# Summary of my research

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Let $G$ be a connected reductive algebraic group over an algebraically closed field of characteristic $p$ and $\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of $G$.

In 1976, Bala and Carter proved that there exists a bijection between the nilpotent orbits in $\mathfrak{g}$ and the conjugacy classes of pairs $(L, P_L)$ where $L$ is a Levi subgroup (of a parabolic subgroup) of $G$ and $P_L$ is a distinguished parabolic subgroup of $L$ if $p = 0$ or $p \gg 0$. It is called the Bala-Carter theorem nowadays.

Let $W$ be the Weyl group and $\Delta$ a set of simple roots of $G$ relative to some maximal torus. There exists a natural bijection between the conjugacy classes of pairs $(L, P_L)$ as above and the $W$-conjugacy classes of pairs $(L_I, P_{I,J})$ where $L_I$ is the standard Levi subgroup corresponding to $I \subset \Delta$ and $P_{I,J}$ is the standard distinguished parabolic subgroup of $L_I$ corresponding to $J \subset I$. Moreover the problem when a parabolic subgroup is distinguished results in arguments of root system, too. This implies the classification of the nilpotent orbits.

Afterward, Pommerening showed that the Bala-Carter theorem can be extended to good characteristic ($p$ is good if $p \geq 7$, in particular). This proof, however, needed to computation case-by-case in each root system.

On the other hand, A. Premet gave a fairly short conceptual proof of the Pommerening's result in 2003. Moreover the proof immediately implies that the existence for good transverse slices to the nilpotent orbits. Premet's proof is based on the Kempf-Rousseau theory. This theory has been used for problems of characteristic 0 case, but has been regarded so useless in prime characteristic.

Suppose that the commutator group $\mathcal{D}G$ of $G$ is simply connected. Let $G_{\mathbb{C}}$ be the semisimple, simply connected algebraic group over $\mathbb{C}$ with the same root system as $G$, and $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G_{\mathbb{C}})$. For any nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}_{\mathbb{C}}$, let $\lambda_{\mathcal{O}}$ be a cocharacter (of a split maximal torus over $\mathbb{Z}$) which is gained by the Jacobson-Morozov theorem. We can regard $\lambda_{\mathcal{O}}$ as a cocharacter of $G$. For any cocharacter $\lambda$ and for any $i \in \mathbb{Z}$, we set

$$\mathfrak{g}(i;\lambda) = \{Z \in \mathfrak{g} \,|\, \mathrm{Ad}(\lambda(\xi))(Z) = \xi^i Z\}.$$

Denote $\mathfrak{g}(2;\lambda_{\mathcal{O}})_{\mathrm{reg}}$ by the unique $C_G(\mathrm{Im}\,\lambda_{\mathcal{O}})$-orbit which is open dense in $\mathfrak{g}(2;\lambda_{\mathcal{O}})$. Premet showed that for any $X \in \mathfrak{g}(2;\lambda_{\mathcal{O}})_{\mathrm{reg}}$, $\lambda_{\mathcal{O}}$ is optimal for $X$. Using this fact, he proved the Bala-Carter theroem.

A main charateristic of the Kempf-Rousseau theory is to construct a good-natured cocharacter for arbitrary nilpotent element directly. It seems that we can prove the Bala-Carter theorem using the Kempf-Rousseau theory mainly and not using the representation theory of Lie algebras over $\mathbb{C}$ as possible. We clear the necessary conditions for this in the master's thesis.