Research plan: Constant anisotropic mean curvature surfaces in spherically symmetric 3-manifolds, and their stability

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0.1. **Present research related to research plan.** The *Morse index* of a constant mean curvature (CMC) surface is defined as the sum of the multiplicities of the negative eigenvalues of its *Jacobi operator* ([15], [16] and [20]), and a CMC surface is *stable* when its area is minimal with respect to all variations preserving volume on each side of the surface. When it is stable, then the Morse index is ≤ 1. So conversely, to show that some such surface is unstable, it is sufficient to show its Morse index is ≥ 2.

The index of both minimal and non-minimal CMC surfaces in the Euclidean 3-space $\mathbb{R}^3$ has been well studied. It is known that the only stable complete minimal surface is a plane [6], and minimal surfaces have finite index if and only if they have finite total curvature [8]. And the index for many minimal surfaces which have finite total curvature has been found ([7], [8], [12], [13]). It is also reported that a CMC surface is stable if and only if it is a round sphere [3], and CMC surfaces without boundary have finite index if and only if they are compact ([11], [18]). Furthermore, the Morse index of Wente tori is described in [14]. In the unit 3-sphere $\mathbb{S}^3$, the totally geodesic spheres have index 1 ([17]). F. Urbano [20] proved that the minimal Clifford torus has index 5 and any other closed (compact without boundary) minimal surface has index ≥ 6. For the closed CMC surfaces in $\mathbb{S}^3$, a similar result to that in [20] is not yet known. But W. Rossman and the author [15] found relatively sharp lower bounds for the index of closed CMC surfaces of revolution, and a numerical method to compute the index of such surfaces is described in [16]. The index and stability of minimal and CMC surfaces in the hyperbolic 3-space $\mathbb{H}^3$ have also been studied ([5], [4], [10]). The stability properties of CMC surfaces of revolution in general simply-connected spherically symmetric 3-spaces, and in the particular case of a positive-definite 3-dimensional slice of Schwarzschild space are described in /citesultana.

The interface between two non-mixing materials may often be represented as a surface. The surface forms into an equilibrium shape for a potential energy which is determined by the forces acting on it. When the surface is liquid, its surface tension is isotropic, i.e. it does not depend on the direction of the surface normal. For more structured materials an anisotropic surface tension is more appropriate. M. Koiso and B. Palmer ([9]) gave the definition of anisotropic mean curvature and also examined the geometry and stability of surfaces with constant anisotropic mean curvature (CAMC) which is a generalization of the idea of constant mean curvature (CMC). In [9], M. Koiso and B. Palmer investigated the analogous class of CAMC surfaces of revolution. The composition of this class of surfaces is strikingly similar to the classical case.

Furthermore, lagrangian submanifold are an important class of geometric objects in a Khler manifold, and the minimal Lagrangian submanifolds are canonical representatives with interesting mathematical structure. Moreover, a compact minimal Lagrangian submanifold immersed in a Khler manifold is Hamiltonian stable when the second variation of its volume is nonnegative under all Hamiltonian deformations. And this hamiltonian stability of Lagrangian submanifolds is studied [1].

0.2. **Proposed plan.** In Osaka City University my host is Professor Y. Ohnita, who works on the Hamiltonian stability and rigidity of Lagrangian submanifolds in complex projective spaces. My initial efforts at Osaka city University would be to acquire a more thorough understanding of the works of Professor Y. Ohnita, and also of other experts in Japan, such as W. Rossman, M. Koiso, S. Nayatani, M. Umehara, K. Yamada, J. Inoguchi, M. Guest and R. Miyaoka. The second phase of my research efforts would then be to apply the tools of my present research to the stability of CAMC surfaces and Lagrangian submanifolds to achieve a deeper understanding of their properties.

To be more specific, I would pursue the following problems:

1 Studying the CAMC surfaces in $\mathbb{R}^3$ and their behavior, and finding the formula for the Jacobi operator from the second variaton formula.
2 Computing the Morse index of CAMC surfaces in $\mathbb{R}^3$.
3 Searching for the CAMC surfaces in other ambient spaces which are spherically symmetric.
4 Having properly defined CAMC surfaces in spherically symmetric 3-spaces, then computing the Morse index and examining the stability of those surfaces.
5 Examining the Willmore stability for minimal surfaces, at least for the tori and Klein bottles which invariant under circle actions.
6 Investigating fundamental questions regarding the Hamiltonian stability and rigidity of Lagrangian submanifolds.

I expect to obtain new results that expand our knowledge about the stability of various surfaces and their connections with more applied fields. It is expected that a number of scientific publications for refereed international journals and conferences would be completed during the period of this project.

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