, the knotting probability $p(D^*)$ is generally different from $p(D)$. For example, the knotting probability $p(D^*)$

[1]Some protein knotting data are listed in "KnotProt"(https://knotprot.cent.uw.edu.pl/).

of the mirror diagram D^* of the diagram D of Fig. 1 is calculated to be

$$p(D^*) = 0.$$

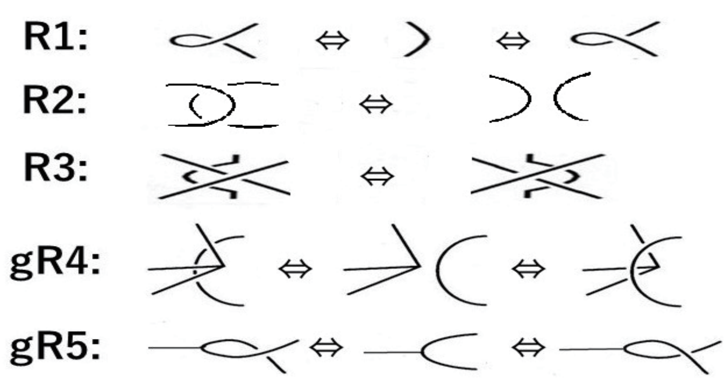

FIGURE 2. Move M_0: Reidemeister moves R_1, R_2, R_3, gR_4, gR_5 for trivalent graph diagrams

2. The chord diagram of an arc diagram and its meaning

A *chord graph* is a trivalent connected graph $(o;\alpha)$ in $\mathbf{R}^3$ consisting of a trivial oriented link o (called a *based loop system*) and the attaching arcs α (called a *chord system*), where some chords of α may meet. A *chord diagram* is a diagram $C = C(o;\alpha)$ of a chord graph $(o;\alpha)$ in a plane P which has only double point singularities with a upper-lower relation by an orthogonal projection. There are three moves M_0, M_1 and M_2 on the chord diagrams which are explained as follows (see [**5, 7, 8, 9**]):

Move M$_0$. This move consists of the Reidemeister moves R_1, R_2, R_3, gR_4, gR_5 as spatial trivalent graphs, illustrated in Fig. 2.

Note that any two arcs in the three arcs together with a vertex or any arc in Fig. 2 can be a part of a based loop although the orientation and the shadow of the based loop are omitted there.

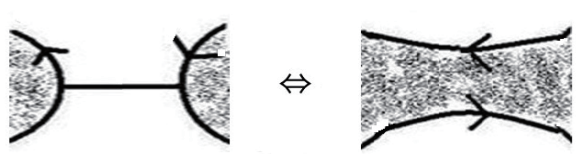

FIGURE 3. Fusion-fission M_1

Move $\mathbf{M}_1$. This move is the *fusion-fission move*, illustrated in Fig. 3, where the fusion operation is done only for a chord between different based loops.

The following lemma is shown in [**5**].

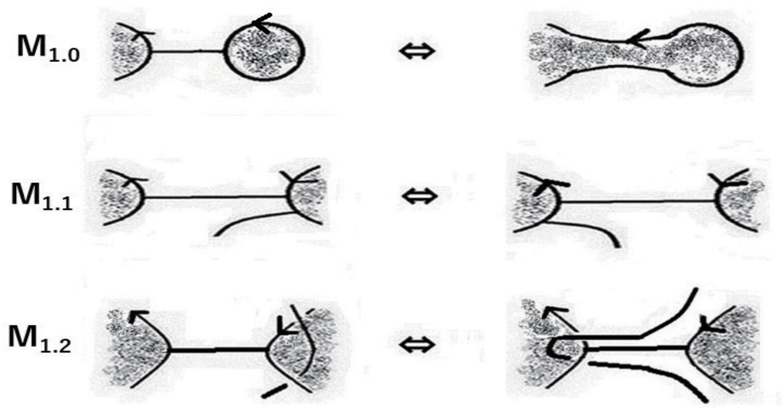

FIGURE 4. Elementary fusion-fission $M_{1.0}$, Chord slide $M_{1.1}$ and Chord pass $M_{1.2}$

Lemma 2.1. Under the use of the move M_0, the move M_1 is equivalent to the combination move of the *elementary fusion-fission move* $M_{1.0}$, the *chord slide move* $M_{1.1}$ and the *chord pass move* $M_{1.2}$ illustrated in Fig. 4. The *birth-death move* illustrated in Fig. 5 is obtained from these moves, unless a closed chord is involved.

FIGURE 5. The birth-death move

As a convention, a closed chord is regarded as a chord with a based loop constructed from the birth-death move.

Move $\mathbf{M}_2$. This move consists of moves on chords, illustrated in Fig. 6.

A chord diagram $C = C(o;\alpha)$ is *regular* if the based loops o bound mutually disjoint disks in the plane P meeting only the chords transversely. A regular chord diagram is *oriented* if every chord is attached to the based loops as in Fig. 7. An *orientable chord diagram* is a chord diagram D which becomes an oriented regular

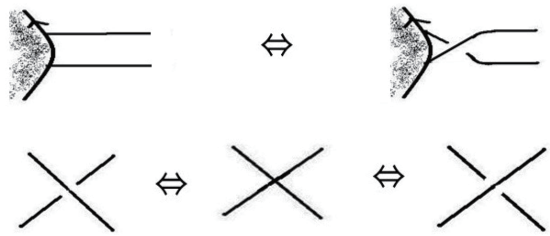

FIGURE 6. Chord moves M_2

chord diagram after applying the moves M0 to D. *Unless otherwise mentioned, a chord diagram means an orientable chord diagram.*

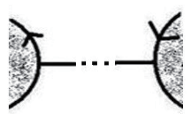

FIGURE 7. A chord in an oriented regular chord diagram

Two chord diagrams C and C' in the plane P are *isomorphic* if there is an orientation-preserving self-homeomorphism $f : P \to P$ sending C to C' which preserves the orientations of the based loops of C and C'.

To explain how to change an arc diagram into an oriented regular chord diagram, let D be an oriented arc diagram with n crossing points. Let $C(D)$ be a chord diagram with $n + 2$ based loops obtained from the diagram D by replacing every crossing point and starting point s and the terminal point t with based loops as in Fig. fig:trans.

In the transformation on the crossing point, the left diagram in the right-hand side is taken, but the right diagram is also admitted by the move M_0 (the Reidemeister moves) given later[2] In the transformations on the end points s and t, the orientation on the based loops o_s and o_t corresponding to the end points s and t, respectively are uniquely chosen so that the chord diagram $C(D)$ is an oriented regular chord diagram. For example, see Fig. 9 for an actual transformation (see Example 4.3 later).

If two based loops are connected by a chord not meeting the other chords, we can replace it by one based loop (which is also called the *chord diagram* of an arc diagram) although a carefulness is needed in the calculation of the knotting probability (see Figs. 19, 21 later).

[2]Once a left or right diagram is chosen at a crossing, the same choice must be done for the other crossings to obtain an oriented regular chord diagram.

