# Result of research

Reiko Shinjo

Any graph can be embedded into the three dimensional Euclidean space $R^3$. However, some graphs can be embedded into the plane $R^2$, which are called planar, and the other graphs can not be, which are called non-planar. We can regard a spatial embedding of planar graph into $R^2 \subset R^3$ as a 'standard' embedding because this embedding is the simplest sort of embedding. On the other hand, it is difficult to decide standard embeddings of non-planar graphs. As a part of the research of a standard embedding, K. Kobayashi introduced a locally unknotted spatial embedding of a graph. Locally unknotted means that there is a set of cycles in the graph $G$ which forms a basis for $H_1(G;\mathbb{Z})$ and the set of knots in the spatial embedding corresponding to the set of cycles bounds a set of disks with disjoint interiors. In [3], we got an idea from his concept and considered a set of connected, compact and orientable surfaces with disjoint interiors bounded by knots in a spatial graph. If each surface has a distinct boundary, we say that the set is a collection of spanning surfaces. Especially if each surface is homeomorphic to a disk, the set is called a collection of spanning disks. T. Endo-T. Otsuki proved that any graph has a locally unknotted spatial embedding. In general, the rank of $H_1(G;\mathbb{Z})$ is not an upper bound of the number of spanning surfaces. Hence we gave the upper bound of the number of spanning surfaces and showed that this upper bound is the least upper bound by constructing a spatial embedding which realizes the upper bound with disks.

We tried to extend the concept of boundary links to spatial graphs in [2]. We defined a spatial embedding of a graph which has a collection of spanning surfaces which corresponds to the set of all knots in the embedding as a boundary spatial embedding and researched. From the result in [3], we have that not every graph has a boundary spatial embedding. Hence we gave a characterization of graphs which have boundary spatial embeddings. Then we classified boundary spatial embeddings of graphs completely up to self pass-equivalence and showed that any two boundary spatial embeddings of a graph are self sharp-equivalent. These are natural extensions of the result concerning boundary links given by L. Cervantes-R. A. Fenn and T. Shibuya.

K. Taniyama showed that two spatial embeddings are spatial-graph-homologous if and only if they have the same Wu invariant. It is known that Wu invariant of a spatial embedding of a graph coincides with linking number if the graph is homeomorphic to a disjoint union of two circles, and it coincides with Simon invariant if $G$ is homeomorphic to a complete graphs on five vertices or a complete bipartite graph on three-three vertices. We showed that two spatial embeddings of a graph are spatial-graph-homologous if and only if all of their linking numbers and Simon invariants coincide. Therefore Homology classification came to be given by the simple calculation, since both linking number and Simon invariant are integral invariants that are easily calculated from a regular diagram of a spatial graph. T. Motohashi-Taniyama showed that two spatial embeddings are spatial-graph-homologous if and only if they are delta-equivalence and Taniyama-A.Yasuhara showed that a delta-move does not change any order 1 finite type invariant of spatial graphs. Therefore we see that linking number and Simon invariant determine all of order 1 finite type invariants of spatial graphs.

By the Classification Theorem of closed 3-braids given by J. S. Birman-W. W. Menasco, it is known that any link $L$ which is a closed $n$-braid ($n = 1, 2$ or $3$) has only a finite number of conjugacy classes of $n$-braid representatives in the $n$-braid group. Concerning 4-braids, an infinite sequences of pairwise non-conjugate 4-braids representing the unknot (resp. $(2,p)$-torus link ($p \geq 2$)) discovered by H. R. Morton (resp. E. Fukunaga). In preprint [6], for any knot (resp. link which satisfies certain conditions) represented as a closed $n$-braid ($n \geq 3$) we gave an infinite sequence of pairwise non-conjugate $(n+1)$-braids representing the knot (resp. the link). In existing results, Morton and Fukunaga used the homomorphism from the 4-braid group to 3-braid group. In addition, Fukunaga used a certain coujugacy invariant. On the other hand, we used Conway polynomial which is a knot invariant. We evaluated third coefficient of the Conway polynomial by using the linking number of a certain links obtained by braids in our sequence. Our result can be obtained by the simple calculation.