Enumerating 3-manifolds with lengths up to 9 by a canonical order

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Abstract

On previous works, we enumerated the prime links with lengths up to 10 and the prime link exteriors with lengths up to 9. In this paper, we make an enumeration of the first 133 closed 3-manifolds which are the 3-manifolds with lengths up to 9 by using the enumeration of the prime link exteriors.

Key words: Lattice point, Length, Link, Exterior, 3-manifold, Table
PACS:

1 Introduction

In [4] we suggested a method of enumerating the prime links, the prime link exteriors and the closed connected orientable 3-manifolds. The idea is to introduce a well-order on the set of links by embedding it into a well-ordered set of lattice points. The set of these lattice points is further embedded in a set of positive rational numbers in [5]. This well-order also naturally induces a well-order on the set of prime link exteriors and eventually induces a well-order on the set of closed connected orientable 3-manifolds. By using this method, the first 28, 26 and 26 lattice points of lengths up to 7 corresponding to the prime links, the prime link exteriors and the closed connected orientable 3-manifolds

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are respectively tabulated without any computer aid in [4]. We enlarged the table of the first 28 lattice points of lengths up to 7 corresponding to the prime links into that of the first 443 lattice points of lengths up to 10 in [7] and enlarged the table of the first 26 lattice points of lengths up to 7 corresponding to the prime link exteriors into that of the first 142 lattice points of lengths up to 9. A tentative goal of this project is to enumerate the lattice points of lengths up to 10 corresponding to the closed connected orientable 3-manifolds by hand to know which invariant is useful in handcalculations. In this paper, we enumerate the first 133 lattice points of lengths up to 9 corresponding to the closed connected orientable 3-manifolds.

2 Definition of a well-order on the set of links

Let $\mathbf{Z}$ be the set of integers, and $\mathbf{Z}^n$ the product of n copies of $\mathbf{Z}$. We put

$$\mathbf{X} = \coprod_{n=1}^{\infty} \mathbf{Z}^n = \{(x_1, x_2, \ldots, x_n) \,|\, x_i \in \mathbf{Z},\ n = 1, 2, \ldots\}.$$

We call elements of $\mathbf{X}$ *lattice points*. For a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{X}$, we put $\ell(\mathbf{x}) = n$ and call it the *length* of $\mathbf{x}$. Let $|\mathbf{x}|$ and $|\mathbf{x}|_N$ be the lattice points determined from $\mathbf{x}$ by the following formulas:

$|\mathbf{x}| = (|x_1|, |x_2|, \ldots, |x_n|)$ and $|\mathbf{x}|_N = (|x_{j_1}|, |x_{j_2}|, \ldots, |x_{j_n}|)$,
where $|x_{j_1}| \leqq |x_{j_2}| \leqq \cdots \leqq |x_{j_n}|$ and $\{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\}$.

We define a well-order (called a *canonical order* [4]) on $\mathbf{X}$ as follows:

Definition 2.1. We define a well-order on $\mathbf{Z}$ by $0 < 1 < -1 < 2 < -2 < 3 < -3 < \cdots$, and for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ we define $\mathbf{x} < \mathbf{y}$ if we have one of the following conditions (1)-(4):

(1) $\ell(\mathbf{x}) < \ell(\mathbf{y})$.
(2) $\ell(\mathbf{x}) = \ell(\mathbf{y})$ and $|\mathbf{x}|_N < |\mathbf{y}|_N$ by the lexicographic order on the natural number order.
(3) $|\mathbf{x}|_N = |\mathbf{y}|_N$ and $|\mathbf{x}| < |\mathbf{y}|$ by the lexicographic order on the natural number order.
(4) $|\mathbf{x}| = |\mathbf{y}|$ and $\mathbf{x} < \mathbf{y}$ by the lexicographic order on the well-order of $\mathbf{Z}$ defined above.

For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{X}$, we put

$$\min|\mathbf{x}| = \min_{1 \leqq i \leqq n} |x_i| \quad \text{and} \quad \max|\mathbf{x}| = \max_{1 \leqq i \leqq n} |x_i|.$$

Let $\beta(\mathbf{x})$ be the $(\max|\mathbf{x}| + 1)$-string braid determined from $\mathbf{x}$ by the identity

$$\beta(\mathbf{x}) = \sigma_{|x_1|}^{\mathrm{sign}(x_1)} \sigma_{|x_2|}^{\mathrm{sign}(x_2)} \cdots \sigma_{|x_n|}^{\mathrm{sign}(x_n)},$$

where we define $\sigma_{|0|}^{\mathrm{sign}(0)} = 1$. We note that $\max|\mathbf{x}| + 1$ is the minimum string number of the braid indicated by the right-hand side of the identity. Let $\mathrm{cl}\beta(\mathbf{x})$ be the closure of the braid $\beta(\mathbf{x})$. Let $\mathbf{L}$ be the set of all links modulo equivalence, where two links are *equivalent* if there is a (possibly orientation-reversing) homeomorphism sending one to the other. Then we have a map

$$\mathrm{cl}\beta : \mathbf{X} \to \mathbf{L}$$

sending $\mathbf{x}$ to $\mathrm{cl}\beta(\mathbf{x})$. By Alexander's braiding theorem, the map $\mathrm{cl}\beta$ is surjective. For $L \in \mathbf{L}$, we define a map

$$\sigma : \mathbf{L} \to \mathbf{X}$$

by $\sigma(L) = \min\{\mathbf{x} \in \mathbf{X} \,|\, \mathrm{cl}\beta(\mathbf{x}) = L\}$. Then σ is a right inverse of $\mathrm{cl}\beta$ and hence is injective. Now we have a well-order on $\mathbf{L}$ by the following definition:

Definition 2.2. For $L, L' \in \mathbf{L}$, we define $L < L'$ if $\sigma(L) < \sigma(L')$.

For a link $L \in \mathbf{L}$, we call $\ell(\sigma(L))$ the *length* of L.