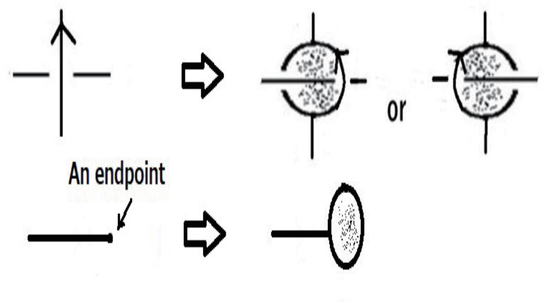

Figure 8. Transformation of an oriented arc diagram into a chord diagram

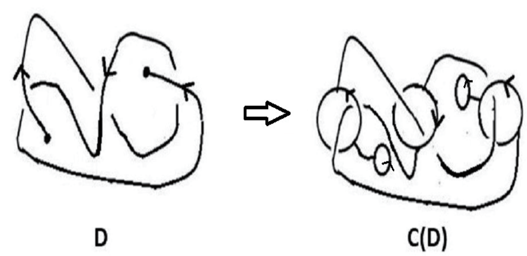

Figure 9. Transformation of an oriented arc diagram into an oriented regular chord diagram

To explain a meaning of a chord diagram and the moves M_0, M_1 and M_2, let

$$A[a,b] = \{(x,t) \in \mathbf{R}^4 |\, x \in A,\ t \in [a,b]\}$$

for a subset $A \subset \mathbf{R}^3$ and an interval $[a,b]$. A ribbon surface-knot is introduced by Yanagawa in [**16**] for a ribbon 2-knot and [**11**, II] for a general ribbon surface-knot. From a chord diagram $C = C(o;\alpha)$, we can construct in a unique way a ribbon surface-knot $F(C) = F(o;\alpha)$ by the following three steps (see Fig. 10 for a ribbon surface-knot constructed from a chord diagram):

(1) Construct the standard oriented S^2-link $S^2(o)$ in $\mathbf{R}^4$ is constructed from the based loops o by capping the cylinder $o[-1,1]$ in the thickened 3-space $\mathbf{R}^3[-1,1]$ with the disks $d[1]$ for a disk d in $\mathbf{R}^3$.

(2) Construct the 1-handles $h(\alpha)$ on the trivial oriented S^2-link $S^2(o)$ with the chords α as the core arcs.

(3) Let $F(o;\alpha)$ be the ribbon surface-knot obtained from $S^2(o)$ by the surgery along the 1-handles $h(\alpha)$.

The following result is obtained in [**5, 7, 9**].

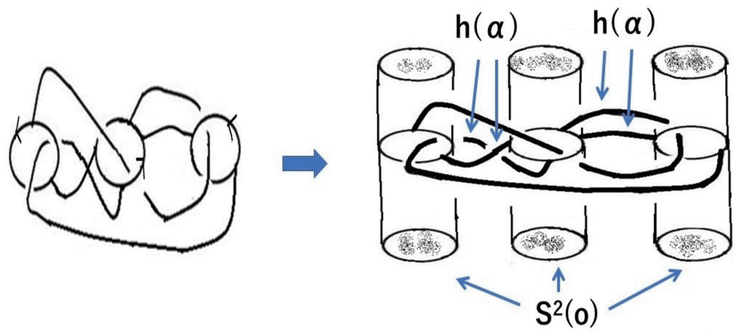

FIGURE 10. A ribbon surface-knot of genus 2 constructed from a chord diagram

Lemma 2.2.
(1) Two ribbon surface-knots $F(o;\alpha)$ and $F(o';\alpha')$ are equivalent (i.e., sent by an orientation-preserving diffeomorphism $\mathbf{R}^4$ to $\mathbf{R}^4$) if and only if a chord diagram $C(o';\alpha')$ is obtained from a chord diagram $C(o;\alpha)$ by a finite number of the moves M_0, M_1 and M_2.

(2) A ribbon surface-knot $F(o;\alpha)$ is a trivial surface-knot (i.e., a surface-knot bounding a handlebody in $\mathbf{R}^4$) if and only if the fundamental group $\pi_1(\mathbf{R}^4 \setminus F(o;\alpha)) \cong \mathbb{Z}$.

For an oriented regular chord diagram $C = C(o;\alpha)$, the finitely presented group $\pi(C)$ is given as in Fig. 11. This presented group $\pi(C)$ is isomorphic to the fundamental group $\pi_1(\mathbf{R}^4 \setminus F(o;\alpha))$ of a ribbon surface-knot $F(o;\alpha)$, which is shown by the group presentation and actually obtained by Yajima in [**15**] for a ribbon torus-knot. It is noted that all the edges attaching to every based loop of o have the same collar in the presentation of $\pi(C)$, so that the set of the based loops is considered as a generating set of the presented group $\pi(C)$.

3. Defining the knotting probability

Let D be an oriented n-crossing arc diagram, and $C(D)$ the chord diagram of D. There are $(n+2)^2$ chord diagrams A obtained from the chord diagram $C(D)$ by joining the loops o_s and o_t with any based loops of $C(D)$ by two chords not passing the other based loops. A chord diagram obtained in this way is called an adjoint chord diagram of $C(D)$ with an *additional chord pair*. The group $\pi(A)$ of an adjoint chord diagram A of the chord diagram $C(D)$ has a group presentation obtained from the group presentation of $\pi(C(D))$ by adding two relations identifying the based loop generators connected by the additional chords.

Note that the ribbon surface-knot $F(C(D))$ of the chord diagram $C(D)$ is a ribbon S^2-knot and the ribbon surface-knot $F(A)$ of an adjoint chord diagram A is a genus 2 ribbon surface-knot. A chord diagram is said to be *unknotted* or *knotted* according to whether it represents a trivial or non-trivial ribbon surface-knot, respectively. By Lemma 2.2 (2), an adjoint chord diagram A of $C(D)$ is unknotted if and only if $\pi(A) \cong \mathbf{Z}$.

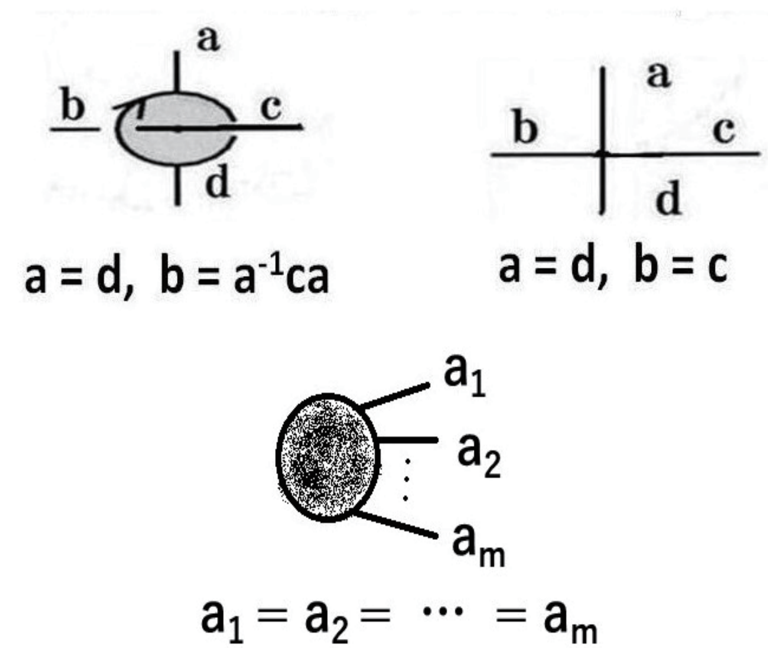

Figure 11. Group relations

The idea of the knotting probability is to measure how many knotted chord diagrams there are among the $(n+2)^2$ adjoint chord diagrams of $C(D)$. Since there are overlaps among them up to canonical isomorphisms, we consider the n^2+2n+2 adjoint chord diagrams A of $C(D)$ by removing some overlaps. The n^2+2n+2 adjoint chord diagrams A of $C(D)$ are classified by the following 4 types (see Fig. 12):

Type I. Here are the 2 adjoint chord diagrams of $C(D)$ which are the adjoint chord diagram with two self-attaching additional chords and the adjoint chord diagram with a self-attaching additional chord on o_s and an additional chord joining o_s with o_t.

