ON PSEUDO-RIBBON SURFACE-LINKS

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ABSTRACT

We first introduce the null-homotopically peripheral quadratic function of a surface-link to obtain a lot of pseudo-ribbon, non-ribbon surface-links, generalizing a known property of the turned spun torus-knot of a non-trivial knot. Next, we study the torsion linking of a surface-link to show that the torsion linking of every pseudo-ribbon surface-link is the zero form, generalizing a known property of a ribbon surface-link. Further, we introduce and algebraically estimate the triple point cancelling number of a surface-link.

Keywords : Surface-knot, Surface-link, Pseudo-ribbon, Ribbon, Torsion linking, Triple point cancelling number, Triple point number

0. Introduction

A *surface-knot* in the 4-space R^4 is a closed connected oriented surface embedded in R^4 by a locally flat PL embedding. A *surface-link* F in R^4 (*with components* F_i $(i = 1, 2, \ldots, r)$) is the union $\cup_{i=1}^r F_i$ where F_i $(i = 1, 2, \ldots, r)$ are mutually disjoint surface-knots in R^4. Let $\rho : R^4 \to R^3$ be the projection sending every point (x, t) (where $x \in R^3$ and $t \in R$) to the point x. The *singularity* of a surface-link F in R^4 is the set

$$\mathcal{S}(F) = \{x \in F| \mid F \cap \rho^{-1}(\rho(x)) \mid \geq 2\}.$$

A surface-link F in R^4 is *generic* if for every point $x \in \mathcal{S}(F)$, we have either

(1) $\rho(x)$ is a double point, that is, there is a 3-ball neighborhood $V_{\rho(x)}$ of $\rho(x)$ in R^3 such that $F \cap \rho^{-1}(V_{\rho(x)})$ consists of two disjoint disks whose images by ρ meet transversely in a line containing $\rho(x)$, or

(2) $\rho(x)$ is a triple point, that is, there is a 3-ball neighborhood $V_{\rho(x)}$ of $\rho(x)$ in R^3 such that $F \cap \rho^{-1}(V_{\rho(x)})$ consists of three disjoint disks every pair of whose images by ρ meet transversely in a line and the resulting three lines meet transversely only at $\rho(x)$.

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[1]Dedicating this paper to Professor Jerome Levine on his 65th birthday.

It is known that every surface-link in R^4 is ambient isotopic to a generic surface-link. In this paper, we concern the following type of surface-link:

Definition 0.1. A surface-link F in R^4 is a *pseudo-ribbon* surface-link if F is ambient isotopic to a generic surface-link F' in R^4 such that $\rho(F')$ has no triple points, namely $|F' \cap \rho^{-1}(\rho(x))| = 2$ for every point $x \in \mathcal{S}(F')$.

In Definition 0.1, we can see further that F' is ambient isotopic to a generic surface F'' such that $\mathcal{S}(F'')$ is $\emptyset$ or a closed 1-manifold, namely the singular surface $\rho(F'')$ has no triple points and no branch points (see J. S. Carter-M. Saito [3], D. Roseman [15]). A ribbon surface-link is a surface-link in R^4 obtained from a trivial S^2-link by a surgery along some embedded 1-handles. Every ribbon surface-link is a pseudo-ribbon surface-link, and conversely every pseudo-ribbon S^2-link is a ribbon S^2-link (see T. Yajima [22]). On the other hand, the turned spun T^2-knot $T(k)$ of a non-trivial knot k (see J. Boyle [1], Z. Iwase [4], C. Livingston [13]) is pseudo-ribbon and non-ribbon (see A. Shima [19]), although the spun T^2-knot $T^0(k)$ is ribbon. In §1, we introduce the null-homotopically peripheral quadratic function of a surface-link which is useful in identifying a non-ribbon surface-link. Using this invariant, we can show in §1 that *any connected sum of the turned spun T^2-knot of a non-trivial knot and any ribbon (or more generally, any pseudo-ribbon) surface-link is a pseudo-ribbon, non-ribbon surface-link.* Next, we discuss the torsion linking ℓ_F of a surface-link F, which is a generalization of the Farber-Levine pairing on S^2-knots as it is mentioned in [7] (see also [8,9,10]). We show that *the torsion linking ℓ_F of every pseudo-ribbon surface-link F in $S^4 = R^4 \cup \{\infty\}$ vanishes* by studying a canonical Seifert hypersurface of F which is constructed by a method analogous to Seifert's algorithm on constructing a Seifert surface of a knot. This result is stated in §2 together with an explanation of the torsion linking ℓ_F and proved in §3. We note that if F is a ribbon surface-link, then this result is known. In fact, the ribbon surface-link F bounds a Seifert hypersurface V such that the torsion part $\mathrm{t}H_1(V;Z) = 0$ (see [11]). Then the vanishing $\ell_F = 0$ follows from a result of M. Sekine [18] showing that the torsion linking ℓ_F is induced from a singular sublinking of the linking $\ell_V : \mathrm{t}H_1(V, \partial V; Z) \times \mathrm{t}H_1(V;Z) \to Q/Z$ defined by the Poincaré duality for every Seifert hypersurface V of F. In §4, we introduce the *triple point cancellation number* $\mathcal{T}(F)$ of a surface-link F which measures a distance to the pseudo-ribbon surface-links. Using the vanishing of the torsion linking of a pseudo-ribbon surface-knot, we shall make an algebraic estimate of $\mathcal{T}(F)$. This method is similar to S. Kamada's argument in [5]. As an application of this estimate, we shall show in §4 that *every surface-link F is concordant to a surface-link F_* with $\mathcal{T}(F_*) = n$ for every previously given integer $n > \mathcal{T}(F)$.*

1. The null-homotopically peripheral quadratic function of a surface-link

By a 2-*chain*, we mean a simplicial 2-chain C with Z_2-coefficients in a 4-manifold W. This 2-chain C is *regular* if $|\partial C|$ is a closed 1-manifold. The *support* of C, denoted by $|C|$ is the union of all simplices in C with non-zero coefficients. For a 2- or 3-submanifold Y of a 4-manifold W, the 2-chain C in W is *Y-proper* if $|C| \cap Y = |\partial C|$. A *disk-chain* is a 2-chain which is obtained from a simplicial map $f : B^2 \to W$ where B^2 is a triangulated disk. Let F be a surface-link in $S^4 = R^4 \cup \{\infty\}$ with components $F_i (i = 1, 2, \ldots, r)$. Let $K(F; Z_2)$ be the subgroup of $H_1(F; Z_2)$ consisting of an element represented by the boundary of an F-proper 2-chain in S^4. We note that every element of $K(F; Z_2)$ is represented by the boundary of an F-proper regular 2-chain in S^4. The *peripheral quadratic function* of a surface-link F is the function

$$\Xi : K(F; Z_2) \to Z_2$$

defined by putting $\Xi(x)$ to be the Z_2-intersection number $\mathrm{Int}_{S^4}(C, \tilde{C})_2$ in S^4 where C and $\tilde{C}$ are F-proper regular 2-chains in S^4 with $x = [\partial C] = [\partial \tilde{C}]$ and $|\partial C| \cap |\partial \tilde{C}| = \emptyset$ such that $\tilde{C}$ is obtained from C by sliding ∂C along F. The function Ξ is well-defined and is a quadratic function with respect to the Z_2-intersection form $\mathrm{Int}_F(\ ,\ )_2$ on F, that is, Ξ has the identity

$$\Xi(x + y) = \Xi(x) + \Xi(y) + \mathrm{Int}_F(x, y)_2$$

for all $x, y \in K(F; Z_2)$. We note that Ξ may be a singular quadratic function for a general surface-link F, although it is always non-singular when F is a surface-knot. Let $\Delta(F; Z_2)$ be the subgroup of $K(F; Z_2)$ generated by the elements represented by the boundaries of all F-proper disk-chains in S^4. For our purpose, we are interested in the restricted quadratic function

$$\xi = \Xi|_{\Delta(F;Z_2)} : \Delta(F; Z_2) \to Z_2,$$

which we call the *null-homotopically peripheral quadratic function* of the surface-link F. For a surface-link F in S^4, let N be a tubular neighborhood of F in S^4, and $E = \mathrm{cl}(S^4 - N)$ be the link-exterior of F. Let D^2 be the unit disk in the complex plane. A *canonical trivialization* of N is an identification $(N, F) = (F \times D^2, F \times 0)$ such that the natural injection $F \times 1 \subset F \times S^1 = \partial N = \partial E \subset E$ induces the trivial composite homomorphism

$$H_1(F \times 1; Z) \to H_1(\partial E; Z) \to H_1(E; Z) \xrightarrow{\gamma} Z,$$

where $\gamma \in \hom(H_1(E; Z), Z) = H^1(E; Z)$ is the epimorphism sending every oriented meridian to $1 \in Z$. The *γ-structure* on the link-exterior E of a surface-link F is the subset of $H^1(E; Z)$ consisting of the γ's under all orientation changes of components of F, whose cardinal number is seen to be 2^r for the component number r of F.