3 A method of a tabulation of prime links and prime link exteriors

Let $\mathbf{L}^{\mathrm{p}}$ be the subset of $\mathbf{L}$ consisting of the prime links, where we consider that the 2-component trivial link is not prime. We use the injection σ for our method of a tabulation of $\mathbf{L}^{\mathrm{p}}$. For $k \in \mathbf{Z}$, let k^n and $-k^n$ be the lattice points determined by

$$k^n = (\underbrace{k, k, \ldots, k}_{n}) \quad \text{and} \quad -k^n = (-k)^n,$$

respectively.

For $\mathbf{x} = (x_1, x_2, \ldots, x_n),\ \mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbf{X}$, let $\mathbf{x}^T$, $-\mathbf{x}$, $(\mathbf{x},\ \mathbf{y})$ and $\delta(\mathbf{x})$ be the lattice points determined by the following formulas:

$$\mathbf{x}^T = (x_n, \ldots, x_2, x_1), \qquad -\mathbf{x} = (-x_1, -x_2, \ldots, -x_n),$$

$$(\mathbf{x},\ \mathbf{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_m),\ \ \delta(\mathbf{x}) = (x'_1, x'_2, \ldots, x'_n),$$

where
$$x'_i = \begin{cases} \mathrm{sign}(x_i)(\max|\mathbf{x}| + 1 - |x_i|) & (x_i \neq 0) \\ 0 & (x_i = 0). \end{cases}$$

A point of our argument on a tabulation of prime links is to define some transformations between lattice points. We make this definition as follows:

Definition 3.1. Let $\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w} \in \mathbf{X},\ k,\ l,\ n \in \mathbf{Z}$ with $n > 0$ and $\varepsilon = \pm 1$. An *elementary transformation* on lattice points is one of the following operations (1)-(12) and their inverses $(1)^{-}$-$(12)^{-}$.

(1) $(\mathbf{x},\ k,\ -k,\ \mathbf{y}) \to (\mathbf{x},\ \mathbf{y})$
(2) $(\mathbf{x},\ k,\ \mathbf{y}) \to (\mathbf{x},\ \mathbf{y})$, where $|k| > \max|\mathbf{x}|,\ \max|\mathbf{y}|$.
(3) $(\mathbf{x},\ k,\ l,\ \mathbf{y}) \to (\mathbf{x},\ l,\ k,\ \mathbf{y})$, where $|k| > |l| + 1$ or $|l| > |k| + 1$.
(4) $(\mathbf{x},\ \varepsilon k^n,\ k+1,\ k,\ \mathbf{y}) \to (\mathbf{x},\ k+1,\ k,\ \varepsilon(k+1)^n,\ \mathbf{y})$, where $k(k+1) \neq 0$.
(5) $(\mathbf{x},\ k,\ \varepsilon(k+1)^n,\ -k,\ \mathbf{y}) \to (\mathbf{x},\ -(k+1),\ \varepsilon k^n,\ k+1,\ \mathbf{y})$, where $k(k+1) \neq 0$.
(6) $(\mathbf{x},\ \mathbf{y}) \to (\mathbf{y},\ \mathbf{x})$

(7) $\mathbf{x} \to \mathbf{x}^T$
(8) $\mathbf{x} \to -\mathbf{x}$
(9) $\mathbf{x} \to \delta(\mathbf{x})$
(10) $(1^n,\ \mathbf{x},\ \varepsilon,\ \mathbf{y}) \to (1^n,\ \mathbf{y},\ \varepsilon,\ \mathbf{x})$, where $\min|\mathbf{x}| \geqq 2$ and $\min|\mathbf{y}| \geqq 2$.
(11) $(k^2,\ \mathbf{x},\ \mathbf{y},\ -k^2,\ \mathbf{z},\ \mathbf{w}) \to (-k^2,\ \mathbf{x},\ \mathbf{w}^T,\ k^2,\ \mathbf{z},\ \mathbf{y}^T)$, where $\max|\mathbf{x}| < k < \min|\mathbf{y}|$, $\max|\mathbf{z}| < k < \min|\mathbf{w}|$ and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ or $\mathbf{w}$ may be empty.
(12) $(\mathbf{x},\ k,\ (k+1)^2,\ k,\ \mathbf{y}) \to (\mathbf{x},\ -k,\ -(k+1)^2,\ -k,\ \mathbf{y}^T)$, where $\max|\mathbf{x}| < k < \min|\mathbf{y}|$ and $\mathbf{x}$ or $\mathbf{y}$ may be empty.

A meaning of the transformations of Definition 3.1 is given by the following lemma (See [4,7]):

Lemma 3.2. If a lattice point $\mathbf{x}$ is transformed into a lattice point $\mathbf{y}$ by an elementary transformation, then we have $\mathrm{cl}\beta(\mathbf{x}) = \mathrm{cl}\beta(\mathbf{y})$ (modulo a split union of a trivial link for (1), (2), (9)).

The outline of a tabulation of prime links is the following (See [4,7] for the details): Let Δ be the subset of $\mathbf{X}$ consisting of 0, 1^m for $m \geqq 2$ and $(x_1, x_2, \ldots, x_n)$, where $n \geqq 4$, $x_1 = 1$, $1 \leqq |x_i| \leqq \frac{n}{2}$, $|x_n| \geqq 2$ and $\{|x_1|, |x_2|, \ldots, |x_n|\} = \{1, 2, \ldots, \max|\mathbf{x}|\}$. Then we have $\sharp\{\mathbf{y} \in \Delta | \mathbf{y} < \mathbf{x}\} < \infty$ for every $\mathbf{x} \in \Delta$ and have $\sigma(\mathbf{L}^{\mathrm{p}}) \subset \Delta$. First, we enumerate the lattice points of Δ under the canonical order and then we omit $\mathbf{x} \in \Delta$ from the sequence if $\mathrm{cl}\beta(\mathbf{x})$ is a non-prime link or a link which has already appeared in the table of prime links. By using Lemma 3.2, we see that if $\mathbf{x}$ is transformed into a smaller one, then $\mathbf{x}$ must be removed from the sequence. We can find most of the omittable lattice points in this way. In [7], we show a table of prime links with lengths up to 10.

