I have been studying moduli spaces of left-invariant (pseudo-)Riemannian metrics on a Lie group G. Here, the moduli space is defined as the orbit space of the action of the group $\mathbb{R}_{>0} \times \mathrm{Aut}(G)$ on the space of all left-invariant metrics $\mathcal{M}_{p,q}(G)$ with signature (p,q) on G. There are three backgrounds to study the moduli spaces:

(B1) Since there are too many left-invariant metrics on a Lie group G, it is hard to determine whether G admits nice left-invariant metrics (*e.g.* left-invariant Ricci solitons) or not. If one describes the moduli space of left-invariant metrics on G, one may be able to determine the existence of nice metrics on G.

(B2) The space $\mathcal{M}_{p,q}(G)$ is a noncompact symmetric space, and the action of $\mathbb{R}_{>0} \times \mathrm{Aut}(G)$ on $\mathcal{M}_{p,q}(G)$ is an isometric action. Isometric actions on noncompact symmetric spaces have been studied actively. The $\mathbb{R}_{>0} \times \mathrm{Aut}(G)$-actions may provide nice examples.

(B3) If G is a three dimensional solvable Lie group, it has been known that a Riemannian metric $\langle , \rangle \in \mathcal{M}_{3,0}(G)$ is a Ricci soliton if and only if the orbit $\mathbb{R}_{>0} \times \mathrm{Aut}(G).\langle , \rangle \subset \mathcal{M}_{3,0}(G)$ is a minimal submanifold. The fact implies that one can study left-invariant metrics on a Lie group G in terms of the geometry of the $\mathbb{R}_{>0} \times \mathrm{Aut}(G)$-actions.

For the background (B1), we have determined the moduli spaces of pseudo left-invariant metrics for each signatures on a particular solvable Lie group G. Also, by applying the description of the moduli spaces, we have shown that any left-invariant metrics on G have constant sectional curvature. These results have been published as Paper 5 in the List.

For the background (B2), I have given some examples of n-dimensional Lie groups G whose $\mathbb{R}_{>0} \times \mathrm{Aut}(G)$-actions on $\mathcal{M}_{n,0}(G)$ are hyperpolar actions with singular orbits. Little is known about hyperpolar actions with singular orbits, and nontrivial examples have been required. These results have been published as Paper 7 in the List.

For the background (B3), I have shown that if the orbit $\mathbb{R}_{>0} \times \mathrm{Aut}(G).\langle , \rangle \subset \mathcal{M}_{n,0}(G)$ is an isolated orbit through a Riemannian metric $\langle , \rangle \in \mathcal{M}_{n,0}(G)$, then the metric $\langle , \rangle$ is a Ricci soliton. An isolated orbit is a minimal submanifold. The result asserts that the previous result "minimal $\Rightarrow$ Ricci soliton" in three dimensional solvable cases holds for general settings if one strengthens the assumption "minimal" to "isolated". These results have been published as Paper 1 in the List.

In Paper 1, I also have obtained a simple characterization for isolated orbits of the proper isometric actions. For a proper isometric action of a Lie group H on a Riemannian manifold X, an orbit $H.p$ is an isolated orbit if and only if there are no nonzero H-invariant normal vector fields on $H.p$. Now I have generalized the essence of isolated orbit to arbitrary submanifolds as follows:

- A Riemannian submanifold Y in a Riemannian manifold X is called an <u>arid submanifold</u> if there exists some proper isometric action on X such that the action preserves Y, and Y does not admit nonzero invariant normal vector fields under the action.