Let $\Delta_E(\partial E; Z_2)$ be the subgroup of $H_1(\partial E; Z_2)$ generated by the elements represented by the boundaries of all ∂E-proper disk-chains in E. Similarly, let

$\Delta_E(F\times 1;Z_2)$ be the subgroup of $H_1(F\times 1;Z_2)$ generated the elements represented by the boundaries of all $F\times 1$-proper disk-chains in E. The quadratic function

$$\xi_E : \Delta_E(F\times 1;Z_2) \to Z_2$$

is defined by a method analogous to the definition of ξ, namely by

$$\xi_E(x) = \mathrm{Int}_E(C,\tilde{C})_2$$

where C and $\tilde{C}$ are $F\times 1$-proper regular 2-chains in E with $x=[\partial C]=[\partial\tilde{C}]$ and $|\partial C|\cap|\partial\tilde{C}|=\emptyset$ such that $\tilde{C}$ is obtained from C by sliding ∂C along $F\times 1$. The following lemma concerns the arguments of J. Boyle [1] and Z. Iwase [4] on T^2-knots.

Lemma 1.1.

(1) The natural composite map

$$k_* : H_1(F\times 1;Z_2)\to H_1(\partial E;Z_2)\to H_1(N;Z_2)\overset{\cong}{\leftarrow} H_1(F\times 0;Z_2)=H_1(F;Z_2)$$

induces an isomorphism

$$k_* : \Delta_E(F\times 1;Z_2)\overset{\cong}{\to}\Delta_E(\partial E;Z_2)\overset{\cong}{\leftarrow}\Delta(F;Z_2).$$

(2) The isomorphism k_* induces an isomorphism from the quadratic function $\xi_E:\Delta_E(F\times 1;Z_2)\to Z_2$ to the quadratic function $\xi:\Delta(F;Z_2)\to Z_2$.

(3) The quadratic function $\xi_E:\Delta_E(F\times 1;Z_2)\to Z_2$ is invariant (up to isomorphisms) under any γ-structure-preserving homeomorphism between the link-exteriors E for all surface-links F. In particular, ξ_E is invariant (up to isomorphisms) under any homeomorphism between the knot-exteriors E for all surface-knots F.

Proof. To see (1), we take a simplicial map $f:B^2\to S^4$ giving an F-proper disk-chain C in S^4. By a general position argument on f and the uniqueness of a regular neighborhood, we may consider that the regular neighborhood $N=F\times D^2$ of F in S^4 meets $|f(B^2)|$ in a singular annulus A such that $L=(\partial N)\cap|f(D^2)|$ is a simple loop bounding a singular disk $\mathrm{cl}(|f(B^2)|-A)$ in E. For the infinite cyclic covering $p:\tilde{E}\to E$ associated with γ, the boundary $\partial E=F\times S^1$ of E lifts to $\partial\tilde{E}=F\times R$ and the loop L lifts to $F\times R$ trivially. Since any component of $p^{-1}(L)$ is homotopic in $\partial\tilde{E}$ to a loop in $F\times 1\subset F\times R=\partial\tilde{E}$, we see that L is homotopic in ∂E to a loop L' in $F\times 1\subset F\times S^1=\partial E$. Since the inclusion $F\times 1\subset N$ is a homotopy equivalence, we see that L' is homotopic to the loop $|\partial C|\times 1$ in $F\times 1$. Thus, we have a simplicial map $f':B^2\to E$ giving an $F\times 1$-proper disk-chain C' in E such that $\partial C'=\partial C\times 1$ in $F\times 1$. Conversely, if we are given an $F\times 1$-proper disk-chain C' in E, then we can construct an F-proper disk-chain C in S^4 with $\partial C'=\partial C\times 1$

by sliding $\partial C'$ along $F \times [0,1] \subset F \times D^2 = N$. This implies that the map k_* is an isomorphism from $\Delta_E(F \times 1; Z_2)$ onto $\Delta(F; Z_2)$ proving (1), and further that k_* is an isomorphism from ξ_E to ξ proving (2). To see (3), let $E' = \mathrm{cl}(S^4 - N')$ be the exterior of a surface-link F' in S^4 where $N' = F' \times D^2$ is a tubular neighborhood of F' in S^4 with the specified trivialization. Let $\gamma' \in H^1(E'; Z)$ be the cohomology class sending each oriented meridian of F' to 1. Assume that there is a γ-structure preserving homeomorphism $h : E' \cong E$. Since by definition, ξ is invariant under all the choices of the oriantations on S^4 and the components of F, we can assume from (2) that h is orientation-preserving and $h^*(\gamma) = \gamma'$.

We need the following sublemma:

Sublemma 1.1.1. Let F_* be a closed oriented surface of a positive genus. Let h be an orientation-preserving auto-homeomorphism of $F_* \times S^1$ such that $h^*(\gamma) = \gamma$ for the Poincaré dual $\gamma \in H^1(F_* \times S^1; Z)$ of the homology class $[F_* \times 1] \in H_2(F_* \times S^1; Z)$. Then h is isotopic to an auto-homeomorphism h' with $h'(F_* \times 1) = F_* \times 1$.

Let F'_* and F_* be any positive genus surface components of F' and F respectively such that $h(F'_* \times S^1) = F_* \times S^1$. Then we see from Sublemma 1.1.1 and a property of γ', γ that the homeomorphism h is isotopic to a homeomorphism $h' : E' \cong E$ such that $h'(F'_* \times 1) = F_* \times 1$ for all F'_* and F_* with $h(F'_* \times S^1) = F_* \times S^1$. The homeomorphism h' induces an isomorphism from the quadratic function $\xi_{E'} : \Delta_{E'}(F' \times 1; Z_2) \to Z_2$ to the quadratic function $\xi_E : \Delta_E(F \times 1; Z_2) \to Z_2$. we have (3) except the proof of Sublemma 1.1.1. □

Proof of Sublemma 1.1.1. Using that the intersection number of $F_* \times 1$ and every 1-cycle in $F_* \times 1$ in $F_* \times S^1$ is 0, we see that $\gamma|_{F_* \times 1} = 0$. For a point $x \in F_*$, we assume that $h(x,1) = (x,1)$ by an isotopic deformation of h. Since $h^*(\gamma) = \gamma$, the automorphism $h_\#$ of $\pi_1(F_* \times S^1, (x,1))$ preserves the subgroup $\pi_1(F_* \times 1, (x,1))$. Let f be an auto-homeomorphism of $(F_* \times 1, (x,1))$ inducing the automorphism $h_\#|_{\pi_1(F_* \times 1, (x,1))}$ up to an conjugation. Then the auto-homeomorphism $h' = f \times 1$ of $F_* \times S^1$ is homotopic to h since $F_* \times S^1$ is a Haken manifold and $h'_\#$ coincides with $h_\#$ up to a conjugation. By F. Waldhausen's result in [21], h is isotopic to h' and $h'(F_* \times 1) = F_* \times 1$. □

Let $\varsigma(F)$ be the Gauss sum

$$GS(\xi) = \sum_{x \in \Delta(F; Z_2)} \exp(2\pi\sqrt{-1}\frac{\xi(x)}{2}).$$

The following theorem is useful to obtain a pseudo-ribbon non-ribbon surface-link:

Theorem 1.2.

(1) For every surface-link F of total genus g, the invariant $\varsigma(F)$ is 0, 1 or $\pm 2^s$ for an integer s with $1 \leq s \leq g$.