Type II. Here are the $2n$ adjoint chord diagrams A of $C(D)$. The $2n$ adjoint chord diagrams of $C(D)$ are given by the additional chord pairs consist of a self-attaching additional chord on o_s (or o_t, respectively) and an additional chord joining o_t (or o_s, respectively) with a based loop except for o_s and o_t.

Type III. Here are the n adjoint chord diagrams A of $C(D)$ where the additional chord pairs consist of an additional chord joining o_s with o_t and an additional chord joining o_s with a based loop except for o_s and o_t.

Type IV. Here are the $n(n-1)$ adjoint chord diagrams A of $C(D)$ where the additional chord pair joins the pair of o_s and o_t with a distinct based loop pair not containing o_s and o_t.

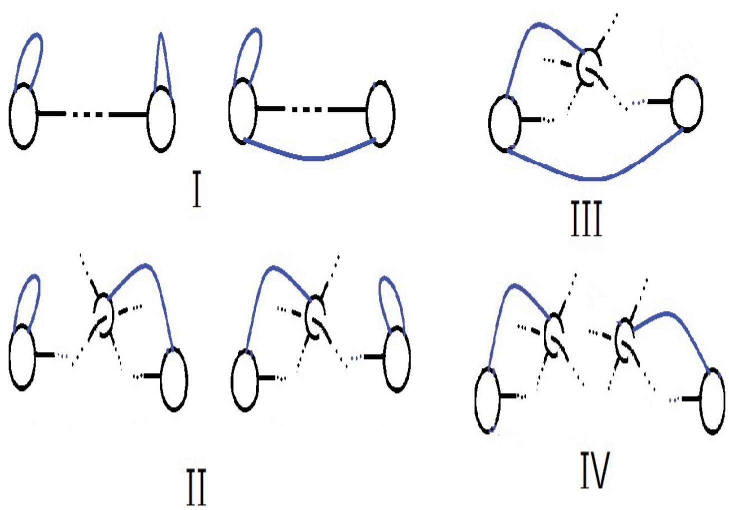

FIGURE 12. Types I, II, III, IV of adjoint chord diagrams

The following lemma shows the reason why it is justified to reduce the $(n+2)^2$ adjoint chord diagrams to the n^2+2n+2 adjoint chord diagrams.

Lemma 3.1. Every adjoint chord diagram of the chord diagram $C(D)$ of any n crossing arc diagram D is deformed into one of the adjoint chord diagrams of types I, II, III and IV of the chord diagram $C(D)$ by the moves M_0, M_1, M_2.

Proof Lemma 3.1. Since $(n+2)^2 - (n^2+2n+2) = 2n+2$, there are $2n+2$ omissions of adjoint chord diagrams. The omitted 2 adjoint chord diagrams of $C(D)$ are the adjoint chord diagram of $C(D)$ where both of the additional chords join o_s and o_t and the adjoint chord diagram with a self-attaching additional chord on o_t and an additional chord joining o_s with o_t which are deformed into the adjoint chord diagram with a self-attaching additional chord on o_s and an additional chord joining o_s with o_t of type I by the moves M_0, M_1, M_2. The omitted n adjoint chord diagrams of $C(D)$ such that the additional chord pair joins o_s and o_t with the same based loop except for o_s and o_t are deformed into the n adjoint chord diagrams of type III by the moves M_0, M_1, M_2. The omitted n adjoint chord diagrams of $C(D)$ where the additional chord pairs consist of an additional chord joining o_s with o_t and an additional chord joining o_t with a a based loop except for o_s and o_t are also deformed into the n adjoint chord diagrams of type III by the moves M_0, M_1, M_2. Thus, the omitted $2n+2$ adjoint chord diagrams are reduced to the adjoint chord diagrams of types I and III.
□

In [**10**], it is shown that every adjoint chord diagram of the chord diagram $C(D)$ of any n crossing arc diagram D is deformed into one of the adjoint chord diagrams of type I, II, III and IV of the chord diagram $C(D)$. Thus, it is justified to reduce the $(n+2)^2$ adjoint chord diagrams to the n^2+2n+2 adjoint chord diagrams.

The *knotting probability* $p(D)$ of an arc diagram D is defined to be the quadruplet

$$p(D) = (p^{\mathrm{I}}(D), p^{\mathrm{II}}(D), p^{\mathrm{III}}(D), p^{\mathrm{IV}}(D))$$

of the following knotting probabilities p^{I}, p^{II}, $p^{\mathrm{III}}(D)$ of types I, II III, IV.

Definition.
(1) Let A_1 and A_2 be the adjoint chord diagrams of type I and assume that there are just k knotted chord diagrams among them. Then the *type I knotting probability* of D is

$$p^{\mathrm{I}}(D) = \frac{k}{2}.$$

Thus, $p^{\mathrm{I}}(D)$ is 0, $\frac{1}{2}$ or 1 for any arc diagram D.

(2) Let $A_i\,(i = 1, 2, \ldots, 2n)$ be the adjoint chord diagrams of type II and assume that there are just k knotted chord diagrams among them. Then the *type II knotting probability* of D is

$$p^{\mathrm{II}}(D) = \frac{k}{2n}.$$

(3) Let $A_i\,(i = 1, 2, \ldots, n)$ be the adjoint chord diagrams of type III and assume that there are just k knotted chord diagrams among them. Then the *type III knotting probability* of D is

$$p^{\mathrm{III}}(D) = \frac{k}{n}.$$

(4) Let $A_i\,(i = 1, 2, \ldots, n(n-1))$ be the adjoint chord diagrams of type IV and assume that there are just k knotted chord diagrams among them. Then the *type IV knotting probability* of D is

$$p^{\mathrm{IV}}(D) = \frac{k}{n(n-1)}.$$

When the orientation of an arc diagram D is changed, all the orientations of the based loops of the chord graph $C(D)$ are changed at once. This means that the knotting probability $p(D)$ does not depend on any choice of orientations of D, and we can omit the orientation of D in figures. See [**10**] for actual calculations of $p(D)$. When one-valued probability is desirable, a suitable average of the knotting probabilities

$$p^{\mathrm{I}}(D),\ p^{\mathrm{II}}(D),\ p^{\mathrm{III}}(D),\ p^{\mathrm{IV}}(D)$$

is considered. The knotting probability $p(D)$ has $p(D) = 1$ if

$$p^{\mathrm{I}}(D) = p^{\mathrm{II}}(D) = p^{\mathrm{III}}(D) = p^{\mathrm{IV}}(D) = 1$$

and otherwise, $p(D) < 1$. The knotting probability $p(D)$ has $p(D) > 0$ if

$$p^{\mathrm{I}}(D) + p^{\mathrm{II}}(D) + p^{\mathrm{III}}(D) + p^{\mathrm{IV}}(D) > 0$$

and otherwise, $p(D) = 0$.

