Research Summary: Constant mean curvature surfaces of revolution in spherically symmetric 3-manifolds, and their stability

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The field of constant mean curvature (CMC) surfaces began in the 19'th century with the works of Riemann, Weierstrass and Enneper. Recently, J. Dorfmeister, F. Pedit, and H. Wu developed an approach for the construction of CMC surfaces, commonly called the DPW method.

A surface of revolution in the Euclidean 3-space $\mathbb{R}^3$ is generated by revolving a given planar curve, the profile curve, about a line in the plane containing it. Surfaces of revolution in the unit 3-sphere $\mathbb{S}^3$ and in the hyperbolic 3-space $\mathbb{H}^3$ are similarly generated by revolving planar curves about a geodesic line, the axis, in the geodesic plane containing this given curve. We define *Delaunay surfaces* as translationally-periodic CMC surfaces of revolution. The profile curves of the CMC surfaces of revolution will periodically have minimal and maximal distances to the axis of revolution. We call the points of minimal distances the *necks*, and the points of maximal distances the *bulges*.

We compute the explicit conformal parametrizations of Delaunay surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ by using the DPW method. We show that these parametrizations are in full agreement with those of more classical approach. In the global study of complete CMC surfaces, Delaunay surfaces (unduloids and nodoids) play an important role as models for the asymptotic behavior and period problems of CMC surfaces, so explicitly understanding how the DPW method makes Delaunay surfaces is valuable. We also compute an explicit area formla for the fundamental pieces of Delaunay surfaces in all of these 3-spaces.

When Delaunay surfaces in $\mathbb{S}^3$ close to become tori, we can study their Morse index. We define the *Morse index* of a closed (compact without boundary) CMC surface $\mathcal{S}$ as the number of negative eigenvalues of its Jacobi operator $\mathcal{L}$, where the function space is the C^∞ functions from $\mathcal{S}$ to the reals $\mathbb{R}$. The *nullity* is the multiplicity of the zero eigenvalue of $\mathcal{L}$. The study of Morse index is important as it is a measure of the degree of instability of a surface. The Morse index of general closed CMC surfaces of revolution in $\mathbb{S}^3$ is still unknown. Hence, we compute lower bounds for the Morse index and nullity of CMC tori of revolution in $\mathbb{S}^3$. In particular, all such tori have index at least five, with index growing at least linearly with respect to the number of the surfaces' bulges, and the index of such tori can be arbitrarily large.

To test the sharpness of the lower bounds in the above theorem, we wish to numerically compute the eigenvalues of the Jacobi operator. Therefore, we study the eigenvalue problems of Jacobi operators and prove a theorem about elliptic Schrodinger operators with symmetric Schrodinger potential functions, defined on a function space over a closed loop. The result is similar to a known result for a function space on an interval with Dirichlet boundary conditions. These theorems provide accurate numerical methods for finding the spectra of those operators over either type of function space. As an application, we numerically compute the Morse index of CMC tori of revolution in $\mathbb{S}^3$, confirming that every such torus has Morse index at least five, and showing that the lower bounds given in the above theorem for this Morse index are close to optimal.

Furthermore, we study the stability properties of CMC surfaces of revolution in general simply-connected spherically symmetric 3-spaces, and in the particular case a positive-definite 3-dimensional slice of Schwarzschild space. We derive their Jacobi operators, and then prove that closed CMC tori of revolution in such spaces are unstable, and finally numerically compute the Morse index of some minimal and closed non-minimal CMC surfaces of revolution in the slice of Schwarzschild space.