Next we enumerate the prime link exteriors with lengths up to 9. Since a knot is determined by its exterior by the Gordon-Luecke Theorem [2], we classify the exteriors of two or more component links. We obtain a table of prime link exteriors, by omitting $7^2_7, 7^2_8, 8^3_7, 8^3_8, 8^2_{16}, 8^2_{15}, 9^2_{43}, 9^2_{44}, 9^2_{49}, 9^3_{13}, 9^3_{14}, 9^3_{19}, 9^3_{18}, 9^3_{17}$ from the table in [7] and replacing the rest of the links with their exteriors because the exteriors of the above 14 links have already appeared (See [8]). So we have the following table of prime link exteriors:

$O < 2^2_1 < 3_1 < 4^2_1 < 4_1 < 5_1 < 5^2_1 < 6^2_1 < 5_2 < 6_2 < 6^3_3 < 6^3_1 < 6_3 < 6^3_2 < 6^2_3 < 7_1 < 6^2_2 < 7^2_1 < 7^2_4 < 7^2_2 < 7^2_5 < 7^2_6 < 6_1 < 7_6 < 7_7 < 7^3_1 < 8^2_1 < 7_3 < 8_2 < 8^3_1 < 8_{19} < 8_{20} < 8_5 < 7_5 < 8_7 < 8_{21} < 8_{10} < 8^3_9 < 8^3_5 < 8_{16} < 8_9 < 8^3_2 < 8_{17} < 8^3_6 < 8^3_{10} < 8^3_4 < 8_{18} < 7^2_3 < 8^2_5 < 8^2_9 < 8^2_8 < 8^2_{12} < 8^2_{13} < 8^2_7 < 8^2_{10} < 8^2_{11} < 8^4_3 < 8^4_2 < 8^4_1 < 8^2_{14} < 8_{12} < 9_1 < 8^2_2 < 9^2_1 < 9^2_{13} < 9^2_{51} < 9^2_{19} < 9^2_{50} < 8^2_3 < 9^2_2 < 9^2_{52} < 9^2_{20} < 9^2_{55} < 9^2_{31} < 9^2_{53} < 9^2_{54} < 8^2_4 < 9^2_{23} < 9^2_{57} < 9^2_{35} < 9^2_{40} < 9^2_5 < 9^2_{14} < 9^2_{21} < 9^2_{34} < 9^2_{37} < 9^2_{59} < 9^2_{29} < 9^2_{39} < 9^2_{61} < 9^2_{41} < 9^2_{42} < 8_6 < 9_{11} < 9_{43} < 9_{44} < 9_{36} < 9_{42} < 7_2 < 8_{14} < 9_{26} < 8_4 < 8^3_3 < 9^3_6 < 9^3_2 < 9^3_8 < 9_{45} < 9_{32} < 9^3_{11} < 8_8 < 9_{20} < 9^3_1 < 7_4 < 8_{11} < 9_{27} < 8_{13} < 8_{15} < 9_{24} < 9_{30} < 9^3_{16} < 9^3_{15} < 9^3_4 < 9^3_{10} < 9^3_{20} < 9^3_{12} < 9^3_{21} < 9_{33} < 9_{46} < 9_{34} < 9_{47} < 9_{31} < 9_{28} < 9_{40} < 9^2_{11} < 9_{17} < 9_{22} < 9^3_5 < 9^3_9 < 9_{29} < 9^2_{12} < 8^2_6 < 9^2_{25}$.

Table 1

4 A method of a tabulation of 3-manifolds

The exteriors of the prime links in Table 1 are all simple manifolds and hence are hyperbolic or special Seifert manifolds. Thus, the fundamental groups of these prime link exteriors are mutually non-isomorphic (cf. Proposition (4.6) of [4]), so that the prime links in Table 1 are π-minimal links, where a π-minimal link means a prime link whose fundamental group is the first appearing group up to isomorphisms in the canonical ordered set of prime links. We make a list of closed connected orientable 3-manifolds by constructing a sequence of 3-manifolds obtained by the 0-surgery of the links in Table 1 and removing the manifolds which have already appeared (See [4]). Let $\chi(L,0)$ denote the manifold obtained by the 0-surgery of a link L. We classify $\chi(L,0)$ for L in Table 1 according to the first homology group $H_1(\chi(L,0))$. There are 10 types of groups 0, $\mathbf{Z}$, $\mathbf{Z}\oplus\mathbf{Z}$, $\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$, $\mathbf{Z}\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2$, $\mathbf{Z}_2$, $\mathbf{Z}_2\oplus\mathbf{Z}_2$, $\mathbf{Z}_3\oplus\mathbf{Z}_3$, $\mathbf{Z}_4$, $\mathbf{Z}_4\oplus\mathbf{Z}_4$ and we have respectively 16, 62, 16, 4, 5, 7, 15, 7, 5, 5 links with these types of groups.

Case 1. $H_1(\chi(L,0)) \cong 0$.

We enumerate the manifolds with $H_1(\chi(L,0)) \cong 0$. The links with this condition are the following:

$2_1^2 < 7_1^2 < 7_2^2 < 8_8^2 < 8_7^2 < 9_{19}^2 < 9_{50}^2 < 9_{52}^2 < 9_{54}^2 < 9_{35}^2 < 9_{21}^2 < 9_{34}^2 < 9_{39}^2 < 9_{42}^2 < 9_{11}^2 < 9_{12}^2$.