(2) If F is a ribbon surface-link of total genus g, then we have $\varsigma(F) = 2^g$.
(3) For any connected sum $F = F_1 \# F_2$ of any surface-links F_1 and F_2, we have

$$\varsigma(F) = \varsigma(F_1)\varsigma(F_2).$$

It appears unknown *whether there is a surface-link F with $\varsigma(F) = -2^s$.*

Proof. If $\Delta(F; Z_2) = 0$, then we have $\varsigma(F) = 1$ since $\xi(0) = 0$. Assume $\Delta(F; Z_2) \neq 0$. Let x_i $(i = 1, 2, \ldots, s)$ and y_j $(j = 1, 2, \ldots, u)$ be a Z_2-basis for $\Delta(F; Z_2)$ with $0 \leq u \leq s \leq g$ such that $\mathrm{Int}_2(x_i, x_{i'}) = \mathrm{Int}_2(y_j, y_{j'}) = 0$ and $\mathrm{Int}_2(x_i, y_j) = \delta_{ij}$ for all i, i', j, j'. Let Δ_i $(i = 1, 2, \ldots, s)$ be the direct summand of $\Delta(F; Z_2)$ with basis x_i, y_i for $i \leq u$ or x_i for $i > u$. Let ξ_i be the restriction of ξ to Δ_i. For every $i \leq u$, we have either $\xi_i(x_i) = 0$ and $\xi_i(x_i + y_i) = \xi_i(y_i) + 1$ or $\xi_i(x_i) = \xi_i(y_i) = \xi_i(x_i + y_i) = 1$, so that $GS(\xi_i) = \pm 2$ by noting $\exp(\pi\sqrt{-1}) = -1$. For every $i > s$, it is direct to see that $GS(\xi_i)$ is 2 or 0 according to whether $\xi_i(x_i)$ is 0 or 1. Since $\Delta(F; Z_2)$ splits into Δ_i $(i = 1, 2, \ldots, s)$ orthogonally with respect to the Z_2-intersection form on F, we have

$$\varsigma(F) = GS(\xi) = GS(\xi_1)GS(\xi_2)\cdots GS(\xi_s) = 0, 1 \,\text{or}\, \pm 2^s,$$

showing (1). To show (2), let F be a ribbon surface-link of total genus g. By [11], F admits a Seifert hypersurface V which is homeomorphic to the connected sum of handlebodies V_i $(i = 1, 2, \ldots, r)$ of total genus g and some copies of $S^1 \times S^2$. Let $O(F; Z_2)$ be the subgroup of $\Delta(F; Z_2)$ generated by a half Z_2-basis of $H_1(F; Z_2)$ which are represented by meridian loops of V_i $(i = 1, 2, \ldots, r)$. It is direct to see that $\xi(x) = 0$ for all $x \in O(F; Z_2)$. We take a Z_2-basis x_i $(i = 1, 2, \ldots, g)$, y_j $(j = 1, 2, \ldots, u)$ of $\Delta(F; Z_2)$ with $0 \leq u \leq g$ such that x_i $(i = 1, 2, \ldots, g)$ are a Z_2-basis of O, and $\mathrm{Int}_2(y_j, y_{j'}) = 0$ and $\mathrm{Int}_2(x_i, y_j) = \delta_{ij}$ for all i, j, j'. As in the argument of the first half, we denote by ξ_i $(i = 1, 2, \ldots, g)$ the restriction of ξ to the direct summand with basis x_i, y_i for $i \leq u$ or x_i for $i > u$. For every $i \leq u$, the identity $\xi_i(x_i) = 0$ implies $\xi_i(x_i + y_j) = \xi_i(y_i) + 1$, so that $GS(\xi_i) = 2$. For every $i > u$, it is direct to see that $GS(\xi_i) = 2$ since $\xi_i(x_i) = 0$. Thus, we have

$$\varsigma(F) = GS(\xi) = GS(\xi_1)GS(\xi_2)\cdots GS(\xi_g) = 2^g,$$

showing (2). To show (3), we first show that $\Delta(F; Z_2) = \Delta(F_1; Z_2) \oplus \Delta(F_2; Z_2)$ under the identification $H_1(F; Z_2) = H_1(F_1; Z_2) \oplus H_1(F_2; Z_2)$. Since $\Delta(F; Z_2) \supset \Delta(F_1; Z_2) \oplus \Delta(F_2; Z_2)$ is obvious, it suffices to show that $\Delta(F; Z_2) \subset \Delta(F_1; Z_2) \oplus \Delta(F_2; Z_2)$. For the infinite cyclic covering $p : \tilde{E} \to E$ associated with γ, we consider an embedding

$$e : F = F \times 1 \stackrel{\subset}{\to} F \times R^1 = \partial\tilde{E} \stackrel{\subset}{\to} \tilde{E}.$$

For F_i $(i = 1, 2)$ instead of F, we have a similar embedding

$$e_i : F_i \to \tilde{E}_i.$$

We regard F as the union $F_1^0 \cup F_2^0$ where F_i^0 is a compact punctured surface of F_i with $\partial F_1^0 = \partial F_2^0$. We note that every (possibly singular) loop $L \subset F$ is homotopic in F to a bouquet

$$B_L = L_1^1 \vee L_2^1 \vee \cdots \vee L_1^r \vee L_2^r$$

such that L_i^j is a (possibly singular) loop in F_i^0 and the base point is sent to a point $b \in e(\partial F_1^0) = e(\partial F_1^0)$ by e. Since $\pi_1(\tilde{E}, b)$ is the free product $\pi_1(\tilde{E}_1, b) * \pi_1(\tilde{E}_2, b)$ and each loop L_i^j represents an element of $\pi_1(\tilde{E}_i, b)$, we see from a result of W. Magnus-A. Karrass-D. Solitar [14;p.182] that if L is null-homotopic in $\tilde{E}$, then some loop L_i^j represents a trivial element of $\pi_1(\tilde{E}_i, b)$. The bouquet obtained from B_L by removing this loop L_i^j is homotopic in F to a bouquet

$$B_{L'} = (L')_1^1 \vee (L')_2^1 \vee \cdots \vee (L')_1^{r-1} \vee (L')_2^{r-1}$$

such that $(L')_i^j$ is a loop in F_i^0 and the base point is sent to the point b by e. Then we note that L is homologous to $L_i^j + B_{L'}$ in F. Since $B_{L'}$ is null-homotopic in $\tilde{E}$, we can conclude by induction on r that L is homologous in F to the sum $L_1 + L_2$ where L_i is the sum of loops in F_i^0 which are null-homotopic in $\tilde{E}_i$. This implies that

$$\Delta(F; Z_2) \subset \Delta(F_1; Z_2) \oplus \Delta(F_2; Z_2)$$

and hence

$$\Delta(F; Z_2) = \Delta(F_1; Z_2) \oplus \Delta(F_2; Z_2).$$

For every element $x_i \in \Delta(F_i; Z_2)$ $(i = 1, 2)$, we have $\mathrm{Int}_2(x_1, x_2) = 0$, so that

$$\xi(x_1 + x_2) = \xi(x_1) + \xi(x_2).$$

By this identity, we have

$$\exp(2\pi\sqrt{-1}\frac{\xi(x_1 + x_2)}{2}) = \exp(2\pi\sqrt{-1}\frac{\xi(x_1)}{2}) \exp(2\pi\sqrt{-1}\frac{\xi(x_2)}{2}),$$

which implies the identity $\varsigma(F) = \varsigma(F_1)\varsigma(F_2)$. $\square$

Let D^2 be the disk, and $\rho_D : D^2 \times [0,1] \to D^2$ the projection to the first factor. Let k be a knot in $D^2 \times [0,1]$ such that $\rho_D(k)$ be a transversely immersed loop in D^2. We consider an unknotted embedding $f^1 : D^2 \times S^1 \to R^3$ with +1-framing. Let

$$\bar{f}^1 : (D^2 \times S^1) \times [0,1] \stackrel{f^1 \times id}{\longrightarrow} R^3 \times [0,1] \subset R^4$$

be the associated embedding. Under the identification

$$(D^2 \times [0,1]) \times S^1 = (D^2 \times S^1) \times [0,1],$$

we obtain the torus $k \times S^1$ in $(D^2 \times S^1) \times [0,1]$. The *turned spun T^2-knot of the knot k* is the T^2-knot $T(k) = \bar{f}^1(k \times S^1)$ in $R^4 \subset R^4 \cup \{\infty\} = S^4$ (see J. Boyle

[1]). From construction, $\rho\mathcal{S}(T(k))$ is a closed 1-manifold which is the product of the double point set of $\rho_D(k)$ by S^1. Hence $T(k)$ is a pseudo-ribbon T^2-knot. When we use an unknotted embedding $f^0 : D^2 \times S^1 \to R^3$ with 0-framing instead of f^1, we obtain the *spun* T^2-knot $T^0(k)$ of the knot k, which is directly seen to be a ribbon T^2-knot. We have the following corollary to Theorem 1.2, generalizing a result of A. Shima [19]:

Corollary 1.3. Any connected sum $F\#T(k)$ of a surface-link F and the turned spun T^2-knot $T(k)$ of a non-trivial knot k is a non-ribbon surface-link.

