Results

(1) Quantum deformations of prehomogeneous vector spaces of commutative parabolic type

This is a joint work with T. Tanisaki and Y. Morita. Prehomogeneous vector spaces of commutative parabolic type are constructed in simple Lie algebras $\mathfrak{g}$ over the complex number field. Their quantum deformations mean the deformation of coordinate algebras as non-commutative algebra with actions of quantized enveloping algebras. These quantum deformations A_q of coordinate algebras of prehomogeneous vector spaces had been constructed by the case study. But our construction does not depend on types of simple Lie algebras.

Let $\mathfrak{p}^+ = \mathfrak{l} \oplus \mathfrak{m}^+$ be a parabolic subalgebra of $\mathfrak{g}$, where $\mathfrak{l}$ is a maximal reductive subalgebra and $\mathfrak{m}^+$ is the nilpotent part. Then the pair $(L, \mathfrak{m}^+)$, where L is the algebraic group corresponding to $\mathfrak{l}$, is a prehomogeneous vector space called of commutative parabolic type. By the Killing form, the coordinate algebra of $\mathfrak{m}^+$ is identified with the enveloping algebra $U(\mathfrak{m}^-)$ of the opposite nilpotent subalgebra $\mathfrak{m}^-$. Our A_q is naturally constructed as a subalgebra $U_q(\mathfrak{m}^-)$ of the quantized enveloping algebra $U_q(\mathfrak{g})$ corresponding to $U(\mathfrak{m}^-)$. Moreover our method contains the construction of the quantum deformation f_q of the fundamental relative invariant f of $(L, \mathfrak{m}^+)$.

By this method I gave the explicit descriptions of A_q and f_q in the case of classical simple Lie algebra $\mathfrak{g}$.

(2) Quantum deformations of b-functions

For a fundamental relative invariant f of $(L, \mathfrak{m}^+)$ there exist the constant coefficient differential operator ${}^t f(\partial)$ and a polynomial $b(s)$ such that ${}^t f(\partial) f^{s+1} = b(s) f^s$. This polynomial $b(s)$ is called the b-function of f. We can define the contravariant symmetric bilinear form on $U(\mathfrak{m}^-)$ by constant coefficient differential operators ${}^t g(\partial)$ corresponding to $g \in U(\mathfrak{m}^-)$. This form is unique up to constant multiple. From the natural bilinear form of $U_q(\mathfrak{g})$, I defined the quantum deformation of the above contravariant form on $U(\mathfrak{m}^-)$ as a nondegenerate symmetric bilinear form on $U_q(\mathfrak{m}^-)$. By this form on $U_q(\mathfrak{m}^-)$ the "quantum differential operator" ${}^t f_q(\partial)$, which is a linear maps on $U_q(\mathfrak{m}^-)$, can be defined, and I showed that there exists $b_q(s) \in \mathbb{C}(q)[q^s]$ satisfying ${}^t f_q(\partial) f_q^s = b_q(s) f_q^s$ for $s \in \mathbb{Z}_{\geq 0}$. I gave the explicit expression of $b_q(s)$ by the calculating in each case. In the case of type A, by Noumi-Umeda-Wakayama the different deformation of $b_q(s)$ from mine have been given.

The b-function of $(L, \mathfrak{m}^+)$ is a special case of a b-function of a relative invariant of a flag manifold G/P. In the case where P is a Borel subgroup, M. Kashiwara determined b-functions by the universal Verma modules. Similarly in the case $\mathrm{Lie}(P) = \mathfrak{p}^+$, I gave the following expression of $b(s)$ of $(L, \mathfrak{m}^+)$ in terms of weights of an irreducible $\mathfrak{g}$-module $V(\mu)$ with highest weight μ (μ is the fundamental dominant weight uniquely determined from $(L, \mathfrak{m}^+)$).

$$b(s) = \prod_{\eta \in Wt(\mu) \setminus \{\mu\}} \left((s\mu + \rho + \mu, s\mu + \rho + \mu) - (s\mu + \rho + \eta, s\mu + \rho + \eta) \right),$$

where $Wt(\mu)$ is the set of highest weights of irreducible $\mathfrak{l}$-submodules of $V(\mu)$ and ρ is the half sum of positive roots of $\mathfrak{g}$. This b-function corresponds to a generator of an ideal of a polynomial ring. By similar approach I gave the quantum b-function of $(L, \mathfrak{m}^+)$ as a generator of an ideal of a Laurent polynomial ring. In the case of type A the above difference between my $b_q(s)$ and Noumi-Umeda-Wakayama's is equal to the difference of the choice of the generator.