For each link L, we calculate $\tau_5(\chi(L,0))$, the 3-manifold invariant of Witten and Reshetikhin-Turaev in Kirby and Melvin's paper [9]:

$$\tau_5(\chi(2_1^2,0)) = C(1+(s^{-2}+2+s^2)\cdot 1),$$
$$\tau_5(\chi(7_1^2,0)) = C(1+(s^{-2}+2+s^2)(2+3s-2s^2+s^3)),$$
$$\tau_5(\chi(7_2^2,0)) = C(1+(s^{-2}+2+s^2)(2s+2s^2-s^3)),$$
$$\tau_5(\chi(8_8^2,0)) = C(1+(s^{-2}+2+s^2)(1+5s^2-5s^3)),$$
$$\tau_5(\chi(8_7^2,0)) = C(1+(s^{-2}+2+s^2)(1+5s-5s^2)),$$
$$\tau_5(\chi(9_{19}^2,0)) = C(1+(s^{-2}+2+s^2)(3-4s+s^2+2s^3)),$$
$$\tau_5(\chi(9_{50}^2,0)) = C(1+(s^{-2}+2+s^2)(2s-3s^2-s^3)),$$
$$\tau_5(\chi(9_{52}^2,0)) = C(1+(s^{-2}+2+s^2)(-3+3s-2s^2+s^3)),$$
$$\tau_5(\chi(9_{54}^2,0)) = C(1+(s^{-2}+2+s^2)(-3s+2s^2-s^3)),$$
$$\tau_5(\chi(9_{35}^2,0)) = C(1+(s^{-2}+2+s^2)(7s-8s^2+4s^3)),$$
$$\tau_5(\chi(9_{21}^2,0)) = C(1+(s^{-2}+2+s^2)(-2+6s-4s^2+2s^3)),$$
$$\tau_5(\chi(9_{34}^2,0)) = C(1+(s^{-2}+2+s^2)(-3+3s+3s^2-4s^3)),$$
$$\tau_5(\chi(9_{39}^2,0)) = C(1+(s^{-2}+2+s^2)(-3+8s-7s^2+s^3)),$$
$$\tau_5(\chi(9_{42}^2,0)) = C(1+(s^{-2}+2+s^2)(-4+5s+5s^2-5s^3)),$$
$$\tau_5(\chi(9_{11}^2,0)) = C(1+(s^{-2}+2+s^2)(-3+8s-7s^2+s^3)),$$

$\tau_5(\chi(9^2_{12},0)) = C(1+(s^{-2}+2+s^2)(-3s+7s^2-6s^3))$,

where $C = \frac{4}{5}\sin^2\frac{\pi}{5}$, $s = e^{\frac{\pi i}{5}}$, and we conclude that the manifolds are different from each other except the following three cases:

$$\tau_5(\chi(9^2_{52},0)) \equiv \tau_5(\chi(9^2_{54},0)),\ \tau_5(\chi(9^2_{34},0)) \equiv \tau_5(\chi(9^2_{12},0)),$$
$$\tau_5(\chi(9^2_{39},0)) \equiv \tau_5(\chi(9^2_{11},0)),$$

where $\alpha \equiv \beta$ means $\alpha = \beta$ or $\alpha = \bar{\beta}$ for complex values α, β and we say α is equivalent to β. For these manifolds, we compute $\tau_7(\chi(L,0))$ in [9]:

$$\tau_7(\chi(9^2_{52},0)) = C(1-2[3]+[3]^2+[2](-3+s-2s^2+s^3+2s^4)),$$
$$\tau_7(\chi(9^2_{54},0)) = C(1-2[3]+[3]^2+[2](5-s+8s^2-3s^3-6s^5)),$$
$$\tau_7(\chi(9^2_{34},0)) = C(1-2[3]+[3]^2+[2](-3+8s+5s^2-6s^3+9s^4-14s^5)),$$
$$\tau_7(\chi(9^2_{12},0)) = C(1-2[3]+[3]^2+[2](-2-8s+15s^2-17s^3+7s^4-6s^5)),$$
$$\tau_7(\chi(9^2_{39},0)) = C(1-2[3]+[3]^2+[2](-3+8s-16s^2+15s^3-12s^4+7s^5)),$$
$$\tau_7(\chi(9^2_{11},0)) = C(1-2[3]+[3]^2+[2](4+8s-9s^2+15s^3-5s^4)),$$

where $C = \frac{4}{7}\sin^2\frac{\pi}{7}$, $[2] = s^{-1}+s$, $[3] = s^{-2}+1+s^2$, $s = e^{\frac{\pi i}{7}}$. Since the values for the above three pairs are not equivalent to each other, we have the enumeration of 3-manifolds in Case 1.

Case 2. $H_1(\chi(L,0)) \cong \mathbf{Z}$.

We enumerate the manifolds with $H_1(\chi(L,0)) \cong \mathbf{Z}$. The links with this condition are the following:

$O < 3_1 < 4_1 < 5_1 < 5_2 < 6_2 < 6_3 < 7_1 < 6_1 < 7_6 < 7_7 < 7_3 < 8_2 < 8_{19} < 8_{20} < 8_5 < 7_5 < 8_7 < 8_{21} < 8_{10} < 8^3_5 < 8_{16} < 8_9 < 8_{17} < 8^3_6 < 8_{18} < 8_{12} < 9_1 < 8_6 < 9_{11} < 9_{43} < 9_{44} < 9_{36} < 9_{42} < 7_2 < 8_{14} < 9_{26} < 8_4 < 9^3_2 < 9_{45} < 9_{32} < 8_8 < 9_{20} < 9^3_1 < 7_4 < 8_{11} < 9_{27} < 8_{13} < 8_{15} < 9_{24} < 9_{30} < 9^3_{10} < 9_{33} < 9_{46} < 9_{34} < 9_{47} < 9_{31} < 9_{28} < 9_{40} < 9_{17} < 9_{22} < 9_{29}$.