Proof. First, we show the Gauss sum invariant $\varsigma(T(k)) = 0$. By the unknotted embedding f^1 with $+1$-framing, we have an embedded disk B in S^4 with $B \cap T(k) = \partial B$ which is an essential loop in $T(k)$ and $\mathrm{Int}_{S^4}(B, \tilde{B})_2 = 1$. Hence the element $x = [\partial B] \in \Delta(T(k); Z_2)$ has $\xi(x) = 1$. Since $H_1(T(k); Z_2)$ has a Z_2-basis consisting of x and the longitude element y of k and the longitude of k represents an infinite order element of $\pi_1(R^3 - k) \cong \pi_1(S^4 - T(k))$, we see that $\Delta(T(k); Z_2) = \{0, x\} \cong Z_2$. Thus, we have $\varsigma(T(k)) = 1 - 1 = 0$. By Theorem 1.2, $\varsigma(F\#T(k)) = \varsigma(F)\varsigma(T(k)) = 0$ and hence $F\#T(k)$ is a non-ribbon surface-link. □

For any integer sequence $g_1 \geq g_2 \geq \cdots \geq g_r \geq 0$ with $g_1 > 0$, we take any pseudo-ribbon (e.g. trivial or ribbon) surface-link with components F_i $(i = 1, 2, \ldots, r)$ such that $\mathrm{genus}(F_1) = g_1 - 1$ and $\mathrm{genus}(F_i) = g_i$ $(i = 2, 3, \ldots, r)$. Then we see from Corollary 1.3 that the connected sum $F_1\#T(k) \cup F_2 \cup \ldots F_r$ is a pseudo-ribbon, non-ribbon surface-link of genera g_i $(i = 1, 2, \ldots, r)$. For a surface-knot, we have a result on the knot-exterior as follows:

Example 1.4 Let F be a surface-knot F with $\varsigma(F) \neq 0$. Since the spun T^2-knot $T^0(k)$ of a non-trivial knot k is a ribbon T^2-knot, we see from Theorem 1.2 that

$$\varsigma(F\#T^0(k)) = 2\varsigma(F) \neq 0.$$

On the other hand, we have $\varsigma(F\#T(k)) = 0$, for $\varsigma(T(k)) = 0$ by Corollary 1.3. By an observation due to F. González-Acuña found in J. Boyle's paper [1], the knot-exteriors of $T(k)$ and $T^0(k)$ are homotopy equivalent. Examining it carefully, we can see also that the knot-exteriors of $F\#T^0(k)$ and $F\#T(k)$ are homotopy equivalent. However, from Lemma 1.1(3), we see that the knot-exteriors of $F\#T^0(k)$ and $F\#T(k)$ are not homeomorphic, generalizing the property between the T^2-knots $T(k)$ and $T^0(k)$ known by J. Boyle [1] and Z. Iwase [4] .

2. The torsion linking of a pseudo-ribbon surface-link

Let $p : \tilde{W} \to W$ be the infinite cyclic covering of a compact oriented 4-manifold W belonging to an element $\gamma \in H^1(W; Z)$. Let A and A' be $\emptyset$ or compact 3-submanifolds of ∂W such that $A' = \mathrm{cl}(\partial W - A)$. Let $\tilde{A} = p^{-1}(A)$ and $\tilde{A}' = p^{-1}(A')$. We briefly explain the torsion linking of $(\tilde{W}, \tilde{A}, \tilde{A}')$ which has been done in [7]. For a Λ-module H, let TH be the Λ-torsion part of H, and $BH = H/TH$. (Unless otherwise stated, abelian groups are regarded as Λ-modules on which t operates as the identity.) Let $\mathrm{t}H$ be the Z-torsion part of H, and $\mathrm{b}H = H/\mathrm{t}H$. Let $E^q(H) = Ext^q_\Lambda(H,\Lambda)$. For a finitely generated Λ-module H, we have a unique maximal finite Λ-submodule DH of H. Then we have a t-anti epimorphism

$$\theta_{A,A'} : DH_1(\tilde{W},\tilde{A};Z) \to E^1(BH_2(\tilde{W},\tilde{A}';Z))$$

which is an invariant of $(\tilde{W},\tilde{A},\tilde{A}')$ or (W,A,A',γ). We denote the kernels of $\theta_{A,A'}$ and $\theta_{A',A}$ by $DH_1(\tilde{W},\tilde{A};Z)^\theta$ and $DH_1(\tilde{W},\tilde{A}';Z)^\theta$, respectively. Let $\mu \in TH_3(\tilde{W},\partial W;Z)$ be the fundamental class of the covering $p:\tilde{W}\to W$, which is characterized by $t\mu=\mu$ and $p_*(\mu)=\gamma\cap[W]$ for the fundamental class $[W]$ of W. Let $\tau H^2(\tilde{W},\tilde{A};Z)$ be the image of the Bockstein coboundary map

$$\delta_{Q/Z} : H^1(\tilde{W},\tilde{A};Z) \to H^2(\tilde{W},\tilde{A};Z).$$

The second duality in [7] is equivalent to the following lemma (see [7; Theorem 6.5]):

Lemma 2.1. The cap product map $\cap\mu : \tau H^2(\tilde{W},\tilde{A};Z) \to \mathrm{t}H_1(\tilde{W},\tilde{A}';Z)$ induces an isomorphism

$$\cap\mu : \mathrm{hom}(DH_1(\tilde{W},\tilde{A};Z)^\theta, Q/Z) \cong DH_1(\tilde{W},\tilde{A}';Z)^\theta.$$

In fact, by Lemma 2.1 we have a t-isometric non-singular bilinear form

$$\ell : DH_1(\tilde{W},\tilde{A};Z)^\theta \times DH_1(\tilde{W},\tilde{A}';Z)^\theta \longrightarrow Q/Z$$

by taking $\ell(x,y) = f_y(x) \in Q/Z$ for $x \in DH_1(\tilde{W},\tilde{A};Z)^\theta$, $y \in DH_1(\tilde{W},\tilde{A}';Z)^\theta$, $f_y \in \mathrm{hom}(DH_1(\tilde{W},\tilde{A};Z)^\theta, Q/Z)$ with $f_y\cap\mu = y$. This bilinear form ℓ is an invariant of $(\tilde{W},\tilde{A},\tilde{A}')$ or (W,A,A',γ) and called the *torsion linking* of $(\tilde{W},\tilde{A},\tilde{A}')$ or (W,A,A',γ). Let F be a surface-link in $S^4 = R^4\cup\{\infty\}$, and E the compact exterior $\mathrm{cl}(S^4-N)$ where N denotes a normal disk bundle of F in S^4. Taking $W=E$, $A=\partial E$, $A'=\emptyset$ and the element $\gamma\in H^1(E;Z)=\mathrm{hom}(H_1(E;Z),Z)$ sending each oriented meridian of F to $1\in Z$, we have, as a surface-link type invariant, the *torsion linking*

$$\ell = \ell_F : DH_1(\tilde{E},\partial\tilde{E};Z)^\theta \times DH_1(\tilde{E};Z)^\theta \longrightarrow Q/Z$$

of the surface-link F. The following theorem is proved in §3:

Theorem 2.2. If F is a pseudo-ribbon surface-link, then the torsion linking ℓ_F vanishes. In other words,

$$DH_1(\tilde{E},\partial\tilde{E};Z)^\theta = DH_1(\tilde{E};Z)^\theta = 0.$$

The linking signature $\sigma(F)$ and the local linking signatures $\sigma_p^i(F)$ are defined as modulo 4 integers in [10] by using the Gauss sum of the quadratic function associated with the torsion linking ℓ_F. The following corollary is direct from Theorem 2.2:

Corollary 2.3. If F is a pseudo-ribbon surface-link, then we have

$$\sigma(F) = \sigma_p^i(F) = 0$$

for all prime numbers p and all positive integers i.