When the orientation of D is changed, all the orientations of the based loops of the chord graph $C(D)$ are changed at once. Thus, the knotting probability $p(D)$ do not depend on any choice of orientations of D, and we can omit the orientation of D in figures.

A chord diagram C is *homotopic* to a chord diagram C' if the based loops of C and C' are unchanged and every chord of C is homotopic to a chord in C' by a plane homotopy relative to the end point of the chord, where a homotopy of a chord linking with a based loop does not granted in a neighborhood of every based loop (see Fig. 13).

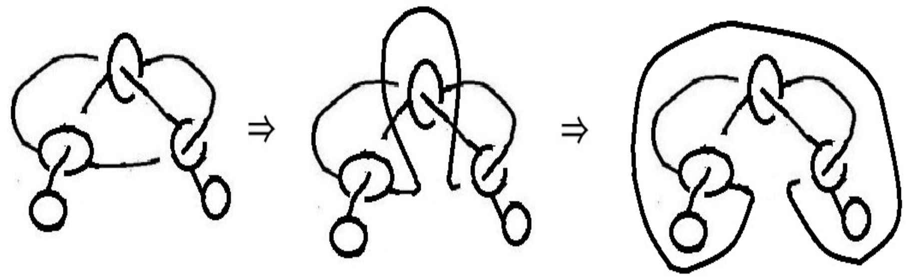

FIGURE 13. A chord homotopy

An arc diagrams D is *homotopy equivalent* to an arc diagram D' if there is a sequence of chord diagrams $C_i\,(i=0,1,\dots,m)$ such that $C_0=C(D)$, $C_m=C(D')$ and C_{i-1} is isomorphic or homotopic to C_i for every i. The homotopy equivalence on an arc diagram is an equivalence relation.

The following result is directly obtained from Lemma 2.2 (1).

Lemma 3.2. The knotting probability $p(D)$ is invariant under the homotopy equivalence of a chord diagram D.

An arc diagram D is closely related to a knotoid which is a planar research object by Turaev [**13**]. In fact, the knotting probability $p(D)$ is invariant under the Reidemeister move R_3 by Lemma 3.2 and the group presentations are identical, but should be distinct from a knotoid since Reidemeister moves R_1, R_2 make the crossing number of D change by which the denominator of $p(D)$ changes in general.

As stated in § 1, an arc diagram D is inbound if the points s and t of D are in the same region in the plane divided by D and the front region of an inbound arc diagram is this region. Any inbound arc diagram D is homotopy equivalent to an inbound arc diagram D' with the infinite front region. Thus, by Lemma 3.2, for every inbound arc diagram D, there is an inbound arc diagram D' such that $p(D')=p(D)$ and the front region of D' is the infinite region.

An *under-closed knot diagram* of an arc diagram D is a knot diagram $\mathrm{cl}_u(D)$ obtained from D by joining the starting point s and the terminal point t with an under-crossing arc.

The following theorem is a main result of this paper.

Theorem 3.3. For the knotting probability $p(D)$ of an arc diagram D, we have the following properties (1)-(4).

(1) The following property (i)-(iv) are mutually equivalent:

(i) $p(D)=0$.

(ii) $p^{\mathrm{I}}(D) = 0$.

(iii) $\pi(C(D)) \cong \mathbf{Z}$.

(iv) Any under-closed knot diagram $\mathrm{cl}_u(D)$ of D represents a trivial knot.

(v) An under-closed knot diagram $\mathrm{cl}_u(D)$ of D represents a trivial knot.

(2) The knotting probability $p(D)$ of D has $p(D) = 1$ if and only if the group $\pi(A)$ is not isomorphic to $\mathbf{Z}$ for every adjoint chord diagram A of the chord diagram $C(D)$.

(3) If D is an inbound arc diagram and D^* is the mirror image of D, then the knotting probability $p(D)$ has the following properties

$$p^{\mathrm{I}}(D) \neq \frac{1}{2}, \quad p^{\mathrm{II}}(D) = p^{\mathrm{III}}(D) \quad \text{and} \quad p(D) = p(D^*).$$

The proof of Theorem 3.3. For (1), the assertion $(i) \Rightarrow (ii)$ is obvious. To see the assertion $(ii) \Rightarrow (iii)$, let A_1 be the adjoint chord diagram of $C(D)$ with two self-attached additional chords chosen which is of type I. Then we have $\pi(A_1) \cong \pi(C(D))$. If $p^{\mathrm{I}}(D) = 0$, then A_1 is unknotted and $\pi(C(D)) \cong \mathbf{Z}$. This means that $(ii) \Rightarrow (iii)$. To see the assertion $(iii) \Rightarrow (iv)$, let $\mathrm{cl}_u(D)'$ be an inbound arc diagram obtained from the under-closed knot diagram $\mathrm{cl}_u(D)$ by cutting a part of the additional under-crossing arc. The presented group $\pi(C(D))$ is canonically isomorphic to the presented group $\pi(C(\mathrm{cl}_u(D)')$ which is canonically isomorphic to the group $\pi(\mathrm{cl}_u(D))$ of the knot diagram $\mathrm{cl}_u(D)$. This result is known as a feature of calculating the fundamental group from a knot diagram in [**1**] and was observed in a research of a knotoid by [**13**]. It is well known by Dehn's lemma that the under-closed knot diagram $\mathrm{cl}_u(D)$ represents a trivial knot if and only if the presented group $\pi(\mathrm{cl}_u(D))$ is isomorphic to $\mathbf{Z}$ (see for example [**3**]). This means that $(iii) \Rightarrow (iv)$. The assertion $(iv) \Rightarrow (v)$ is obvious. To see the assertion $(v) \Rightarrow (i)$, assume that an under-closed knot diagram $\mathrm{cl}_u(D)$ represents a trivial knot. Then $\pi(C(D)) \cong \pi(\mathrm{cl}_u(D)) \cong \mathbf{Z}$ by the argument on the assertion $(iii) \Rightarrow (iv)$. Let A be any adjoint chord diagram of $C(D)$. Since the group $\pi(A)$ is a quotient group of $\pi(C(D))$, we have $\pi(A) \cong \mathbf{Z}$. By Lemma 2.2 (2), the adjoint chord diagram A is unknotted, meaning that $(v) \Rightarrow (i)$. This shows (1).

