Plan of the study (Yosuke Saito)

We use the following notations. For a complex number q satisfying $|q|<1$ and $x\in\mathbb{C}$, set $(x;q)_\infty:=\prod_{n\geq 0}(1-xq^n)$. For $x\in\mathbb{C}\backslash\{0\}$, set $\Theta_q(x):=(q;q)_\infty(x;q)_\infty(qx^{-1};q)_\infty$ (theta function). By setting $D_x=x\dfrac{\partial}{\partial x}$ (Euler derivative), we define $E_k(x;q):=-{D_x}^k\log\Theta_q(x)$ ($k\in\mathbb{Z}_{>0}$). For complex numbers q, p satisfying $|q|<1$, $|p|<1$ and $x\in\mathbb{C}$, set $(x;q,p)_\infty:=\prod_{m,n\geq 0}(1-xq^mp^n)$. For $x\in\mathbb{C}\backslash\{0\}$, set $\Gamma_{q,p}(x):=\dfrac{(qpx^{-1};q,p)_\infty}{(x;q,p)_\infty}$ (elliptic gamma function).

Let N be an positive integer, β be a complex number, and p be a complex number satisfying $|p|<1$. The Hamiltonian of the elliptic Calogero-Moser system $H_N^{\mathrm{CM}}(\beta,p)$ is defined by

$$H_N^{\mathrm{CM}}(\beta,p):=\sum_{i=1}^N{D_{x_i}}^2-\beta(\beta-1)\sum_{1\leq i\neq j\leq N}E_2(x_i/x_j;p).$$

Then the following fact is known: the function $\Psi_N(x;\beta,p):=\prod_{1\leq i\neq j\leq N}\Theta_p(x_i/x_j)^{\beta/2}$ satisfies

$$H_N^{\mathrm{CM}}(\beta,p)\Psi_N(x;\beta,p)=\{2N\beta D_p+C_N(\beta,p)\}\Psi_N(x;\beta,p),\ \cdots(*)$$

where $C_N(\beta,p)$ is a complex number determined by N, β, p. It is remarkable that the derivative $D_p=p\dfrac{\partial}{\partial p}$ is in the right hand side of $(*)$. This means that the elliptic Calogero-Moser system has a solution which involves the infinitesimal deformation of the elliptic modulus p.

Let N be a positive integer, q, p be complex numbers satisfying $|q|<1$, $|p|<1$, and t be a complex number satisfying $t\in\mathbb{C}\backslash\{0\}$. The Hamiltonian of the elliptic Ruijsenaars system $H_N^{\mathrm{R}}(q,t,p)$ is defined by

$$H_N^{\mathrm{R}}(q,t,p):=\sum_{i=1}^N\prod_{j\neq i}\left(\frac{\Theta_p(tx_i/x_j)\Theta_p(qt^{-1}x_i/x_j)}{\Theta_p(x_i/x_j)\Theta_p(qx_i/x_j)}\right)^{\frac{1}{2}}T_{q,x_i},$$

where $T_{q,x}$ is the q-shift operator which is defined by $T_{q,x}f(x)=f(qx)$. Then the function $\Psi_N(x;q,t,p):=\prod_{1\leq i\neq j\leq N}\left(\dfrac{\Gamma_{q,p}(tx_i/x_j)}{\Gamma_{q,p}(x_i/x_j)}\right)^{1/2}$ satisfies

$$H_N^{\mathrm{R}}(q,t,p)\Psi_N(x;q,t,p)=t^{\frac{-N+1}{2}}\sum_{i=1}^N\prod_{j\neq i}\frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)}\Psi_N(x;q,t,p).\cdots(**)$$

It is known that by setting $t=q^\beta$ and by taking the limit $q\to 1$ appropriately, the equation $(**)$ degenerates to the equation $(*)$. Thus it is probable that the equation $(**)$ contains a certain difference deformation of the elliptic modulus p. By standing the point of view, the author will study the elliptic Ruijsenaars system.