3. Constructing a canonical Seifert hypersurface for a pseudo-ribbon surface-link

We assume that the singularity image $\rho\mathcal{S}(F)$ in R^3 of a pseudo-ribbon surface-link F in R^4 consists of mutually disjoint simple loops C_i $(i = 1, 2, \ldots, r)$. Let N_i be a regular neighborhood of C_i in $\rho(F)$, so that we have a homeomorphism

$$h_i : (X, v) \times S^1 \cong (N_i, C_i) \quad (i = 1, 2, \ldots, r),$$

where X denotes a cone over a four-point set with v as the vertex. We regard X as the subgraph of the 1-skelton $J^{(1)}$ of a bouquet J of two 2-simplices at a vertex such that the complement graph is the union of two disjoint 1-simplices I and I'. In this case, v is the vertex of the bouquet J. Then h_i extends to an embedding $\bar{h}_i : J \times S^1 \longrightarrow R^3$. We note that there are two choices on regarding X as such a subgraph of $J^{(1)}$. Our choice is made to satisty the condition that

$$P = \mathrm{cl}(\rho(F) - \cup_{i=1}^r N_i) \cup (\cup_{i=1}^r \bar{h}_i((I \cup I') \times S^1)$$

is an orientable 2-manifold with an orientation induced from $\rho(F) - \cup_{i=1}^r N_i$. The 2-manifold P is referred to as a 2-manifold obtained from $\rho(F)$ by *orientation-preserving cut* along the C_i's. Let P_j $(j = 1, 2, \ldots, s)$ be the components of P such that the compact 3-manifold V_j in R^3 bounded by P_j satisfies the condition that $P_j \subset V_{j'}$ implies $j < j'$.

Let T be a four-sided disk, and I_0 a proper interval in T splitting T into two four-sided disks. We identify the quotient space T/I_0 with J so that the quotient map $q : T \longrightarrow J = T/I_0$ is a half-twist band projection with $q(I_0) = v$. For a subset A of R^3 and an interval $[a, b]$ granting $a = b$, we denote the subset $\{(x, t) | x \in A, t \in [a, b]\}$ of R^4 by $A[a, b]$. We choose real numbers t_j $(j = 1, 2, \ldots, s)$ so that $t_1 < t_2 < \cdots < t_s$. After an ambient deformation of F, we have a Seifert hypersurface V for F so that

$$V = (\coprod_{j=1}^s V_j[t_j]) \cup (\coprod_{i=1}^r \bar{h}_i^*(T \times S^1),$$

where $\bar{h}_i^*$ denotes an embedding $\bar{h}_i^* : T \times S^1 \longrightarrow R^4$ such that the composite

$$\rho\bar{h}_i^* : T \times S^1 \xrightarrow{\bar{h}_i^*} R^4 \xrightarrow{\rho} R^3$$

is equal to the composite

$$T \times S^1 \xrightarrow{q\times 1} J \times S^1 \xrightarrow{\bar{h}_i} R^3.$$

By construction, we have

$$\rho(V) = (\coprod_{j=1}^{s} V_j) \cup (\coprod_{i=1}^{r} \bar{h}_i(J \times S^1)).$$

Further, we can assume that $V \subset \rho(V)[t_1, t_s]$. We call this hypersurface V a *canonical Seifert hypersurface* of the pseudo-ribbon surface-link F. Let E_V be the compact oriented 4-manifold obtained from the exterior E of F in $S^4 = R^4 \cup \{\infty\}$ by splitting along $V \cap E (\cong V)$. Let $V^{\pm} (\cong \pm V)$ be the two copies of V in $\partial E_V \subset E_V$. Let $i^{\pm} : V \cong V^{\pm} \subset E_V$ be the composite injections. The following theorem is a key to our argument:

Theorem 3.1. For a canonical Seifert hypersurface V of a pseudo-ribbon surface-link F, the induced homomorphisms

$$i_*^{\pm} : \mathrm{t}H_1(V; Z) \longrightarrow \mathrm{t}H_1(E_V; Z)$$

on the torsion part of the first integral homology are trivial.

To prove this theorem, we need some preliminaries. First we show the following lemma:

Lemma 3.2. For the inclusion $k : \coprod_{j=1}^{s} V_j[t_j] \subset V$, we have

$$\mathrm{t}H_1(V; Z) \subset \mathrm{image}[H_1(\coprod_{j=1}^{s} V_j[t_j]; Z) \xrightarrow{k_*} H_1(V; Z)].$$

Proof. Let $A_i = \bar{h}_i^*(I \times S^1)$ $(i = 1, 2, \ldots, r)$ be proper annuli with any orientations in V. Then we have

$$\mathrm{Int}([A_i], x) = 0$$

for all $x \in \mathrm{t}H_1(V; Z)$ with respect to the intersection form

$$\mathrm{Int} : H_2(V, \partial V; Z) \times H_1(V; Z) \longrightarrow Z.$$

This means that every element $x \in \mathrm{t}H_1(V;Z)$ is represented by an embedded closed oriented 1-manifold L_x with $L_x \cap A_i = \emptyset$ for all i, so that L_x is isotopically deformed into $\coprod_{j=1}^{s} V_j[t_j]$. $\square$

We also need the following lemma:

Lemma 3.3. For every polyhedron V in R^3, we have $\mathrm{t}H_1(V;Z) = 0$.

Proof. We may assume that V is compact and connected. Further, we may assume that V is a compact connected 3-submanifold of R^3 by taking a regular neighborhood of V instead of V. For any elements $x \in \mathrm{t}H_1(V, \partial V; Z)$ and $y \in \mathrm{t}H_1(V;Z)$, we can represent x and y by disjoint closed oriented 1-manifolds L_x and L_y in V. Then there are a non-zero integer m and a 2-chain c_y in V such that $\partial c_y = mL_y$ and the torsion linking

$$\ell_V : \mathrm{t}H_1(V, \partial V; Z) \times \mathrm{t}H_1(V;Z) \longrightarrow Q/Z$$

is computed by the identity $\ell_V(x,y) = \mathrm{Int}(L_x, c_y)/m \pmod 1$. Since the linking number $\mathrm{Link}(L_x, L_y) \in Z$ in R^3 is defined, we have

$$\mathrm{Int}(L_x, c_y)/m = \mathrm{Link}(L_x, mL_y)/m = \mathrm{Link}(L_x, L_y) \in Z$$

and hence $\ell_V(x,y) = 0 \in Q/Z$. Using that the torsion linking ℓ_V is non-singular, we obtain $\mathrm{t}H_1(V, \partial V; Z) = \mathrm{t}H_1(V;Z) = 0$. $\square$

By using Lemmas 3.2 and 3.3, Theorem 3.1 is proved as follows:

Proof of Theorem 3.1. We regard $[t_1, t_s] \subset R^1 \cup \{\infty\} = S^1$. Let $f : \rho(V) \times S^1 \longrightarrow R^4 \subset R^4 \cup \{\infty\} = S^4$ be an embedding sending $\rho(V) \times [t_1, t_s]$ to $\rho(V)[t_1, t_s]$ identically. We represent any element $x \in \mathrm{t}H_1(V;Z)$ by a closed oriented 1-manifold L_x in $\coprod_{j=1}^{s} V_j[t_j]$. Let $L_j[t_j] = L_x \cap V_j[t_j]$ for a closed 1-manifold L_j or $\emptyset$ in V_j. We take a point $t_0 \in S^1 - [t_1, t_s]$. For any j with $L_j \neq \emptyset$, we further take a subarc $\alpha_j^+ \subset S^1$ with $\partial \alpha_j^+ = \{t_0, t_j\}$ so that $L_j \times \alpha_j^+$ meets $V_j[t_j]$ from the positive side of $V_j[t_j]$. Then the image $f(\coprod_{j=1}^{s} L_j \times \alpha_j^+)$ is a disjoint union of annuli which is contained in E_V and whose boundary consists of L_x in V^+ and $L'_x = f(\coprod_{j=1}^{s} L_j \times t_0)$ in $f(\rho(V) \times t_0) \subset E_V$. Since

$$H_1(\rho(V) \times S^1; Z) \cong H_1(\rho(V); Z) \oplus H_1(S^1; Z),$$

we see from Lemma 3.3 that $[L_x] = 0 \in H_1(\rho(V) \times S^1; Z)$. Using that the natural homomorphism $H_1(\rho(V) \times t_0; Z) \longrightarrow H_1(\rho(V) \times S^1; Z)$ is injective, we see that L'_x bounds a 2-chain c' in $f(\rho(V) \times t_0)$. Thus, $L_x \subset V^+$ bounds a 2-chain $f(\coprod_{j=1}^{s} L_j \times \alpha_j^+) + c'$ in E_V, which means that $i_*^+(x) = 0$. Similarly, $i_*^-(x) = 0$. $\square$

