Plan of my research for the future

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Plan for the near future : The subject of my research is translated into the words of knot theory as "Study on Seifert surfaces of hyperbolic knots from the view point of Kleinian group theory". A hyperbolic knot is a knot whose complement admits a complete hyperbolic structure of finite volume. The group of a hyperbolic knot is identified with a Kleinian group via the holonomy representation of the complete hyperbolic structure. Since the fundamental group of an incompressible Seifert surface for a knot is naturally identified with a subgroup of the knot group, it is naturally determines a Kleinian surface group. Jorgensen studied deeply the combinatorial structure of the Ford domains of quasifuchsian (once-)punctured torus groups. As a consequence, the relationship has been concretely understood between hyperbolic and topological structures of a punctured torus bundle over the circle (e.g. the figure-8 knot complement). The method used here is expected to be applied to manifolds other than surface bundles. In particular, it will be applied to such a manifold with two "boundary components" that is obtained from a hyperbolic knot complement by cutting along a Seifert surface. The study of the deformation space of (complete) hyperbolic structures of such a manifold using the combinatorial structures of Ford domains will bring us a deeper understanding of topological property of knots. The kind of study of hyperbolic manifolds that uses combinatorial structures is getting known to match well the "coarse geometry" which holds an important place in the theory of Kleinian groups.

In what follows I list what I would like to study in a few years from now. Given a hyperbolic knot in a closed 3-manifold and a punctured torus T with the knot as boundary, we obtain a 3-manifold with two "boundary components" $T_{\pm}$ by cutting the closed manifold along T. Then the quasi-isometric deformation space of the hyperbolic structures with parabolic subgroups corresponding to the knot is identified with the product of two copies of the Teichmüller space of the punctured torus. It is expected that one can define an invariant which is similar to the angle invariant used in the Jorgensen theory. The problem would be the effect of the manifold connecting $T_{\pm}$ because the structure of punctured torus groups is well-known. In a short range of time, I would like to study (i) the manifolds obtained from the product of the punctured torus and the interval by performing Dehn surgery along a simple closed curve in the level-surface, and (ii) the handle-number-1 manifolds with two copies of the punctured torus as "boundary".

To study (i), from two copies of a manifold obtained by a doubly cusped group we obtain a manifold with two thrice-punctured spheres as "boundary components" and with a rank-2 cusp by gluing the manifolds along thrice-punctured spheres in the convex core boundary. Since the relationship between hyperbolic Dehn surgery and the Ford domains has been clarified by my study, we can describe the combinatorial structure of the Ford domain of the hyperbolic structure on the manifold considered in (i) which is contained as a geometrically finite boundary group in the deformation space. By this construction, we get a base-point which is necessary for the geometric continuity method. It is expected that an argument similar to the Jorgensen theory will be valid if we only consider deformation satisfying a symmetric condition. Something really new will be required when we consider deformation without symmetry.

To study (ii), the starting point seems to find a nice parametrization of the $PSL(2,\mathbb{C})$-representation space of the fundamental group of a manifold. We cannot expect an easy application of the parametrization of punctured torus groups studied by Bowditch for this case, though we can in the study of (i) because the fundamental group of the manifold has a very simple presentation. The key seems to understand the structure of the curve complex of the manifold and the relationship between the curve complex and the representation space.

Prospect for the future: I expect in the future to study the geometry of the "space of all hyperbolic 3-manifolds" by using the "coarse geometry" mentioned above. If we restrict the geometry to the knot complements, we can define a topology on the "space of knots" which is natural from geometric structures and which is different from the distance on the knots obtained from knot-diagrams. In the surface theory one has such rich mathematical objects as the moduli space and the Teichmüller space even if the topological structure of the underlying object is fixed because of the non-existence of rigidity. I think it is important to study the space of 3-manifolds to reveal the relationship between topological and geometric structures by such a method mentioned here.