2 Present and future Research

I will list now some main topics of research I'm currently interested in, and plan to work on in the visible future. My choice of problems is not fixed, and will also depend on my interaction with other mathematicians I expect to meet. After every topic I will also briefly explain the expected results of its investigation.

2.1 Hyperbolic volume

There have been so far several situations, in which the hyperbolic volume exhibits a relation to a combinatorily defined knot invariant. The most important one is Kashaev's conjecture on values of colored Jones polynomials, popularized by H. Murakami.

Another correspondence was observed by Brittenham, namely that the volume is bounded on alternating knots of given genus. His bound can be improved by inequalities of Lackenby-Agol-Thurston involving an invariant of knot diagrams called twist number. These bounds are also related to conjectures of Dunfield, namely that the volume linearly approximates a logarithm of the determinant of alternating knots. I'm currently interested in obtaining and improving such inequalities.

2.2 Trivalent graphs and enumeration of knots by genus

There is also a relation between the Brittenham approach and the enumeration problems of knots of given genus, and the sl_N weight system of trivalent graphs known from the theory of Vassiliev invariants.

One can express the maximal volume of knots of given canonical genus by links L_G associated to planar trivalent graphs G similarly to Habiro's claspers. The sl_N weight system of G then is related to both the hyperbolic volume and the enumeration of knots by genus, and also to the enumeration of 1-vertex triangulations of oriented surfaces carried out by my collaborator A. Vdovina.

2.3 Weight system-volume-conjectures

The form of the relation between the sl_N weight system W_N of G and volume of L_G is not yet clear, but calculations suggest that definitely something is going on beyond accidental coincidences. I hope in the future to deepen my understanding of hyperbolic volumes, in particular to understand better these relations. I hope also to find out whether and what is a relation of these coincidences to the Volume conjecture. For example, can one calculate the colored Jones polynomials of L_G, and establish the relation modulo the Volume conjecture? Can one gain insight into the Volume conjecture from these relations? This is also linked to understanding the, in particular multiplicative, structure of the sl_N weight systems of G. Few facts are known, including the multiplicative character of Vogel's algebra and Bar-Natan's version of the 4-Color-Theorem. A new observation from the work of Bacher and Vdovina on 1-vertex triangulations is that the linear term of the sl_N weight systems vanishes in Euler characteristic < -1. Their work also implies bounds on the number of linear monomials in the calculation of W_N, that in turn bound the asymptotical growth of the number of alternating knots of given genus, which I seek to improve. I hope to progress on at least some of these many interrelated problems in the future.

2.4 Gauss sum invariants

There are still opportunities left in the application of Gauss sum invariants to positive knots and related knot classes, most naturally, to improve the existing inequalities. More importantly, a computational project is to implement Fiedler's new character Gauss sum invariants (which take as input not a single diagram, but a sequence of diagrams of knots in the solid torus), in the hope to distinguish knot orientation with them, after the success on braids. This will obviously give a huge impetus on the theory of Gauss sum invariants.

2.5 Number theoretic properties of knot invariants

One of my original mathematical interests was number theory (my specialization turned into a different direction by the influence I experienced during my studies). I'm interested in situations in which knot invariants can be studied from the point of view of some elementary number theoretic properties. A series of problems I intend to work on is related to determinants of achiral knots with particular properties, for example unknotting number one. These determinants are sums of two squares. It would be interesting to study which such numbers occur in which situations. Number theoretic properties of the determinant have also applications to unknotting numbers and knot distance, and maybe I can find more such applications.

2.6 Non-trivial Jones polynomial problem

I seek further generalizations of the non-triviality result for the Jones polynomial, for example to arborescent knots. I also try to prove that there are achiral knots of any odd crossing number at least 15.

2.7 Mutation and the colored Jones polynomial

In joint work with Toshifumi Tanaka, we found examples of knots with the same polynomial invariants and hyperbolic volume, with variously coinciding 2-cable polynomials and colored Jones polynomials, which are not mutants. In particular, we show that there exists an infinite family of pairs of hyperbolic knots with equal colored Jones polynomial, which are not mutants. This answers a question of Przytycki. We would like to understand further the related phenomena, and construct infinite series of examples with specific properties.

2.8 5-moves and Montesinos links

Extending work of Ishiwata, I determined the 5-move equivalence classes of Montesinos links up to mutation; one obtains from this a Jones and Kauffman polynomial test for a Montesinos link. To remove 'up to mutation', I must determine what of the Montesinos links $(1/2,...,1/2,2/5,...,2/5)$ and their mutants (permute the $1/2, 2/5$ s) are 5-move equivalent. A 5-move preserves the 5-Burnside group of the link group, and maps 1-to-1 the homomorphisms to a fixed 5-group, so I explore group theoretic ways to accomplish this.

2.9 Other topics

I'm also working on some problems of braids, for example the question of Rudolph whether (strongly) quasipositive knots have (strongly) quasipositive braid representations of minimal strand number. I expect counterexamples but they are not easy to construct. (I had in the past some counterexamples for positive braid representations and, jointly with M. Hirasawa, minimal genus band representations.)