For (2), the group $\pi(A)$ of an adjoint chord diagram A is the group obtained from the presented group $\pi(C(D))$ by identifying the generator of o_s with a based loop generator and the generator of o_t with a based loop generator. Thus, there is an adjoint chord diagram A with $\pi(A) \cong \mathbf{Z}$ (which is an unknotted chord diagram by Lemma 2.2 (2)) if and only if $p(C(D)) < 1$.

For (3), let A_1 and A_2 be the type I adjoint chord diagrams of $C(D)$. Removing a self-attached additional chord from A_1 or A_2 does not change the group $\pi(A_1)$ or $\pi(A_2)$ up to isomorphisms, respectively. For an inbound arc diagram D, removing the additional chord joining o_s with o_t from A_1 or A_2 does not change the group $\pi(A_1)$ or $\pi(A_2)$ up to isomorphisms, respectively. Thus, both the groups $\pi(A_1)$ and $\pi(A_2)$ are isomorphic to the group $\pi(C(D))$, meaning that $p(D)$ is 0 or 1 and $p(D) \neq \frac{1}{2}$.

To see that $p^{\mathrm{II}}(D) = p^{\mathrm{III}}(D)$, let $A(2)_i \,(i = 1, 2, \ldots, 2n)$ be the type II adjoint chord diagrams of $C(D)$, and $A(3)_i \,(i = 1, 2, \ldots, n)$ the type III adjoint chord diagrams of $C(D)$. Let $A(2)'_i\,(i = 1, 2, \ldots, 2n)$ be the chord diagrams obtained from $A(2)_i\,(i = 1, 2, \ldots, 2n)$ by removing the self-attached additional chord. Let $A(3)'_i\,(i = 1, 2, \ldots, n)$ be the chord diagrams of $C(D)$ obtained from $A(3)_i\,(i = 1, 2, \ldots, n)$ by removing the additional chord joining o_s with o_t. We have $\pi(A(2)_i) \cong \pi(A(2)'_i)$ for all i. Since D is inbound, we also have $\pi(A(3)_i) \cong \pi(A(3)'_i)$ for all i. The set $A(2)'_i\,(i = 1, 2, \ldots, 2n)$ is equal to the double of the set $A(3)'_i\,(i = 1, 2, \ldots, n)$, meaning that $p^{\mathrm{II}}(D) = p^{\mathrm{III}}(D)$.

To see that $p(D) = p(D^*)$, assume by Lemma 3.2 that the front region of D is the infinite region. Consider the half plane

$$\mathbf{R}^2_{\geq 0} = \{(x, 0, z) \in \mathbf{R}^2 |\, z \geq 0\}$$

in the half 3-space

$$\mathbf{R}^3_{\geq 0} = \{(x, y, z) \in \mathbf{R}^3 |\, z \geq 0\}.$$

The arc diagram D is considered as a diagram in the half plane $\mathbf{R}^2_{\geq 0}$ the points s and t of D in the boundary line

$$\partial \mathbf{R}^2_{\geq 0} = \{(x, 0, 0) \in \mathbf{R}^2\},$$

which is a diagram of a proper arc L in the upper half 3-space $\mathbf{R}^3_{\geq 0}$. As it is explained in [**5**], Yajima's construction in [**15**] says that the ribbon S^2-knot $F(C(D))$ corresponds to a ribbon structure of the spun 2-knot $\Sigma(L)$ of L in $\mathbf{R}^4$ associated to the spinning construction

$$(x, y, z) \longrightarrow (x, y, z\cos\theta, z\sin\theta), \quad 0 \leq \theta \leq 2\pi.$$

Then we see that the based loop system of the chord system $C(D)$ of D naturally corresponds to the trivial S^2-link system of the ribbon structure of $S(L)$. Let L^* be the mirror image of L in the half 3-space

$$\mathbf{R}^3_{\leq 0} = \{(x, y, z) \in \mathbf{R}^3 |\, z \leq 0\}$$

obtained by the π-rotation of L. The additional chord α_i of every adjoint chord diagram A_i of $C(D)$ is regarded as an arc joining s (or t) with an upper crossing point of D and hence α_i is a core of a 1-handle attaching to the spun 2-knot $\Sigma(L)$. By a deformation of α_i through $\Sigma(L)$, we obtain a core α'_i of a 1-handle attached to $\Sigma(L)$ regarded as an arc joining s (or t) with the corresponding upper crossing point of D (see Fig. 14).

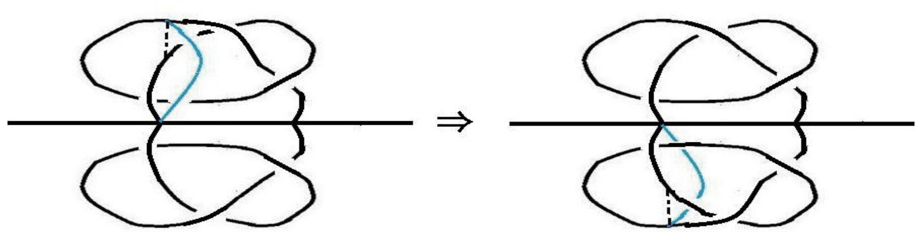

FIGURE 14. Deforming a core arc of the 1-handle obtained an additional chord

This means that the ribbon surface-knot $F(A_i)$ of every adjoint chord diagram A_i of $C(D)$ is equivalent to the ribbon surface-knot $F(A_i^*)$ of the corresponding adjoint chord diagram A_i^* of $C(D^*)$. Then the type I, II, III or IV of A_i is equal to the type of A_i^* and $F(A_i)$ is trivial if and only if $F(A_i^*)$ is trivial, for every i. Hence we have $p(D) = p(D^*)$. This completes the proof of Theorem 3.3. □

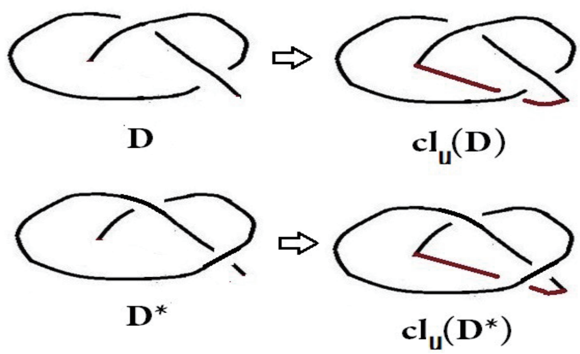

FIGURE 15. The under-closed knot diagrams $\mathrm{cl}_u(D)$ and $\mathrm{cl}_u(D^*)$ obtained from the arc diagrams D and D^* by adding an under crossing arc, respectively

For example, for the under-closed knot diagrams $\mathrm{cl}_u(D)$ and $\mathrm{cl}_u(D^*)$ of an arc diagram D and the mirror image D^* in Fig. 15, the diagram $\mathrm{cl}_u(D)$ is a diagram of the trefoil knot and the diagram $\mathrm{cl}_u(D^*)$ is a diagram of a trivial knot, so that $p(D) > 0$ and $p(D^*) = 0$ by Theorem 3.3 (1). A precise computation argument of $p(D)$ and $p(D^*)$ will be done in Example 4.2. This example gives also an example of an arc diagram with $p^{\mathrm{I}}(D) = \frac{1}{2}$ and $p^{\mathrm{II}}(D) \neq p^{\mathrm{III}}(D)$. See also Example 4.3 later for another similar example.