We see that $\chi(6_3,0) \cong \chi(9^3_2,0)$, $\chi(6_2,0) \cong \chi(9^3_1,0)$ and $\chi(8^3_5,0) \cong \chi(9_{46},0)$. So we omit $\chi(9^3_2,0)$, $\chi(9^3_1,0)$ and $\chi(9_{46},0)$ from the sequence. For the rest of the links, we can see, by calculating the Alexander polynomials or Alexander modules, that the manifolds are different from each other except the following two cases:

$$\Delta(\chi(O,0)) = \Delta(\chi(9^3_{10},0)),\ \Delta(\chi(9_{28},0)) = \Delta(\chi(9_{29},0)).$$

However, we have

$$\chi(O,0) \not\cong \chi(9^3_{10},0),\ \chi(9_{28},0) \not\cong \chi(9_{29},0)$$

by the following discussion. For the first case, we transform the framed link 9^3_{10} with coefficient 0 into a framed knot K with coefficient 0 by the Kirby calculus on handle slides. We see that K is a non-trivial knot by $c_0(K;x) = 1+(x-1)^2(x^2+2x-3)$, where $c_0(K;x)$ is a coefficient polynomial of the HOMFLY polynomial $P(K;\ell,m)$ and is easily computed (See the section 1 of [3]). Then we have $\chi(9^3_{10},0) \cong \chi(K,0) \not\cong \chi(O,0)$ by Gabai's positive answer to the Property R conjecture [1]. For the second case, we substitute the fifth roots of unity for the Jones polynomials of 9_{28} and 9_{29} and we have

$$J_{9_{28}}(\omega) = -5 - 10\omega + 3\omega^2 - 12\omega^3, \; J_{9_{29}}(\omega) = -3 + 9\omega - 6\omega^2 + 6\omega^3,$$

where ω is any one of the fifth roots of unity. We see that $J_{9_{28}}(\omega) \neq J_{9_{29}}(\omega')$ for any of the fifth roots of unity ω, ω' and we have $\chi(9_{28}, 0) \not\cong \chi(9_{29}, 0)$ by Kirby and Melvin's theorem [9],p.530. This completes the enumeration of 3-manifolds in Case 2.

Case 3. $H_1(\chi(L, 0)) \cong \mathbf{Z} \oplus \mathbf{Z}$.

We enumerate the manifolds with $H_1(\chi(L, 0)) \cong \mathbf{Z} \oplus \mathbf{Z}$. The links with this condition are the following:

$5^2_1 < 7^2_4 < 7^2_6 < 7^2_3 < 8^2_{12} < 8^2_{13} < 8^2_{10} < 8^4_3 < 8^4_1 < 9^2_{13} < 9^2_{55} < 9^2_{31} < 9^2_5 < 9^2_{37} < 9^2_{41} < 9^2_{25}$.

We see that $\chi(5^2_1, 0) \cong \chi(8^4_1, 0)$ and $\chi(8^2_{12}, 0) \cong \chi(8^2_{10}, 0)$, so we omit $\chi(8^4_1, 0)$ and $\chi(8^2_{10}, 0)$ from the sequence. The latter homeomorphism can be shown as follows: for $L = 8^2_{12}$ or 8^2_{10}, we have $S^2 \times S^1$ as the 0-surgery space of one component of L and we can move the other 0-framed component of 8^2_{12} to that of 8^2_{10} within $S^2 \times S^1$, so we have the homeomorphism. For the rest of the links, we compute the two-variable Alexander polynomials:

$\Delta(\chi(5^2_1, 0)) = 1,$ $\quad \Delta(\chi(7^2_4, 0)) = 1 + t_1^2,$
$\Delta(\chi(7^2_6, 0)) = 1 - t_1 + t_1^2,$ $\quad \Delta(\chi(7^2_3, 0)) = 2,$
$\Delta(\chi(8^2_{12}, 0)) = (1 - t_1)^2,$ $\quad \Delta(\chi(8^2_{13}, 0)) = 1 - 3t_2 + t_2^2,$
$\Delta(\chi(8^4_3, 0)) = 0,$ $\quad \Delta(\chi(9^2_{13}, 0)) = 1 + t_1^2 + t_1^4,$
$\Delta(\chi(9^2_{55}, 0)) = 1-t_1+t_1^2,$ $\quad \Delta(\chi(9^2_{31}, 0)) = 1-t_1+t_1^2-t_1^3+t_1^4,$
$\Delta(\chi(9^2_5, 0)) = (1+t_1t_2)^2,$ $\quad \Delta(\chi(9^2_{37}, 0)) = (1+t_1^2)(1-t_1+t_1^2),$
$\Delta(\chi(9^2_{41}, 0)) = 1 - t_2 + 3t_1t_2 - t_1^2t_2 + t_1^2t_2^2,$ $\quad \Delta(\chi(9^2_{25}, 0)) = 1 - 4t_1 + t_1^2,$

and conclude that the manifolds are different from each other except the case

$$\Delta(\chi(7^2_6, 0)) = \Delta(\chi(9^2_{55}, 0)),$$

by using the following lemma [4,8].

Lemma 4.1. Let M, M' be closed connected orientable 3-manifolds with $H_1(M) \cong H_1(M') \cong \mathbf{Z}^r$ and $\Delta_M(t_1, t_2, \ldots, t_r)$, $\Delta_{M'}(t_1, t_2, \ldots, t_r)$ their Alexander polynomials. If there is a homeomorphism $h : M \to M'$, then there is an automorphism ψ of the multiplicative free abelian group $\langle t_1, t_2, \ldots, t_r \rangle$ with basis t_i $(i = 1, 2, \ldots, r)$ such that

$$\Delta_M(t_1, t_2, \ldots, t_r) = \pm t_1^{s_1} t_2^{s_2} \cdots t_r^{s_r} \Delta_{M'}(\psi(t_1), \psi(t_2), \ldots, \psi(t_r))$$

for some integers s_i $(i = 1, 2, \ldots, r)$.

