Research results

Takanori Yasuda

Let G be a connected semisimple group over a number field k. Writing $\mathbb{A}$ for the adele ring of k, $G(k)\backslash G(\mathbb{A})$ becomes a locally compact space, and has a $G(\mathbb{A})$-invariant measure. By this we can consider the space $L^2(G(k)\backslash G(\mathbb{A}))$ of square-integrable functions on $G(k)\backslash G(\mathbb{A})$. The representation on it induced by the right regular action of $G(\mathbb{A})$ has an orthogonal decomposition,

$$L^2(G(k)\backslash G(\mathbb{A})) = L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A})) \oplus L^2_{\mathrm{cont}}(G(k)\backslash G(\mathbb{A})),$$

where $L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}))$ is the maximal completely reducible closed subspace and $L^2_{\mathrm{cont}}(G(k)\backslash G(\mathbb{A}))$ is its orthogonal complement. I considered two invariant subspaces of $L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}))$ which are closely related to each other for a specific G. I shall explain these spaces concretely. As G we take the unitary group of the rank 2 hyperbolic hermitian space over a quaternion division algebra R over k. It is an inner form of $Sp(2)$. We say that an element ϕ of $L^2(G(k)\backslash G(\mathbb{A}))$ is L^2-cusp form if the constant terms of ϕ along all the proper k-parabolic subgroup vanish. It is known that the space $L^2_0(G(k)\backslash G(\mathbb{A}))$ of L^2-cusp forms of G is contained in $L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}))$.

The first invariant space of my concern is the *residual spectrum* of G. The residual spectrum is the orthogonal complement of $L^2_0(G(k)\backslash G(\mathbb{A}))$ in $L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}))$. I determined the irreducible decomposition of the residual spectrum of G completely when k is totally real. $GL(n)$, $Sp(2)$, $U(2,2)$ are examples of which the irreducible decomposition of the residual spectrum are completely determined. There are other examples such that the irreducible decomposition of their residual spectrum are partially determined. These examples are all quasisplit algebraic group. Since Langlands' spectral theory of Eisenstein series is applied to determine the residual spectrum we need to know an analytic behavior of Eisenstein series. In case of a quasisplit group we can know the analytic behavior of Eisenstein series from the Langlands-Shahidi theory. However G is not quasisplit group. I reduced the problem of the analytic behavior of Eisenstein series for G to that for $Sp(2)$ using the Jacquet-Langlands correspondence. Some irreducible components of the residual spectrum of G are determined by the Langlands classification, and the others are constructed by the theta lifts of the trivial representation of the unitary groups of rank 1 skew-hermitian spaces.

The second invariant space of my concern is the space of *CAP forms* of G. This is a subspace of $L^2_0(G(k)\backslash G(\mathbb{A}))$. We say that an L^2-cusp form ϕ is CAP form if there exists an element ϕ' of an irreducible component of the residual spectrum such that ϕ and ϕ' share the same absolute values of Hecke eigenvalues at almost all places of k. I constructed many examples of CAP forms of G. In case of $GSp(2)$, Piatetski-Shapiro constructed the Saito-Kurokawa representations as examples of CAP forms, and Soudry determined the other CAP forms. I constructed the inner form version of the Saito-Kurokawa representations and the examples given by Howe and Piatetski-Shapiro. The method of construction is as follows. Since a pair of G and the unitary group $U(V)$ of rank 2 skew-hermitian space V of determinant 1 over R becomes a reductive dual pair, we can consider the Weil representation of $U(V) \times G$. On the other hand, $U(V)$ has a structure close to the product of the multiplicative group of two quaternion algebras. Hence an L^2-cusp form of $U(V)$ is obtained by the form of the product of L^2-cusp forms of the multiplicative groups of two quaternion algebras. When one of these L^2-cusp forms is a constant function, its theta lift becomes a CAP form of G.

There exists a CAP form which does not satisfy the multipliticity one property. This phenomenon does not occur in case of $Sp(2)$. This is caused by the failure of the Hasse principle. That is, there exist two skew-hermitian spaces which are not isometric globally but isometric locally. These produce different spaces, but which are isomorphic as representations. The multiplicity obtained in this way coincides with Arthur's multiplicity formula.