The *connected sum* $D \cdot D'$ of two oriented arc diagrams D and D' is an oriented arc diagram obtained by joining the terminal point t of D with the starting point v'_s of the arc diagram D' embedded in a disk neighborhood of the terminal point t of D in the plane.

Corollary 3.4. For the connected sums on arc diagrams $D_i\,(i = 1, 2, 3)$, we have (1) and (2).

(1) $p(D_1 \cdot D_2) > 0$ if and only if $p(D_1) > 0$ or $p(D_2) > 0$.

(2) If $p(D_i) > 0\,(i = 1, 2, 3)$, then $p(D_1 \cdot D_2 \cdot D_3) = 1$.

The proof of Corollary 3.4. Note that $p(D) > 0$ if and only if $\pi(C(D)) \not\cong \mathbf{Z}$. (1) is obtained from the fact that $\pi(C(D_1 \cdot D_2)) \cong \mathbf{Z}$ if and only if $\pi(C(D_1)) \cong \mathbf{Z}$ and $\pi(C(D_2)) \cong \mathbf{Z}$. For (2), let $\pi(C(D_i)) \not\cong \mathbf{Z}\,(i = 1, 2, 3)$. Let A be any adjoint chord diagram of the chord diagram $C(D_1 \cdot D_2 \cdot D_3)$. Then we see that there is an epimorphism from the group $\pi(A)$ onto the group $\pi(C(D_i))$ for some i depending on choices of an additional chord pair. Thus, $\pi(A) \not\cong \mathbf{Z}$.

For the group $\pi(C(D))$ of the chord diagram $C(D)$, let $\pi(C(D))'$ and $\pi(C(D)''$ be the first and second commutator subgroups of $\pi(C(D))$. The quotient group $\pi(C(D))/\pi(C(D))' \cong \mathbf{Z}$ and the quotient group $M(C) = \pi(C(D)'/\pi(C(D)''$ forms a finitely generated Λ-module, called the *module* of an arc diagram D, where Λ denotes the integral Laurent polynomial ring $\mathbf{Z}[t, t^{-1}]$ identical to the integral group ring $\mathbf{Z}[\mathbf{Z}]$. The following corollary gives a new criterion for an arc diagram D with $p(D) = 1$.

Corollary 3.5. If the module $M(D)$ of an arc diagram D cannot be generated by two elements over Λ, then $p(D) = 1$.

The proof of Corollary 3.5. Let A be an adjoint chord diagram of $C(D)$. The *Lambda*-module $\pi(A)'/\pi(A)''$ for the group $\pi(A)$ is a quotient Λ-module of $M(D)$ by two elements, which is not 0 by assumption. Thus, $\pi(A) \not\cong \mathbf{Z}$. □

By Corollary 3.5, we see that $p(D) = p(D^*) = 1$ for the inbound arc diagram D obtained from any diagram of any prime knot with Nakanishi index ≥ 3 by removing any open arc not meeting the crossing points (see [**3**] for the Nakanishi index).

4. Some examples

Here are some examples useful in computing the knotting probability of an arc diagram. For an inbound arc diagram D, note by Theorem 3.3 (3) that $p^{\mathrm{I}}(D) = 0$ or 1, $p^{\mathrm{II}}(D) = p^{\mathrm{III}}(D)$ and $p(D) = p(D^*)$.

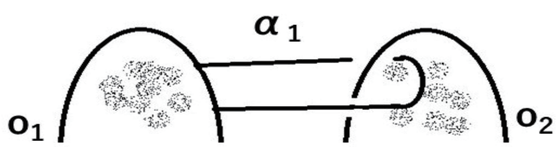

Figure 16. A chord diagram of 2 based loops with a special chord

Example 4.1.
(i) The chord diagram $C(D)$ of an arc diagram D is deformed into a chord diagram with just one based loop by a finite number of the moves M_0, M_1 and M_2, then the knotting probability $p(D) = 0$ by Theorem 3.3 (1) since $\pi(C(D) \cong \mathbf{Z}$.

(ii) A *monotone arc diagram* is an arc diagram such that a point starting from s or t meets the upper crossing point first at every crossing. If D is a monotone arc diagram, then the knotting probability $p(D) = 0$. In fact, the chord diagram $C(D)$ is deformed by the moves M_0, M_1 and M_2 into a chord diagram with just one based loop. Then the result is obtained from (i).

(iii) If the chord diagram $C(D)$ of an arc diagram D is deformed into a chord diagram of just two based loops o_1 and o_2 and chords containing one chord α_1 in 16 by a finite number of the moves M_0, M_1 and M_2, then the knotting probability

$p(D) = 0$. This is because the generators on o_1 and o_2 are commutative and hence the group $\pi(C(D)) \cong \mathbf{Z}$.

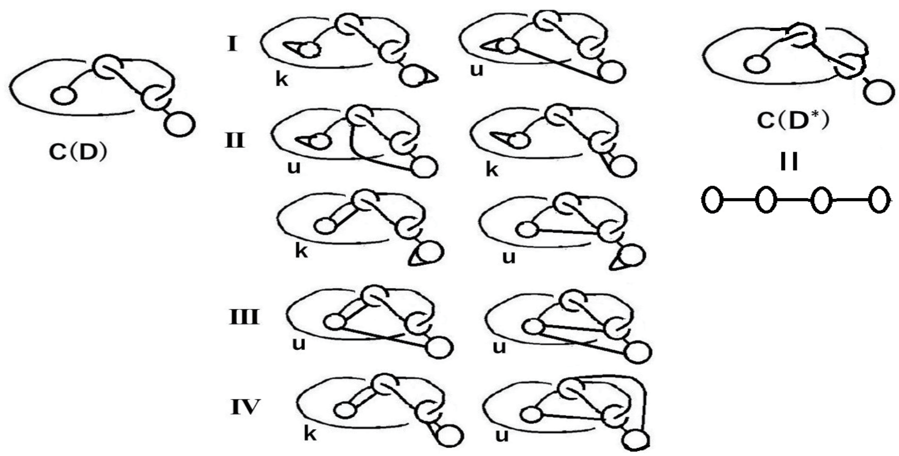

FIGURE 17. Calculations on the arc diagram D and the mirror image D^* in Fig. 1, where "u"= an unknotted chord diagram and "k"= a knotted chord diagram

Example 4.2. Among all the arc diagram with at most 2 crossings, the arc diagrams with non-zero knotting probabilities are homotopy equivalent to the non-inbound arc diagram D with 2 crossings in Fig. 1 which has the the knotting probability

$$p(D) = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2})$$

with $p^{\mathrm{II}}(D) = \frac{1}{2}$ and $p^{\mathrm{III}}(D) = 0$. The calculation on $p(D)$ is done by the adjoint chord diagrams of $C(D)$ in Fig. 17. The other chord diagrams except for the chord diagram $C(D)$ of this arc diagram D are checked to be deformed into chord diagrams without crossing points by the moves M_0, M_1 and M_2. In particular, the chord diagram $C(D^*)$ of the mirror diagram D^* of D in Fig. 1 has $p(D^*) = 0$ which is seen from Fig. 17.

