Abstract of future research

For the details of some notations, refer to abstract of present research.

1. **de la Vallée Poussin mean**: At this stage, we don't know that the estimate

$$\|(f - v_n(f))w\|_{L^p(\mathbb{R})} \leq CT^{1/4}(a_n)E_{p,n}(w; f). \tag{A}$$

is sharp or not. We may decrease more the power of T by technique of proof. And we also may weaken the condition of an Erdős-type weight, that is $T(a_n) \leq c\,(n/a_n)^{2/3}$. We also show L^p boundedness of derivatives of the de la Vallée Poussin mean. One of these is the following: Suppose that w belongs to $\mathcal{F}_\lambda(C^4+)$ which is a smooth subclass of $\mathcal{F}(C^2+)$. If $T^{(2j+1)/4}fw \in L^p(\mathbb{R})$, then for $2 \leq p \leq \infty$,

$$\|v_n(f)^{(j)}w\|_{L^p(\mathbb{R})} \leq C\left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4}fw\|_{L^p(\mathbb{R})} \tag{B}$$

and for $1 \leq p \leq 2$,

$$\|v_n(f)^{(j)}w\|_{L^p(\mathbb{R})} \leq C\left(\frac{n}{a_n}\right)^j a_n^{(2-p)/2p}\|T^{(2j+1)/4}fw\|_{L^2(\mathbb{R})}$$

for all $1 \leq j \leq k$ and $n \in \mathbb{N}$. We use duality of L^1-norm and Riesz-Thorin interpolation theorem to prove L^p boundedness of the de la Vallée Poussin mean. But, unfortunately, we cannot use duality of L^1-norm because T remains in the proof and it is unbounded. So we could know (B) holds true or not for $1 \leq p \leq 2$. We would like to find the way to break through obstructions by unboundedness of T.

2. **Lagrange interpolation polynomials and Laguerre-type weights**: We are studying convergence condition of the Lagrange interpolation polynomials $L_n(f)(t)$ with weight w_ρ. Here, f is a continuous function f on $\mathbb{R}$. We need to find the condition such that

$$\lim_{n\to\infty} \|(L_n(f) - f)w_\rho\|_{L^p(\mathbb{R})} = 0 \tag{C}$$

for $1 < p < \infty$. We already showed (C) in L^2-case. It is also shown the similarities for the cases $2 < p < \infty$, but these conditions are very complicated. Moreover, not continuous for p. To solve these problem, first step of the solution is the following: For $1 < \lambda < \infty$, to find a continuous function $g_p(x)$ with 2 variables (t, p) such that $g_p(t) = 1$ $(1 < p \leq \lambda,\ x \in \mathbb{R})$, $\lim_{p\to\lambda+0} g_p(t) = 1$ and for every $1 < p < \infty$,

$$\lim_{n\to\infty} \|(L_n[F] - F)g_p w_\rho\|_{L_p(\mathbb{R})} = 0$$

and some conditions for existence of $g_p(t)$. In addition, as an application of above subject, we will study the case of $\mathbb{R}^+ := [0, \infty)$. This study have a connection with the theory of Laguerre polynomials. A weight w_ρ is an analogy of Laguerre weight xe^{-x}. To advance this study, first, we are going to define a relevant class of weights on $\mathbb{R}^+$ in response to $\mathcal{F}(C^2+)$ on $\mathbb{R}$ by symmetry of $\mathcal{F}(C^2+)$ and $W_\rho(x) := x^\rho \exp(-R(x)),\ x \geq 0$ is a weight in this class. We put $x = t^2$. Then We can transform the theory on $\mathbb{R}^+$ into the theory on $\mathbb{R}$. To applicate to Laguerre-type, we have to show relations between $\mathbb{R}^+$ and $\mathbb{R}$ of orthogonal polynomials, Lagrange interpolation polynomials and MRS number, etc.