Research plan (April 2021 ∼)

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The study of elliptic operators in the framework of global setting brought deep understandings in the manifold theory and analysis. In my coming research activity, I am going to study global phenomena of mainly sub-elliptic operators including elliptic cases.

Laplace-Beltrami operator, Dirac operator and signature operator are defined by means of geometric structures of manifolds, the first one is a second order elliptic differential operator and the last two are first order elliptic differential operators, respectively. In the study here sub-Laplacians and sub-Riemannian structures are the main subjects. A sub-Laplacian is a sub-elliptic second order differential operator and is defined by a sub-Riemannian structure.

This geometric structure (= sub-Riemannian structure) requires that there exists a bracket generating sub-bundle in the tangent bundle.

The opposite structure, that is foliation structure, was studied since many years ago. On the other hand until recently it was not so much studied of such structure from geometric and global analytic point of view, so that I am expecting that to pursue the research of this subject has an enough meaning even by comparing with Riemannian manifold theory.

Contact manifolds and nilpotent Lie groups are well known examples of manifolds carrying a "good" sub-Riemannian structure and it happens often that the total space of a Riemannian submersion has both structures, foliation and sub-Riemannian. Hence there are ample examples of such manifolds to be studied.

Although we can define a transversely elliptic operator on foliated manifolds, there are no natural differential operator associated to the foliation structure. On the other hand there is an intrinsically defined second order differential operator (we call it sub-Laplacian) on sub-Riemannian manifolds reflecting the structure. Hence, again there must be enough meaning to study the sub-Laplacian in contrast with the Laplacian for the Riemannian case.

Moreover, since this operator satisfies the "*sub-elliptic estimate*" (= sub-ellipticity) proved by Hörmander, the basic structure of the spectrum is same with elliptic operator cases. However there is non trivial characteristic variety so that it is not enough to treat them in the topological framework(K-theory). This may include difficulty in the study and at the same time we can expect the possibility of new phenomena other than obtained by the property of ellipticity. So, I am going to continue the research on these topics under the direction to find a new phenomena which will be not included in the elliptic operator theory, and together including the problems whether named classical manifolds have this structure and analytic properties of this type operator, like Weyl low or an explicit construction of the heat kernel.

Although in this academic year 2020 our activities were restricted by the pandemic crisis and so I could not invite nor visit my long standing coworkers in Europe, I expect it will be over within the first half of this year and I can continue joint research work with them.

More concretely, under the title

"From elliptic operators to sub-elliptic operators"

I am planing the research on the following explicit problems (a) $\sim$ (e) in the coming several academic years:

(a) Nilpotent Lie groups are basic examples of sub-Riemannian manifolds carrying a "good" sub-Riemannian structure (i.e., equi-regular sub-Riemannian structure). Among them I had been studying a class of groups attached to Clifford algebras (which I call "pseudo H-type algebras and groups") in these years. As the next step, I am going to classify "*integral lattices*" and study various properties on spectral zeta functins on these nilmanifolds.

In particular, I am going to study the inverse spectral problem of the sub-Laplacian from the point of a classical famous problem relating with the Riemann zeta function which must be interesting since the residues of the spectral zeta function of the sub-Laplacian relate with the values of Riemann zeta function at integral points.

(b) I was expecting that "Weinstein's eigenvalue theorem" must be valid also for sub-Laplacians under similar assumptions (=Maslov quantization condition), and I had established a complete proof of such the theorem. However important is to find concrete examples of such Lagrangian submanifolds satisfying the condition for this case.

So I will try to find various examples of such Lagrangian submanifolds, especially examples other than tori, since in such cases these are well-known as common hyper surfaces of first integrals in the completely integrable geodesic (or bicharacteristic) flow cases.

(c) Construction of a Calabi-Yau structure on the cotangent bundle or the punctured cotangent bundle (= zero section removed) and a construction of a quantization operator ($\approx$ Bargmann type transformation) by the method of "pairing polarizations"

The structure guarantees the existence of bi-characteristic flow invariant measure required in the Malsov quatization condition, if the structure is bi-characteristic flow invariant.

The manifolds carrying such a structure are highly restricted and we know that the punctured cotangent bundle of the Cayley projective plane $T_0^*(P^2\mathbb{O})$ has a Kähler structure and moreover Calabi-Yau structure. So within this year I shall complete the problem for this case and consider its applications.

(d) I had started a study of a sub-Laplacian on a manifold with conic singularities from a point of view of symbolic calculus of pseudo-differential operators.

(e) Although Lie groups, especially compact Lie groups carry various sub-Riemannian structures, it is not clear in general whether a compact symmetric space has such a structure. So I will try to study when or which symmetric spaces have such a structure and study a possible classification.