By computing $\tau_5(\chi(L, 0))$ for $L = 7^2_6,\ 9^2_{55}$, we have $\chi(7^2_6, 0) \not\cong \chi(9^2_{55}, 0)$. In fact

$$\tau_5(\chi(7^2_6, 0)) = C\left(1 + (q^{-1} + 2 + q)(7 + 3q + q^2 + 5q^3)\right),$$
$$\tau_5(\chi(9^2_{55}, 0)) = C\left(1 + (q^{-1} + 2 + q)(-1 + 2q - 3q^2 + 3q^3)\right),$$

where $C = \frac{8}{5}\sin^2\frac{\pi}{5}$, $q = e^{\frac{2\pi i}{5}}$. We note here that some double covering spaces of $\chi(7^2_6, 0)$ and $\chi(9^2_{55}, 0)$ associated with the monodromy homomorphisms sending the meridians of trivial components of 7^2_6 and 9^2_{55} to $1 \in \mathbf{Z}_2$ and the meridians of the other components of 7^2_6 and 9^2_{55} to $0 \in \mathbf{Z}_2$ are homeomorphic.

Case 4. $H_1(\chi(L,0)) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$.

The links with this condition are the following:

$$6^3_2 < 9^3_{12} < 9^3_{21} < 9^3_9.$$

Their three-variable Alexander polynomials are

$$\Delta(\chi(6^3_2,0)) = 1, \ \ \Delta(\chi(9^3_{12},0)) = (1-t_1)^2,$$
$$\Delta(\chi(9^3_{21},0)) = 0, \ \ \Delta(\chi(9^3_9,0)) = 1 - t_1 + t_1^2,$$

and we see that the manifolds are different from each other by using Lemma 4.1.

Case 5. $H_1(\chi(L,0)) \cong \mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

The links with this condition are the following:

$$8^3_9 < 8^3_{10} < 8^3_4 < 9^3_8 < 9^3_{20}.$$

We see that $\chi(8^3_4,0) \cong \chi(9^3_{20},0)$ and remove $\chi(9^3_{20},0)$ from the sequence. This can be shown as follows. We write $8^3_4 = K_1 \cup K_2 \cup K_3$, where $K_1 \cup K_2$ is trivial and $|\mathrm{lk}(K_1,K_3)| = |\mathrm{lk}(K_2,K_3)| = 2$. First we transform the framed link 9^3_{20} with coefficient 0 into a 3-component framed link $L' = K'_1 \cup K'_2 \cup K'_3$ with coefficient 0 by the Kirby calculus on a handle slide so that $K'_1 \cup K'_2$ is trivial and $|\mathrm{lk}(K'_1,K'_3)| = |\mathrm{lk}(K'_2,K'_3)| = 2$. We can move K_3 to K'_3 within $(S^2 \times S^1)\sharp(S^2 \times S^1) \cong \chi(K_1 \cup K_2, 0) \cong \chi(K'_1 \cup K'_2, 0)$ and have $\chi(8^3_4,0) \cong \chi(9^3_{20},0)$.

For each link L except 9^3_{20}, we compute the first homology groups of the seven double covering spaces of $\chi(L,0)$ and we distinguish the manifolds.

Case 6. $H_1(\chi(L,0)) \cong \mathbf{Z}_2$

There are 7 links with this condition:

$$6^3_3 < 6^3_1 < 7^3_1 < 8^3_3 < 9^3_{16} < 9^3_{15} < 9^3_4.$$

We see that $\chi(8^3_3,0) \cong \chi(9^3_{16},0)$ and $\chi(9^3_{16},0)$ is omitted from our sequence. For the rest of the manifolds, we compute the first homology groups of the double covering spaces and we have the manifolds except $\chi(7^3_1,0)$ and $\chi(9^3_{15},0)$ are mutually distinct. We, however, see $\chi(7^3_1,0) \cong \chi(4_1,2)$, $\chi(9^3_{15},0) \cong \chi(5_1,2)$ and the linkings of the first homology groups of the double covering spaces of $\chi(4_1,2)$ and $\chi(5_1,2)$ are $\pm\frac{2}{5}$ and $\pm\frac{1}{5}$ respectively, which are non-isomorphic by [6], so we distinguish $\chi(7^3_1,0)$ from $\chi(9^3_{15},0)$.

Case 7. $H_1(\chi(L,0)) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$

There are 15 links with this condition:

$4^2_1 < 6^2_3 < 7^2_5 < 8^2_9 < 8^2_{11} < 8^4_2 < 8^2_{14} < 9^2_1 < 9^2_2 < 9^2_{23} < 9^2_{57} < 9^2_{14} < 9^2_{59} < 9^2_{29} < 8^2_6$.

We see that $\chi(4_1^2, 0) \cong \chi(8_2^4, 0)$ and we omit $\chi(8_2^4, 0)$ from our sequence. For each link L except 8_2^4, we compute the first homology groups of the three double covering spaces of $\chi(L, 0)$, and we conclude that the manifolds are different from each other.

Case 8. $H_1(\chi(L, 0)) \cong \mathbf{Z}_3 \oplus \mathbf{Z}_3$

There are 7 links with this condition:

$$6_1^2 < 6_2^2 < 8_5^2 < 9_{51}^2 < 8_3^2 < 9_{20}^2 < 9_{40}^2.$$

M. Shimozawa proves that $\chi(6_1^2, 0) \cong \chi(9_{51}^2, 0)$ by using the Kirby-Rolfsen moves, so we omit $\chi(9_{51}^2, 0)$ from our sequence. We show his proof in Appendix. For each link L except 9_{51}^2, we compute the first homology groups of four $\mathbf{Z}_3$-covering spaces of $\chi(L, 0)$ and we distinguish the manifolds.

Case 9. $H_1(\chi(L, 0)) \cong \mathbf{Z}_4$.

We have 5 links with this condition:

$$8_1^3 < 8_2^3 < 9_6^3 < 9_{11}^3 < 9_5^3.$$

By computing the first homology groups of the double covering spaces of the manifolds, we see that the manifolds are different from each other.

