An abstract of the results of my research

This is a survay about a monodromy and Lefschetz pencil. So I explay here it.

We consider a smooth n dimensional projective variety X and a family $\{X_t\}_{t\in\mathbb{P}^1}$ of hyperplane sections of X. Moreover we consider a family containing singular hyperplane sections whose singurality is an only ordinary double point. This family is called a Lefschetz pencil. And we see that the generic pencil is the Lefshetz pencil.

We consider a singular hyperplane section and smooth hyperplane sections around it. Then we get a vanishing cycle $\delta \in H^{n-1}(X_p, \mathbb{Z})$ on a smooth hyperplane section determined by an ordinary doble point. The Picard-Lefschetz fomula says a relation between this vanishing cycle and a monodromy representation.

Let $X_1, \cdots, X_m$ singular hyperplane sections with an only ordinary doble point determined by $q_1, \cdots, q_m \in \mathbb{P}^1$ and p a different point from $q_1, \cdots, q_m$. Then we get vanishing cycles $\delta_1, \cdots, \delta_m \in H^{n-1}(X_p, \mathbb{Z})$ determined from ordinary double points. From the Picrd-Lefschetz formula we see that their vanishing cycles are conjugate under the monodromy representation.

And we consider the Gysin homomorphism $j_* : H^{n-1}(X_p, \mathbb{Z}) \to H^{n+1}(X, \mathbb{Z})$ induced by an inclusion map $j : X_p \hookrightarrow X$. Then a weak Lefschetz's theorem says that kernel of the Gysin homomorphism j_* is generated by their vanishing cycles. We call it a vanishing cohomology and denote it by $H^{n-1}(X_p, \mathbb{Z})_{van}$.

Then we see that the monodromy action

$$\rho : \pi_1(\mathbb{P}^1 - \{q_1, \cdots, q_m\}, p) \to \mathrm{Aut} H^{n-1}(X_p, \mathbb{Q})_{van}$$

is irreducible. In Hodge theory's terms this result says that there exists no non-trivial local subsystem of the local system with stalk $H^{n-1}(X_p, \mathbb{Q})_{van}$.