# Results

(1) Quantum deformations of prehomogeneous vector spaces of commutative parabolic type:

This is a joint work with T. Tanisaki and Y. Morita. Prehomogeneous vector spaces of commutative parabolic type are constructed in simple Lie algebras $\mathfrak{g}$ over the complex number field. Their quantum deformations mean the deformation of coordinate algebras as non-commutative algebra with actions of quantized enveloping algebras. Let $\mathfrak{p}^+ = \mathfrak{l} \oplus \mathfrak{m}^+$ be a parabolic subalgebra of $\mathfrak{g}$, where $\mathfrak{l}$ is a maximal reductive subalgebra and $\mathfrak{m}^+$ is the nilpotent part. Then the pair $(L, \mathfrak{m}^+)$, where $L$ is the algebraic group corresponding to $\mathfrak{l}$, is a prehomogeneous vector space called of commutative parabolic type. The coordinate algebra of $\mathfrak{m}^+$ is identified with the enveloping algebra $U(\mathfrak{m}^-)$ of the opposite nilpotent subalgebra $\mathfrak{m}^-$. The quantum deformation $A_q$ of the coordinate algebra is naturally constructed as a subalgebra $U_q(\mathfrak{m}^-)$ of the quantized enveloping algebra $U_q(\mathfrak{g})$ corresponding to $U(\mathfrak{m}^-)$. Moreover this method contains the construction of the quantum deformation $f_q$ of the fundamental relative invariant $f$ of $(L, \mathfrak{m}^+)$. These constructions do not depend on types of simple Lie algebras.

By this method I gave the explicit descriptions of $A_q$ and $f_q$ in the case where $\mathfrak{g}$ is a classical simple Lie algebra.

(2) Quantum deformations of $b$-functions:

For a fundamental relative invariant $f$ of $(L, \mathfrak{m}^+)$ there exist the constant coefficient differential operator ${}^t f(\partial)$ and a polynomial $b(s)$ such that ${}^t f(\partial) f^{s+1} = b(s) f^s$. This polynomial $b(s)$ is called the $b$-function of $f$.

The $b$-function $b(s)$ of $(L, \mathfrak{m}^+)$ is a special case of a $b$-function of an invariant on a flag manifold $G/P$. In the case where $P$ is a Borel subgroup, M. Kashiwara determined $b$-functions by the universal Verma modules. Similarly in the case $\mathrm{Lie}(P) = \mathfrak{p}^+$, I gave the following expression of $b(s)$ of $(L, \mathfrak{m}^+)$ in terms of weights of the irreducible $\mathfrak{g}$-module $V(\mu)$ with highest weight $\mu$, where $\mu$ is the fundamental dominant weight corresponding to the simple root $\alpha$ uniquely determined from $(L, \mathfrak{m}^+)$.

$$b(s) = \prod_{\eta \in Wt(\mu) \setminus \{\mu\}} \left( (s\mu + \rho + \mu, s\mu + \rho + \mu) - (s\mu + \rho + \eta, s\mu + \rho + \eta) \right),$$

where $Wt(\mu)$ is the set of highest weights of irreducible $\mathfrak{l}$-submodules of $V(\mu)$ and $\rho$ is the half sum of positive roots of $\mathfrak{g}$. The $b$-function $b(s)$ corresponds to a generator of an ideal of a polynomial ring defined by a universal Verma module. The differential operator used in the above correspondence can be regarded as a central element of the enveloping algebra $U(\mathfrak{l})$. This relation is the Capelli identity.

By the quantum universal Verma module, the quantum $b$-function $b_q(s)$ of $(L, \mathfrak{m}^+)$ is regarded as a generator of an ideal of a Laurent polynomial ring, and

$$b_q(s) = \prod_{\eta \in Wt(\mu) \setminus \{\mu\}} \left[ \frac{(s\mu + \rho + \mu, s\mu + \rho + \mu) - (s\mu + \rho + \eta, s\mu + \rho + \eta)}{2m_\eta} \right]_q.$$

Here $m_\eta$ is the positive integer defined by $(\mu, \mu - \eta) = m_\eta \frac{(\alpha,\alpha)}{2}$, and $[t]_q = \frac{q^t - q^{-t}}{q - q^{-1}}$. By the choice of generator this expression of $b_q(s)$ is up to $q_\alpha^{ns}$ multiple ($q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$, $n \in \mathbb{Z}$). Clearly the zero points of $b(s)$ coincide with those of $b_q(s)$. The proof of these results from universal Verma modules do not depend on types of simple Lie algebras $\mathfrak{g}$.