Case 10. $H_1(\chi(L, 0)) \cong \mathbf{Z}_4 \oplus \mathbf{Z}_4$.

$$8_1^2 < 8_2^2 < 9_{53}^2 < 8_4^2 < 9_{61}^2$$

hold this condition and we see that they are mutually distinct by computing the first homology groups of the three double covering spaces.

We unify the cases 1 to 10 and show an enumeration of the closed connected orientable 3-manifolds with lengths up to 9 in Table 2. In the table, the M-line denotes the order of the manifold $\chi(L, 0)$ given by the link L with $\sigma(L) = \mathbf{x}$ (See Section 2 for the definition of σ).

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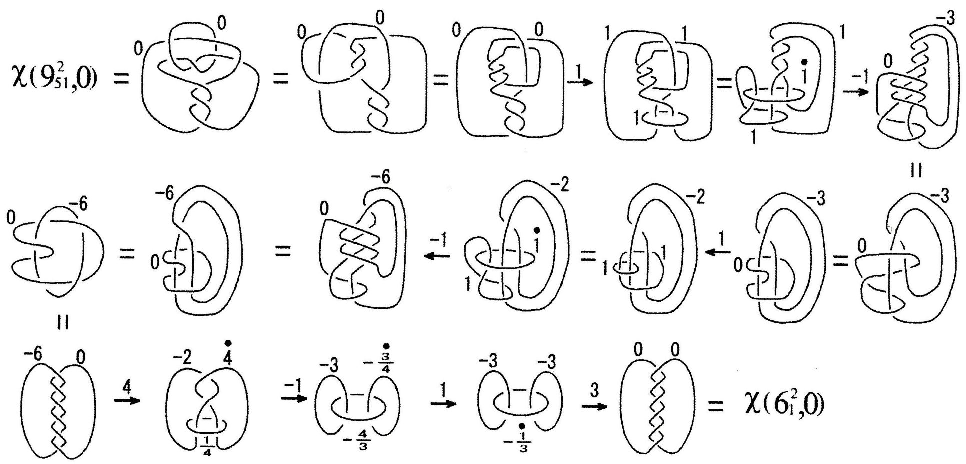
Appendix

We show M. Shimozawa's proof of the fact that $\chi(6_1^2, 0) \cong \chi(9_{51}^2, 0)$. In the figure, $\dot{\alpha}$ and $\underrightarrow{\beta}$ (or $\underleftarrow{\beta}$) for a rational number α and an integer β mean the β right-hand twists about the component labeled $\dot{\alpha}$.



