Research plan

Takanori Ayano

I will construct the concrete theory of Abelian functions for higher genus curves.

1. Degeneration of the inverse function of the hyperelliptic integrals.

Let $\alpha_1, \ldots, \alpha_5$ be five distinct complex numbers, V be the hyperelliptic curve of genus 2 defined by $y^2 = (x-\alpha_1)\cdots(x-\alpha_5)$, and α be a complex number such that $\alpha \neq \alpha_i$ for any i. Let

$$u = \int_{(\alpha_1,0)}^{(x,y)} -\frac{x-\alpha}{2y}dx.$$

We can regard x and y as functions of u. Since $y(u)$ can be expressed in terms of $x(u)$ and $x'(u)$, by substituting this expression into the defining equation of V, we can obtain the differential equation satisfied by $x(u)$. In this research, we will derive the recurrence formulae with respect to the coefficients of the series expansion of $x(u)$ around $u = 0$. When we take $\alpha_4, \alpha_5 \to \alpha$, the curve V becomes a singular curve. This singular curve is birationally equivalent to the elliptic curve E defined by $y^2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$. In this research, we will give relationships between the limit of $x(u)$ under $\alpha_4, \alpha_5 \to \alpha$ and the Weierstrass elliptic function associated with E. Further, we will study whether the convergence of $x(u)$ under $\alpha_4, \alpha_5 \to \alpha$ becomes the uniform convergence.

2. We will decompose a meromorphic function on $\mathbb{C}^2$ into a product of a holomorphic function and a meromorphic function.

Let $f(x)$ be a polynomial of degree 5, V be the hyperellptic curve of genus 2 defined by $y^2 = f(x)$, and $\sigma(u) = \sigma(u_1, u_3)$ be the sigma funciton associated with V. The function σ is a holomorphic function on $\mathbb{C}^2$. Let $\sigma_i = \frac{\partial}{\partial u_i}\sigma$ and $\wp_{i,j} = \frac{\partial}{\partial u_i \partial u_j}(-\log\sigma)$. Via the Abel-Jacobi map, the rational function x on V can be identified with the meromorphic functions $g(u) = \wp_{1,1}(2u)/2$ and $h(u) = -\sigma_3(u)/\sigma_1(u)$. Then there exists a meromorphic function k on $\mathbb{C}^2$ such that the relation $g - h = \sigma k$ holds on $\mathbb{C}^2$. In this research, we will give an explicit expression of k in terms of σ.

3. Relationships between Abelian functions of genus 2 and elliptic functions.

Let $f(x)$ be a polynomial of degree 3 such that $f(x)$ has no multiple root and $f(0) \neq 0$, V be the hyperelliptic curve of genus 2 defined by $y^2 = f(x^2)$, E_1 be the elliptic curve defined by $y^2 = f(x)$, E_2 be the elliptic curve defined by $y^2 = x^3 f(1/x)$, and $\mathrm{Jac}(V), \mathrm{Jac}(E_1), \mathrm{Jac}(E_2)$ be the Jacobians of V, E_1, E_2, respectively. Then we can construct morphisms $\varphi_1 : V \to E_1$ and $\varphi_2 : V \to E_2$. By these morphisms, the holomorphic maps

$$\psi_1 \;:\; \mathrm{Jac}(V) \to \mathrm{Jac}(E_1), \quad \psi_2 \;:\; \mathrm{Jac}(V) \to \mathrm{Jac}(E_2)$$

are induced. In this research, we will express ψ_k explicitly. By pulling back the Weierstrass elliptic functions associated with E_k by ψ_k, we can obtain an Abelian function f_k associated with V. In this research, we will express f_k in terms of the fundamental Abelian functions $\wp_{i,j}$. Further, we will pull back the addition formula of the Weierstrass elliptic function by the map ψ_k and derive the addition formula of f_k.