Example 4.3. The arc diagram D with 3 crossings in Fig. 18 has

$$p(D) = (\frac{1}{2}, \frac{1}{3}, 0, \frac{1}{6}), \quad p(D^*) = 0.$$

The calculation on $p(D)$ is done in Fig. 18.

Example 4.4. A *reduced alternating arc diagram* is an inbound arc diagram obtained from a reduced alternating knot diagram with crossing number ≥ 3 by removing any open arc not meeting the crossing points (see [**3**]). If D is a reduced alternating arc diagram, then the knotting probability $p(D)$ of D has $p(D) > 0$. This is obtained from Theorem 3.3 (1) because the reduced alternating knot diagram $\bar{D}$ is a knot diagram of a non-trivial knot.

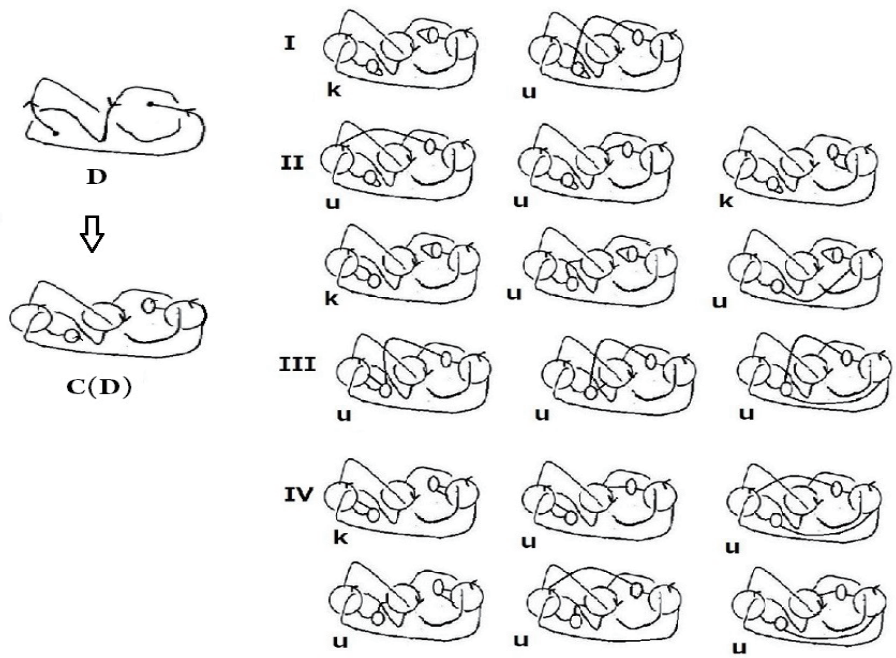

FIGURE 18. Calculations on the arc diagram D and the mirror image D^* in Fig. 18

Example 4.5. Let (a,b) be a coprime integral pair with $a,b \geq 2$ Let $T(a,b)$ be a torus knot diagram obtained by the closure of the braid $(\sigma_1\sigma_2\ldots\sigma_{a-1})^b$ (see for example [**3**]). The *torus arc diagram* of type (a,b) is the inbound arc diagram $D(a,b)$ obtained from $T(a,b)$ by removing an open arc in the outer most string of the closure strings. Then the knotting probability $p(D(a,b))$ of the torus arc diagram $D(a,b)$ of type (a,b) has $0 < p(D(a,b)) < 1$. For an actual calculation of $p(D(a,b))$, it is easier to consider the knotting probability $p(D(a,b)^*)$ of the mirror image $D(a,b)^*$ of $D(a,b)$ by Theorem 3.3 (3). For example see Fig. 19 for the torus arc diagram $D(3,q)^*$, where an unknotted adjoint chord diagram is shown. For the general torus arc diagram $D(a,b)^*$, an unknotted adjoint chord diagram is constructed by a similar attachment of the additional chords. For example, we have the following computation results:

$$
\begin{aligned}
p(D(2,3)) = p(D(2,3)^*) &= (1, \frac{1}{3}, \frac{1}{3}, 0), \\
p(D(3,2)) = p(D(3,2)^*) &= (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{6}),
\end{aligned}
$$

whose calculations on $D(2,3)^*$ and $D(3,2)^*$ are made from Fig. 20 and Fig. 21, respectively.

Example 4.6. An inbound arc diagram with n bridges is an inbound arc diagram obtained from a knot diagram with n bridges by removing an open upper bridge arc (see [**3**]). Then every inbound arc diagram D with at most 3 bridges (see [**3**]) has

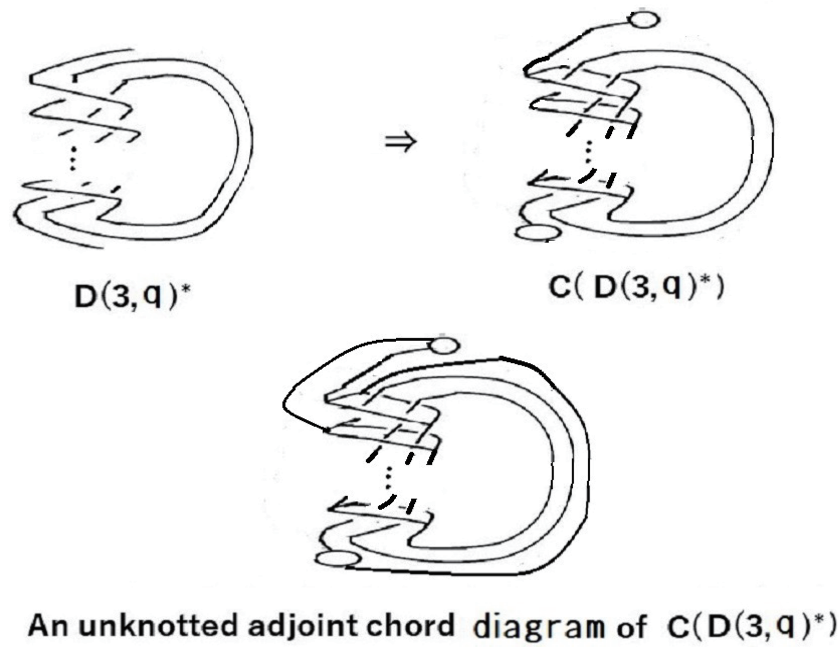

Figure 19. Attaching the additional chords to the torus arc diagram $D(3,q)^*$ to obtain an unknotted adjoint chord diagram

the knotting probability $p(D) < 1$. For this purpose, let D have 3 bridges. Consider an adjoint chord diagram A of the chord diagram $C(D)$ or the mirror image $C(D)^*$ obtained by joining o_s and o_t with two based loops near the two bridge chords. Then the chord diagram A is deformed into a chord diagram with one based loop by the moves M_0, M_1, M_2, which shows that $p(D) < 1$ by by Theorem 3.3 (3). For example, for the *figure-eight arc diagram* D in Fig. 22, we have

$$p(D) = p(D^*) = (1, \frac{1}{4}, \frac{1}{4}, 0).$$

Example 4.6. The arc diagram D with 6 crossings illustrated in Fig. 23 is an arc diagram with the knotting probability

$$p(D) = 1.$$

However, the mirror image D^* of D has

$$p(D^*) = 0.$$

The arc diagram D with 9 crossings illustrated in Fig. 24 is an inbound arc diagram with the knotting probability

$$p(D) = p(D^*) = 1.$$

These result are obtained as direct applications of Corollary 3.4 (2).

