Abstract of present research

Our study is polynomial approximation theory as an application of potential theory. On $\mathbb{R}$, every polynomial $P(t)$ blows up as $|t| \to \infty$, we must multiply a weight function $w(t)$. Then, for $1 \le p \le \infty$ and $fw \in L^p(\mathbb{R})$, is there exist a sequence of polynomials $\{P_n\}$ such that

$$\lim_{n\to\infty} \|(f - P_n)w\|_{L^p(\mathbb{R})} = 0 \tag{A}$$

holds? We assume that an exponential weight w belongs to relevant class $\mathcal{F}(C^2+)$. Let w be $w(t) = \exp(-Q(t))$. We consider a function $T(t) := tQ'(t)/Q(t)$, $(t \neq 0)$. If T is bounded, then w is called a Freud-type weight, and otherwise, w is called an Erdős-type weight. In this study, we consider Erdős-type weights.

1. **Convergence of the de la Vallée Poussin mean**: The de la Vallée Poussin mean $v_n(f)$ of f is defined by $v_n(f)(t) := \frac{1}{n}\sum_{j=n+1}^{2n} s_j(f)(t)$, where $s_m(f)(x)$ is the partial sum of Fourier series of f for orthogonal polynomials with respect to w. The degree of approximation for f defined by $E_{p,n}(w; f) := \inf_{P\in\mathcal{P}_n} \|(f - P)w\|_{L^p(\mathbb{R})}$. Here, $\mathcal{P}_n$ is the set of all polynomials of degree at most n. We assume that $w \in \mathcal{F}(C^2+)$ and suppose that $T(a_n) \le c\,(n/a_n)^{2/3}$ for some $c > 0$. Here, the notation a_n is called MRS number. Then there exists a constant $C \ge 1$ such that for every $n \in \mathbb{N}$ and when $fw \in L^p(\mathbb{R})$,

$$\|(f - v_n(f))w\|_{L^p(\mathbb{R})} \le CT^{1/4}(a_n)E_{p,n}(w; f). \tag{B}$$

   We show the following two conditions such that the right side of (B) converge to 0 as $n \to \infty$. First, if w belongs to smooth subclass $\mathcal{F}_\lambda(C^3+)$ and $T^{1/4}fw \in L^p(\mathbb{R})$, second, if f be an absolutely continuous function with $f'w \in L^p(\mathbb{R})$. Then the de la Vallée Poussin mean $v_n(f)$ for a function f gives one of the concrete example of (A). Moreover, if f is more smoother function, $v_n(f)$ is not only a good approximation polynomial for f, but also its derivatives give an approximation for f'.
2. **Uniform convergence of the Fourier partial sum**: By the way, we show the condition uniformly convergence of $s_n(f)$ for a weight in a class $\mathcal{F}_\lambda(C^3+)$: Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$. Suppose that f is continuous and has a bounded variation on any compact interval of $\mathbb{R}$. If f satisfies $\int_{\mathbb{R}} w(x)|df(x)| < \infty$, then

$$\lim_{n\to\infty} \left\|(f - s_n(f))\frac{w}{T^{1/4}}\right\|_{L^\infty(\mathbb{R})} = 0.$$

3. **Lagrange interpolation polynomials**: Let w_ρ be $w_\rho(t) := |t|^\rho w(t)$ for $\rho > 0$. For $f \in C(\mathbb{R})$, we write the Lagrange interpolation polynomial $L_n(f)(t)$ with nodes $\{t_{j,n,\rho}\}_{j=1}^n$, where $\{t_{j,n,\rho}\}_{j=1}^n$ are the zeros of n-th orthogonal polynomial with respect to w_ρ. We show the condition such that $L_n(f)(t)$ converges to f with w_ρ in L^2-norm: Let $f \in C(\mathbb{R})$ and $\beta > 1/2$. If $|(1 + t^2)^{\beta/2}w_\rho(x)f(x)| \to 0$ as $|t| \to \infty$, then we have

$$\lim_{n\to\infty} \|(L_n(f) - f)w_\rho\|_{L^2(\mathbb{R})} = 0. \tag{C}$$

   Moreover, for $1 < p < 2$, let $\beta > 1/p$ and $|(1 + t^2)^{\beta/2}T^{1/2}(t)\Phi^{-1/4}(t)w_\rho(t)f(t)| \to 0(|t| \to \infty)$(where, $\Phi(t) := (1 + Q(t))^{-2/3}T^{-1}(t)$). Then we have (C) in the case of $1 < p < 2$.