The following corollary is direct from Theorem 3.1 since the infinite cyclic covering space $\tilde{E}$ is constructed from the copies $((E_V)_i;(V^+)_i,(V^-)_i)$ $(i \in Z)$ of the triplet $(E_V;V^+,V^-)$ by pasting $(V^-)_{i-1}$ to $(V^+)_i$ for all i:

Corollary 3.4. For any canonical Seifert hypersurface V of a pseudo-ribbon surface-link F, every lift $\tilde{i}: V \longrightarrow \tilde{E}$ of the natural injection $i: V \longrightarrow E$ induces the trivial homomorphism

$$\tilde{i}_* = 0 : \mathrm{t}H_1(V;Z) \longrightarrow \mathrm{t}H_1(\tilde{E};Z).$$

By using Corollary 3.4, Theorem 2.2 is proved as follows:

Proof of Theorem 2.2. We consider the following commutative diagram:

$$\begin{array}{ccc}
\tau H^2(\tilde{E},\partial\tilde{E};Z) & \xrightarrow{\cap\mu} & \mathrm{t}H_1(\tilde{E};Z) \\
\tilde{i}^* \Big\downarrow & & \Big\uparrow \tilde{i}_* \\
\mathrm{t}H^2(V,\partial V;Z) & \xrightarrow[\cong]{\cap[V]} & \mathrm{t}H_1(V;Z).
\end{array}$$

In this diagram, we have

$$\tilde{i}_* = 0 : \mathrm{t}H_1(V;Z) \longrightarrow \mathrm{t}H_1(\tilde{E};Z)$$

by Corollary 3.4 and hence

$$\cap\mu = 0 : \tau H^2(\tilde{E},\partial\tilde{E};Z) \longrightarrow \mathrm{t}H_1(\tilde{E};Z).$$

By Lemma 2.1, we have

$$\cap\mu = 0 : \hom(DH_1(\tilde{E},\partial\tilde{E};Z)^\theta, Q/Z) \cong DH_1(\tilde{E};Z)^\theta,$$

which implies $DH_1(\tilde{E},\partial\tilde{E};Z)^\theta = DH_1(\tilde{E};Z)^\theta = 0$. □

Here is another corollary to Theorem 3.1.

Corollary 3.5. For any canonical Seifert hypersurface V of a pseudo-ribbon surface-link F, the natural homomorphism $j_* : \mathrm{t}H_1(V;Z) \longrightarrow \mathrm{t}H_1(V,\partial V;Z)$ is trivial.

Proof. From the boundary isomorphism $\bar{\partial} : H_2(S^4, V\times I;Z) \cong H_1(V\times I;Z)$ and the excision isomorphism $H_2(E_V, V^+\cup V^-;Z) \cong H_2(S^4, V\times I;Z)$, the composite of the natural homomorphisms

$$\mathrm{t}H_2(E_V,\partial E_V;Z) \xrightarrow{\partial} \mathrm{t}H_1(\partial E_V;Z) \xrightarrow{i'_*} \mathrm{t}H_1(V\times I;Z)$$

is an isomorphism. Since $\partial E_V = \partial(V \times I)$, the Poincaré duality implies that the composite of the natural homomorphisms

$$\hom(\mathrm{t}H_1(E_V;Z),Q/Z) \xrightarrow{i_*^{\#}} \hom(\mathrm{t}H_1(\partial E_V;Z),Q/Z) \xrightarrow{\partial'^{\#}} \hom(\mathrm{t}H_2(V \times I, \partial(V \times I);Z),Q/Z)$$

is an isomorphism and hence the composite of the natural homomorphisms

$$\mathrm{t}H_2(V \times I, \partial(V \times I);Z) \xrightarrow{\partial} \mathrm{t}H_1(\partial E_V;Z) \xrightarrow{i_*} \mathrm{t}H_1(E_V;Z)$$

is an isomorphism by applying $\hom(\ \ ,Q/Z)$ to the homomorphisms above. Further, composing a suspension isomorphism

$$\sigma : \mathrm{t}H_1(V,\partial V;Z) \cong H_2((V,\partial V) \times (I,\partial I);Z) = \mathrm{t}H_2(V \times I, \partial(V \times I);Z)$$

to this composite isomorphism, we obtain an isomorphism

$$\theta = i_*\partial\sigma : \mathrm{t}H_1(V,\partial V;Z) \cong \mathrm{t}H_1(E_V;Z).$$

For the natural homomorphism $j_* : \mathrm{t}H_1(V;Z) \longrightarrow \mathrm{t}H_1(V,\partial V;Z)$, the composite $j_*\theta : \mathrm{t}H_1(V;Z) \longrightarrow \mathrm{t}H_1(E_V;Z)$ is equal to the map $i_*^+ - i_*^-$ which is the zero map. Thus, $j_* = 0$. □

We note that Corollary 3.5 does not mean that $\mathrm{t}H_1(V;Z) = 0$. It is unknown whether *every pseudo-ribbon surface-link F admits a Seifert hypersurface V with* $\mathrm{t}H_1(V;Z) = 0$.

4. The triple point cancelling number of a surface-link

The *triple point number* of a surface-link F in R^4, denoted by $\mathbb{T}(F)$ is the minimum on the triple point number of the singular surface $\rho(F')$ for all generic surface-links F' ambient isotopic to F. In this section, we shall discuss a similar but distinct concept on a surface-link F. Let F' be a generic surface-link ambient isotopic to F. By an ambient deformation of F' without changing $\rho(F')$, we can consider that the set $F' \cap \rho^{-1}(B_x)$ for a 3-ball neighborhood B_x of every triple point $x \in \rho(F')$ in R^3 is the union $D_1[t_1] \cup D_2[t_2] \cup D_3[t_3]$ where D_i is a proper disk in B_x and $t_1 < t_2 < t_3$. Then we make an orientation-preserving cut on $D_i \cup D_{i+1} \subset B_x$ for $i = 1$ or 2 to obtain from F' a new generic surface-link F'_1 in R^4 (see J. S. Carter-M. Saito [2,Figure N]). When we compare $\rho(F'_1)$ with $\rho(F')$, $\rho(F'_1)$ has the triple points decreased by one point and the branch points increased by two points. We call the operation $F' \Rightarrow F'_1$ a *triple point cancelling operation* on $\rho(F')$. The *triple point cancelling number* of the singular surface $\rho(F')$ is the minimum of the number of triple point cancelling operations on $\rho(F')$ needed to obtain a pseudo-ribbon surface-link F'_* in R^4.

Definition 4.1. The *triple point cancelling number* of a surface-link F in R^4, denoted by $\mathcal{T}(F)$ is the minimum on the triple point cancelling number of the singular surface $\rho(F')$ for all generic surface-links F' ambient isotopic to F.

If we compare $\mathbb{T}(F)$ to the crossing number of a classical knot, then we could compare $\mathcal{T}(F)$ to the unknotting number of a classical knot. The following lemma is useful to understand a triple point cancelling operation:

Lemma 4.2. Let F'_* be a surface-link obtained by doing m triple point cancelling operations on $\rho(F')$ for a generic surface-link F' ambient isotopic to a surface-link F in R^4. Then F'_* is ambient isotopic to a surface-link F_* obtained from F by an embedded surgery along m mutually disjoint 1-handles on F. Conversely, if F_* is a surface-link obtained from F by an embedded surgery along m mutually disjoint 1-handles on F, then F_* and F are respectively ambient isotopic to generic surface-links F'_* and F' such that F'_* is a surface-link obtained by doing m triple point cancelling operations on $\rho(F')$.