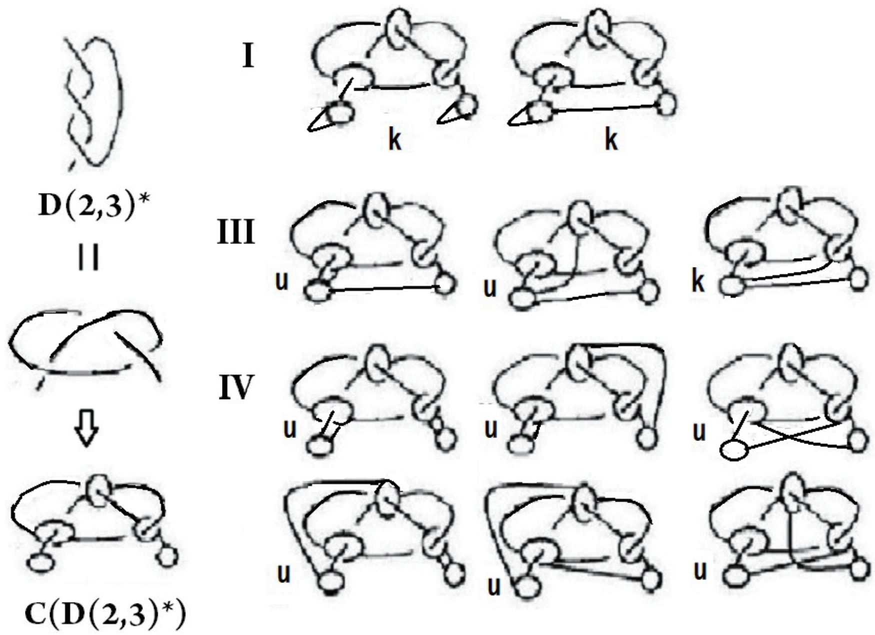

FIGURE 20. Calculations on $D(2,3)^*$

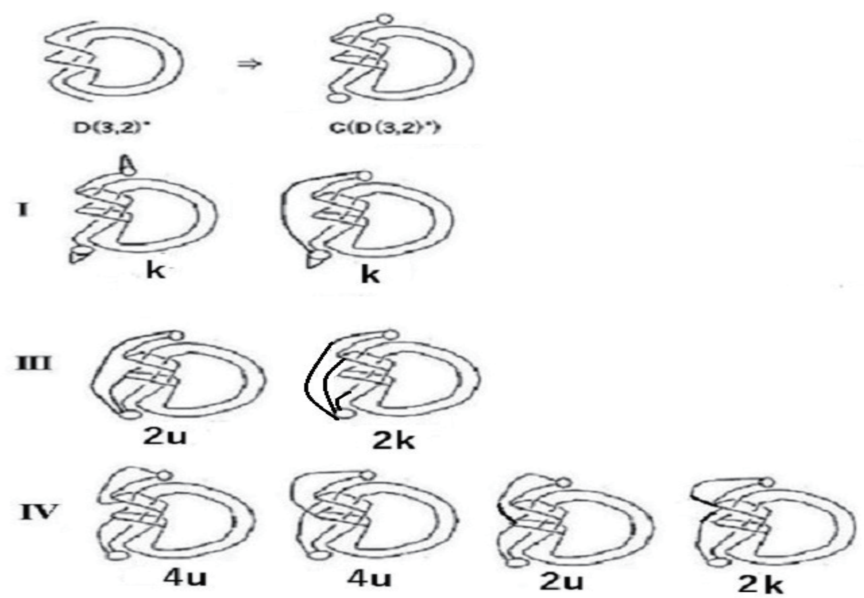

FIGURE 21. Calculations on $D(3,2)^*$

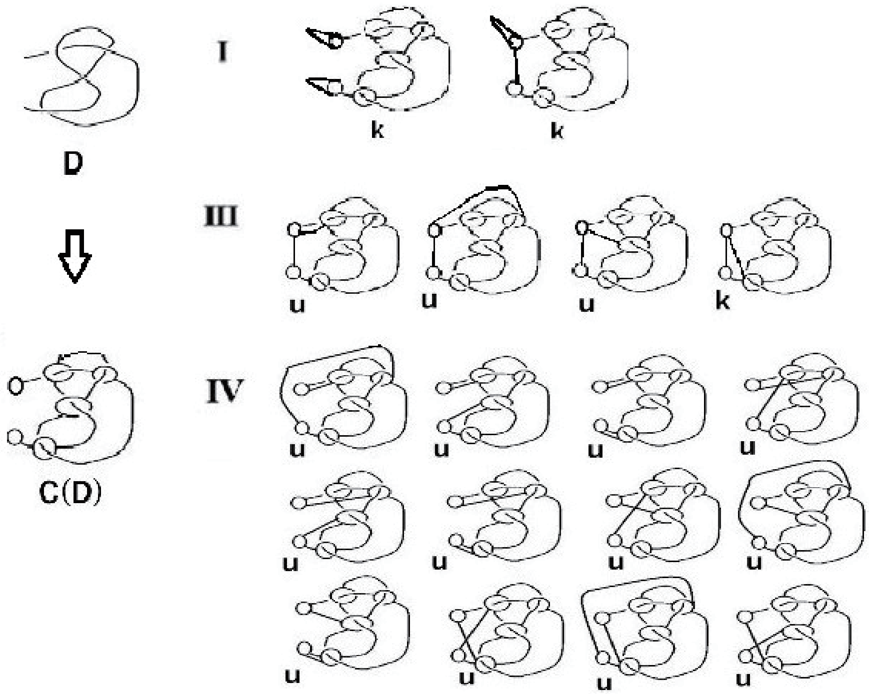

FIGURE 22. A calculation on the figure-eight arc diagram D

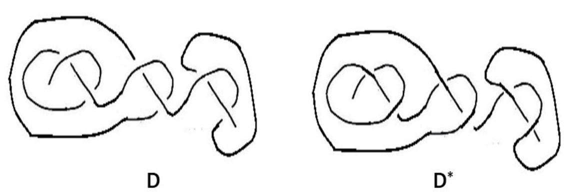

FIGURE 23. A 6 crossing arc diagram D with knotting probability $p(D) = 1$ and the mirror image D^* with $p(D^*) = 0$.

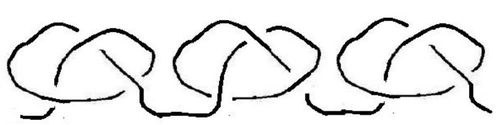

FIGURE 24. A 9 crossing arc diagram D with knotting probability $p(D) = p(D^*) = 1$.

Acknowledgements. This work was in part supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics).

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