M	$\mathbf{x}$	L	M	$\mathbf{x}$	L	M	$\mathbf{x}$	L
1	0	O	46	$(1^2,-2^2,1^2,-2^2)$	8^3_4	91	$(1^3,2,1^2,-3,2,-3)$	9_{43}
2	1^2	2^2_1	47	$(1,-2,1,-2,1,-2,1,-2)$	8_{18}	92	$(1^3,2,-1^2,-3,2,-3)$	9_{44}
3	1^3	3_1	48	$(1^3,2,-1,-3,2,-3)$	7^2_3	93	$(1^3,-2,1^2,3,-2,3)$	9_{36}
4	1^4	4^2_1	49	$(1^3,-2,1,3,-2,3)$	8^2_5	94	$(1^3,-2,-1^2,3,-2,3)$	9_{42}
5	$(1,-2,1,-2)$	4_1	50	$(1^2,-2,1^2,3,-2,3)$	8^2_9	95	$(1^3,2,-1,2,3,-2,3)$	7_2
6	1^5	5_1	51	$(1^2,-2,1,-2,3,-2,3)$	8^2_8	96	$(1^3,2,-1,2,-3,2,-3)$	8_{14}
7	$(1^2,-2,1,-2)$	5^2_1	52	$(1^2,-2,1,3,-2^2,3)$	8^2_{12}	97	$(1^3,-2,1,-2,3,-2,3)$	9_{26}
8	1^6	6^2_1	53	$(1,-2,1,-2,1,3,-2,3)$	8^2_{13}	98	$(1^3,-2,1,-2,-3,2,-3)$	8_4
9	$(1^3,2,-1,2)$	5_2	54	$(1^2,-2,1,3,-2,3^2)$	8^2_7	99	$(1^3,2,-1,-3,2^2,-3)$	8^3_3
10	$(1^3,-2,1,-2)$	6_2	55	$(1,-2,1,3,-2^3,3)$	8^2_{11}	100	$(1^3,-2,1,3,-2^2,3)$	9^3_6
11	$(1^2,2,1^2,2)$	6^3_3	56	$(1,2^2,1,3,2^2,3)$	8^4_3	101	$(1^2,-2,1^2,3,-2^2,3)$	9^3_8
12	$(1^2,-2,1^2,-2)$	6^3_1	57	$(1,-2,3,-2,1,-2,3,-2)$	8^2_{14}	102	$(1^2,2,-1,2,1,3,-2,3)$	9_{45}
13	$(1^2,-2,1,-2^2)$	6_3	58	$(1,-2,1,3,-2,-4,3,-4)$	8_{12}	103	$(1^2,-2,1,-2,1,3,-2,3)$	9_{32}
14	$(1,-2,1,-2,1,-2)$	6^3_2	59	1^9	9_1	104	$(1^2,-2,1,3,-2,1,3,-2)$	9^3_{11}
15	$(1,-2,1,3,-2,3)$	6^2_3	60	$(1^6,2,-1,2)$	8^2_2	105	$(1^3,2,-1,-3,2,-3^2)$	8_8
16	1^7	7_1	61	$(1^6,-2,1,-2)$	9^2_1	106	$(1^3,-2,1,3,-2,3^2)$	9_{20}
17	$(1^4,2,-1,2)$	6^2_2	62	$(1^5,-2,1^2,-2)$	9^2_{13}	107	$(1^2,2,-1,2^2,3,-2,3)$	7_4
18	$(1^4,-2,1,-2)$	7^2_1	63	$(1^4,-2,1^3,-2)$	9^2_{19}	108	$(1^2,2,-1,2^2,-3,2,-3)$	8_{11}
19	$(1^3,-2,1^2,-2)$	7^2_4	64	$(1^4,-2,-1^3,-2)$	9^2_{50}	109	$(1^2,-2,1,-2^2,3,-2,3)$	9_{27}
20	$(1^3,-2,1,-2^2)$	7^2_2	65	$(1^5,2,-1,2^2)$	8^2_3	110	$(1^2,-2,1,-2^2,-3,2,-3)$	8_{13}
21	$(1^2,-2,1^2,-2^2)$	7^2_5	66	$(1^5,-2,1,-2^2)$	9^2_2	111	$(1^2,-2,1,3,2^3,3)$	8_{15}
22	$(1^2,-2,1,-2,1,-2)$	7^2_6	67	$(1^4,2,-1^2,2^2)$	9^2_{52}	112	$(1^2,-2,1,3,-2^3,3)$	9_{24}
23	$(1^2,2,-1,-3,2,-3)$	6_1	68	$(1^4,-2,1^2,-2^2)$	9^2_{20}	113	$(1^2,-2^2,1,-2,3,-2,3)$	9_{30}
24	$(1^2,-2,1,3,-2,3)$	7_6	69	$(1^4,2,-1,2,-1,2)$	9^2_{55}	114	$(1^2,2^2,1,-3,2^2,-3)$	9^3_{15}
25	$(1,-2,1,-2,3,-2,3)$	7_7	70	$(1^4,-2,1,-2,1,-2)$	9^2_{31}	115	$(1^2,-2^2,1,3,-2^2,3)$	9^3_4
26	$(1,-2,1,3,-2^2,3)$	7^3_1	71	$(1^3,2,1^3,2^2)$	9^2_{53}	116	$(1,-2,1,-2,1,-2,3,-2,3)$	9^3_{10}
27	1^8	8^2_1	72	$(1^3,2,-1^3,2^2)$	9^2_{54}	117	$(1,-2,1,-2,1,3,-2^2,3)$	9^3_{12}
28	$(1^5,2,-1,2)$	7_3	73	$(1^3,-2,1^3,2^2)$	8^2_4	118	$(1,-2,1,-2,1,-3,2^2,-3)$	9^3_{21}
29	$(1^5,-2,1,-2)$	8_2	74	$(1^3,-2,1^3,-2^2)$	9^2_{23}	119	$(1,-2,1,-2^2,1,3,-2,3)$	9_{33}
30	$(1^4,-2,1^2,-2)$	8^3_1	75	$(1^3,2,-1^2,2,-1,2)$	9^2_{57}	120	$(1,-2,1,-2,3,-2,1,-2,3)$	9_{34}
31	$(1^3,2,1^3,2)$	8_{19}	76	$(1^3,-2,1^2,-2,1,-2)$	9^2_{35}	121	$(1,-2,1,-2,-3,-2,1,-2,-3)$	9_{47}
32	$(1^3,2,-1^3,2)$	8_{20}	77	$(1^2,-2,1^2,-2,1^2,-2)$	9^2_{40}	122	$(1^2,-2,1,-2,3,-2,3^2)$	9_{31}
33	$(1^3,-2,1^3,-2)$	8_5	78	$(1^4,-2,1,-2^3)$	9^2_5	123	$(1^2,-2,1,3,-2^2,3^2)$	9_{28}
34	$(1^4,2,-1,2^2)$	7_5	79	$(1^4,-2^2,1,-2^2)$	9^2_{14}	124	$(1,-2,1,3,-2,1,3,-2,3)$	9_{40}
35	$(1^4,-2,1,-2^2)$	8_7	80	$(1^3,-2,1^2,-2^3)$	9^2_{21}	125	$(1^2,-2,1,3,-2,-4,3,-4)$	9^2_{11}
36	$(1^3,2,-1^2,2^2)$	8_{21}	81	$(1^3,-2,1,-2,1,-2^2)$	9^2_{34}	126	$(1,-2,1,-2^3,3,-2,3)$	9_{17}
37	$(1^3,-2,1^2,-2^2)$	8_{10}	82	$(1^3,-2,1,-2^2,1,-2)$	9^2_{37}	127	$(1,-2,1,-2,3,-2^3,3)$	9_{22}
38	$(1^3,2,-1,2,-1,2)$	8^3_9	83	$(1^3,2^2,1^2,2^2)$	9^2_{59}	128	$(1,-2,1,3,-2^4,3)$	9^3_5
39	$(1^3,-2,1,-2,1,-2)$	8^3_5	84	$(1^3,-2^2,1^2,-2^2)$	9^2_{29}	129	$(1,-2^2,1,-2,3,-2^2,3)$	9^3_9
40	$(1^2,-2,1^2,-2,1,-2)$	8_{16}	85	$(1^2,-2,1^2,-2,1,-2^2)$	9^2_{39}	130	$(1,-2^2,3,-2,1,-2,3,-2)$	9_{29}
41	$(1^3,-2,1,-2^3)$	8_9	86	$(1^2,2,-1,2,1^2,2^2)$	9^2_{61}	131	$(1,-2,1,-2,3,-2,-4,3,-4)$	9^2_{12}
42	$(1^3,-2^2,1,-2^2)$	8^3_2	87	$(1^2,-2,1,-2,1^2,-2^2)$	9^2_{41}	132	$(1,-2,1,-2,-3,2,4,-3,4)$	8^2_6
43	$(1^2,-2,1,-2,1,-2^2)$	8_{17}	88	$(1^2,-2,1,-2,1,-2,1,-2)$	9^2_{42}	133	$(1,-2,1,3,-2^2,-4,3,-4)$	9^2_{25}
44	$(1^2,-2,1,-2^2,1,-2)$	8^3_6	89	$(1^4,2,-1,-3,2,-3)$	8_6			
45	$(1^2,2^2,1^2,2^2)$	8^3_{10}	90	$(1^4,-2,1,3,-2,3)$	9_{11}			

Table2