Proof. The proof of the first half part is obvious from the definition of a triple point cancelling operation. To prove the second half part, we use three 2-spheres $S_i \subset R^3[t_i]$ $(i = 1, 2, 3)$ with $t_1 < t_2 < t_3$ such that the singularity image $\rho\mathcal{S}(S_1 \cup S_2 \cup S_2)$ is homeomorphic to a suspension of a three point set and hence has just two triple points. Let B^3 be a 3-ball in R^3 such that $D_i = S_i \cap B^3[t_i]$ is a disk with $\rho(\mathrm{cl}(S_i - D_i))$ $(i = 1, 2, 3)$ mutually disjoint disks in R^3. We find a 1-handle $h \subset B^3[t_1, t_2]$ on $D_1 \cup D_2 \cup D_3$ connecting D_1 and D_2 such that $\rho(h)$ induces a triple point cancelling operation on $\rho(S_1 \cup S_2 \cup S_3)$. Let h_i $(i = 1, 2, \ldots, m)$ be mutually disjoint 1-handles on F to produce a surface-link F_*. Then F is ambient isotopic to a generic surface-link F' such that $F' \cap B^3_i[t_1, t_3] = F' \cap \rho^{-1}(B^3_i)$ and $(B^3_i[t_1, t_3]; F' \cap B^3_i[t_1, t_3], h_i)$ is $[t_1, t_3]$-level-preservingly homeomorphic to $(B^3[t_1, t_3]; D_1 \cup D_2 \cup D_3, h)$ for some m mutually disjoint 3-balls B^3_i $(i = 1, 2, \ldots, m)$ in R^3. The surface-link F_* is ambient isotopic to a surface-link F'_* obtained by doing m triple point cancelling operations on $\rho(F')$. □

We use the following result later:

Corollary 4.3. Let F be an S^2-knot obtained from any non-trivial 2-bridge knot by the 2-twist spinning. Then $\mathcal{T}(F) = 1$.

Proof. Since $DH_1(\tilde{E}; Z) = H_1(\tilde{E}; Z) \cong \Lambda/(p, t+1)$ for an integer $p \geq 3$ (see M. Teragaito [20]) and $BH_2(\tilde{E}, \partial\tilde{E}; Z) = 0$, the torsion linking ℓ_F is not zero by [7]. Hence we have $\mathcal{T}(F) \geq 1$ by Theorem 2.2. On the other hand, the S^2-knot F has a Seifert hypersurface V homeomorphic to a punctured Lens space. Then there is a 1-handle h on F such that $h \subset V$ with $\mathrm{cl}(V - h)$ is a solid torus, so that the surface F_1 obtained from F by the embedded surgery along h is a trivial T^2-surface which is a pseudo-ribbon surface-knot. Hence $\mathcal{T}(F) \leq 1$ and $\mathcal{T}(F) = 1$. □

The following remark concerns the difference $\mathbb{T}(F) - \mathcal{T}(F)$:

Remark 4.4. For a surface-link F, $\mathcal{T}(F) = 0$ if and only if $\mathbb{T}(F) = 0$ if and only if F is pseudo-ribbon by the definitions. S. Satoh observed that the difference $\mathbb{T}(F) - \mathcal{T}(F)$ is positive for every non-pseudo-ribbon surface-link F. In fact, we have $\mathbb{T}(F) \geq \mathcal{T}(F) > 0$ by the definitions. If $\mathbb{T}(F) = \mathcal{T}(F) > 0$, then we have a generic surface-link F' with $\mathcal{T}(F') = \mathbb{T}(F') = 1$ by taking $\mathcal{T}(F) - 1$ times of triple point cancelling operations. Then we see from a result of S. Satoh [16] that we find a simple double line connecting to the triple point and a branch point in $\rho(F')$, so that we can eliminate the triple point by moving this branch point along this double line, meaning that F' is a pseudo-ribbon surface-link, contradicting to $\mathcal{T}(F') = 1$. Hence $\mathbb{T}(F) - \mathcal{T}(F) > 0$. S. Satoh and A. Shima [17] showed that $\mathbb{T}(S(3_1)) = 4$ for the S^2-knot $S(3_1)$ obtained from the trefoil knot 3_1 by the 2-twist spinning. By Corollary 4.3, we have $\mathcal{T}(S(3_1)) = 1$, so that $\mathbb{T}(S(3_1)) - \mathcal{T}(S(3_1)) = 3$. *There are open questions asking whether $\mathbb{T}(F) - \mathcal{T}(F) \geq 3$ for every non-pseudo-ribbon surface-knot F and whether there is a surface-knot F such that $\mathbb{T}(F) - \mathcal{T}(F)$ is greater than any previously given positive integer.*

The inequality $\mathcal{T}(F_1 \# F_2) \leq \mathcal{T}(F_1) + \mathcal{T}(F_2)$ holds for any surface-knots F_1 and F_2, and the equality does not appear to hold in general (see T. Kanenobu [6]). *It is also an open question whether there is such an example.*

From now, we shall establish an estimate of the triple point cancelling number of a general surface-link. Let $E = \mathrm{cl}(S^4 - N)$ be the knot-exterior of a surface-knot F in S^4 where $N = F \times D^2$ is a tubular neighborhood of F in S^4 with the specified trivialization. Let V_0 be the handlebody such that $\partial V_0 = F$. Let M_ϕ be the closed 4-manifold obtained from the exterior E and $V_0 \times S^1$ by attaching the boundaries by a homeomorphism $\phi : \partial E = F \times S^1 \to \partial V_0 \times S^1$ which preserves the S^1-factor. Then M_ϕ is a closed connected oriented 4-manifold with $H_1(M_\phi; Z) \cong Z$, which we call a Z^{H_1}-*manifold*. We use the concept of exactness of Z^{H_1}-manifold in [8,9] in our argument.

Lemma 4.5. Let F be a pseudo-ribbon surface-knot in S^4. Then there exists an attachment ϕ such that the Z^{H_1}-manifold M_ϕ is a spin exact Z^{H_1}-manifold.

Proof. Let V be a canonical Seifert hypersurface for F in S^4. Let C be the image of the boundary homomorphism $\partial : H_2(V, F; Z) \longrightarrow H_1(F; Z)$. By Corollary 3.5, we have a subgroup $\bar{C}$ of $H_1(F; Z)$ such that $\bar{C} \supset C$ and the natural monomorphism $H_1(F; Z)/C \longrightarrow H_1(V; Z)$ induces a monomorphism

$$H_1(F; Z)/\bar{C} \longrightarrow \mathrm{b}H_1(V; Z).$$

Then $\bar{C}$ is a self-orthogonal complement of $H_1(F; Z)$ with respect to the intersection form $\mathrm{Int} : H_1(F; Z) \times H_1(F; Z) \longrightarrow Z$. Let x_i, y_i be a Z-basis for $H_1(F; Z)$ such that x_i $(i = 1, 2, \ldots, g)$ is a Z-basis for $\bar{C}$ and $\mathrm{Int}(x_i, x_j) = \mathrm{Int}(y_i, y_j) = 0$ and $\mathrm{Int}(x_i, y_j) = \delta_{ij}$ for all i, j. Let K_i^x and K_i^y be simple loops on F such that $K_i^x \cap K_j^x = K_i^y \cap K_j^y = K_i^x \cap K_j^y = \emptyset$ for all i, j with $i \neq j$ and $K_i^x \cap K_i^y$ is one point for all i. Let V_0 be a handlebody with $\partial V_0 = F$ such that K_i^x $(i = 1, 2, \ldots, g)$ bound mutually disjoint meridian disks D_i $(i = 1, 2, \ldots, g)$ in V_0. Let $\bar{V} = V \cup V_0$ be a closed oriented 3-manifold pasting F with these data which produces a Z^{H_1}-manifold M_ϕ. From the homology exact sequence of the pair $(\bar{V}, V)$, we obtain a natural isomorphism $\mathrm{b}H_1(V; Z) \cong H_1(\bar{V}; Z)$. When we regard V as V^+ in E_V, we see from Theorem 3.1 that the simple loop $K_i^x \subset V^+$ bounds a compact oriented surface F_i in E_V. Let $\bar{F}_i = F_i \cup D_i$ $(i = 1, 2, \ldots, g)$ be closed oriented surfaces in M_ϕ. Let $T_i = K_i^y \times S^1 \subset F \times S^1 = \partial E \subset M_\phi$ $(i = 1, 2, \ldots, g)$. Then the closed oriented surfaces $\bar{F}_i$ and T_i $(i = 1, 2, \ldots, g)$ form a Z-basis for $H_2(M_\phi; Z)$ with $\mathrm{Int}(\bar{F}_i, \bar{F}_j) = \mathrm{Int}(T_i, T_j) = 0$ and $\mathrm{Int}(\bar{F}_i, T_j) = \delta_{ij}$ for all i, j with respect to the intersection form $\mathrm{Int} : H_2(M_\phi; Z) \times H_2(M_\phi; Z) \longrightarrow Z$. [To see that $\mathrm{Int}(\bar{F}_i, \bar{F}_j) = 0$, we note that K_i^x $(i = 1, 2, \ldots, g)$ represent torsion elements in $H_1(V; Z)$, which implies that for each i there is a non-zero integer m_i such that $m_i \bar{F}_i$ is homologous to a cycle $C_i + C_i'$ in M_ϕ where C_i is a cycle in $\bar{V}$ and C_i' is a cycle in $\mathrm{int}E_V \subset S^4$. Then we have

$$m_i m_j \mathrm{Int}(\bar{F}_i, \bar{F}_j) = \mathrm{Int}(m_i \bar{F}_i, m_j \bar{F}_j) = 0$$

showing $\mathrm{Int}(\bar{F}_i, \bar{F}_j) = 0$.] In particular, M_ϕ is spin. Using a collar of $\bar{V}$ in M_ϕ, we take mutually disjoint closed oriented surfaces $\bar{F}_i'$ $(i = 1, 2, \ldots, g)$ in M_ϕ such that $\bar{F}_i' \cap \bar{V} = \emptyset$ and $\bar{F}_i'$ is homologous to $\bar{F}_i$ in M_ϕ. Since the normal disk bundle N_i' of $\bar{F}_i'$ in M_ϕ is trivial, the leaf V^* of M_ϕ obtained by taking connected sums of $\bar{V}$ and $\partial N_i'$ $(i = 1, 2, \ldots, g)$ in M_ϕ satisfies the condition that $\mathrm{t}H_1(V^*; Z) = 0$ and the image of the natural homomorphism $H_2(V^*; Z) \longrightarrow H_2(M_\phi; Z)$ is a self-orthogonal complement with respect to the intersection form $\mathrm{Int} : H_2(M_\phi; Z) \times H_2(M_\phi; Z) \longrightarrow Z$. Let M_ϕ' be the 4-manifold obtained from M_ϕ by splitting along V^*. Then by [8;(4.7.2)] $\mathrm{t}H_1(V^*; Z) = 0$ implies $\mathrm{t}H_1(M_\phi'; Z) = \mathrm{t}H_2(M_\phi, V^*; Z) = 0$. We see from [9] that V^* is an exact leaf of M_ϕ and thus M_ϕ is exact. □

For a finitely generated Λ-module H, let $e(H)$ denote the minimal number of Λ-generators of H. By convention, $e(0) = 0$. We show the following theorem, which improves and generalizes Kamada's estimate in [5]:

Theorem 4.6. Let F be a surface-link with r components and total genus g. Then for every Λ-submodule H' of $H = H_1(\tilde{E}, \partial \tilde{E}; Z)$ such that $D = H/H'$ is a $(t-1)$-divisible finite Λ-module, there is a Λ-submodule D' of D such that

$$e(D') \leq \mathcal{T}(F) \quad \text{and} \quad e(E^2(D/D')) \leq g + e(H') - r + 1 + \mathcal{T}(F).$$

Proof. Let $m = e(H')$. Let F' be a surface-link obtained from F by an embedded surgery along m mutually disjoint 1-handles representing Λ-generators for

H'. Let E' be the compact exterior of F' in S^4. Then we see that $H_1(\tilde{E}'; Z) \cong H/H' = D$, so that F' is a surface-knot of genus $g' = g + m - (r-1)$. Since the 1-handle surgery can be done apart from the triple points of $\rho(F)$, we obtain a pseudo-ribbon surface-knot F'' of genus

$$g'' = g' + \mathcal{T}(F) = g + m - (r-1) + \mathcal{T}(F)$$

from F' by $\mathcal{T}(F)$ times of triple point cancelling operations. Let E'' be the knot-exterior of F''. Then we have $H_1(\tilde{E}''; Z) \cong D/D'$ for a Λ-submodule D' of D with $e(D') \leq \mathcal{T}(F)$. By Theorem 2.2, we have a t-anti isomprphism

$$D/D' = H_1(\tilde{E}''; Z) = DH_1(\tilde{E}''; Z) \cong E^1(BH_2(\tilde{E}'', \partial\tilde{E}''; Z)).$$

Let M_ϕ be an exact Z^{H_1}-manifold obtained from F'' by Lemma 4.5. By an argument in [8], we have a Λ-isomorphism

$$E^1(BH_2(\tilde{M}_\phi; Z)) \cong E^1(BH_2(\tilde{E}'', \partial\tilde{E}''; Z)).$$

Since M_ϕ is exact, we see from [9] that there is a splitting

$$BH_2(\tilde{M}_\phi; Z) \cong X \oplus \Lambda^{g''}$$

for a torsion-free Λ-module X with $E^0E^0(X) = \Lambda^{g''}$, so that

$$E^1(BH_2(\tilde{E}'', \partial\tilde{E}''; Z)) \cong E^1(X)$$

and we have a t-anti isomorphism

$$E^2(D/D') = E^2(H_1(\tilde{E}''; Z)) \cong E^2E^1(X).$$

Using a natural Λ-epimorphism $\Lambda^{g''} \cong E^0E^0(X) \to E^2E^1(X)$ (see [7]), we have

$$e(E^2(D/D')) = e(E^2E^1(X)) \leq g''. \qquad \square$$

A surface-link F is *concordant* to a surface-link F' if there is a proper locally-flat embedding $f : F \times [0,1] \to S^4 \times [0,1]$ such that $f(F \times 0) = F \times 0$ and $f(F \times 1) = F' \times 1$. Since the triple point cancelling number of every trivial surface-link is zero, the following corollary also implies that *every positive integer is the triple point cancelling number of a surface-link with any previously given genera of the components.*

Corollary 4.7. For every surface-link F and every integer $m \geq \mathcal{T}(F)$, there is a surface-link F_* such that F_* is concordant to F and $\mathcal{T}(F_*) = m$.

Proof. Let S_n be the n-fold connected sum of any S^2-knot in Corollary 4.3. Then we have $\mathcal{T}(S_n) \leq n$. Let F_n be any connected sum of F and S_n and $F_0 = F$. Then F_n is concordant to F, since every S^2-knot is concordant to the trivial S^2-knot (see M. A. Kervaire [12]). Let E_n be the link-exterior of F_n in S^4. Let $H_n = H_1(\tilde{E}_n, \partial\tilde{E}_n; Z)$. Then we have

$$H_n = H_0 \oplus [\Lambda/(p, t+1)]^n.$$

In Theorem 4.6, we take $H = H_n$ and $H' = H_0$. Then by Theorem 4.6, there is a finite Λ-submodule D' of $D = H/H' = [\Lambda/(p, t+1)]^n$ such that $e(D') \leq \mathcal{T}(F_n)$ and $e(E^2(D/D')) \leq g + e(H_0) - r + 1 + \mathcal{T}(F_n)$. Since t acts on D as the (-1)-multiple map, we have a Λ-isomorphism

$$E^2(D/D') = \hom(D/D', Q/Z) \cong D/D'.$$

Thus, we have

$$n = e(D) \leq e(D') + e(D/D') \leq 2\mathcal{T}(F_n) + g + e(H_0) - r + 1,$$

where g and r denote the total genus and the component number of F, respectively. Using that g, $e(H_0)$ and r are independent of n, we see that

$$\lim_{n\to+\infty} \mathcal{T}(F_n) = +\infty.$$

Using that $\mathcal{T}(F_{n+1}) \leq \mathcal{T}(F_n)+1$ for all n, we find an integer n such that $\mathcal{T}(F_n) = m$ for every integer $m \geq \mathcal{T}(F)$. □

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