# RESEARCH RESULTS

AIRI ASO

Twisted Alexander polynomial is a generalization of Alexander polynomial, which is one of the classical invariants of knots, and is defined for a knot and a representation of the fundamental group of the knot complement. Twisted Alexander polynomial was introduced by Lin [1], and Wada defined it for arbitrary finitely presentable groups and its representations [2] in 1990's. Wada showed that the twisted Alexander polynomial can distinguish Kinoshita-Terasaka knot and Conway's 11 crossing knots, whose Alexander polynomials are trivial [2].

It is known that there are relations between twisted Alexander polynomials and the properties of knots, e.g. the genus and the fiberedness of knots. More precisely, for a knot $K$ and a nonabelian $\mathrm{SL}(2,\mathbb{F})$-representation $\rho : \pi_1(S^3 \backslash K) \to \mathrm{SL}(2,\mathbb{F})$ of $\pi_1(S^3 \backslash K)$, the degree of the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ (i.e. the difference of the the highest degree and the lowest degree of $\Delta_{K,\rho}(t)$) is less than or equal to the number obtained from the genus of $K$, and if $K$ is fibered $\Delta_{K,\rho}(t)$ is a monic polynomial.

We say that a knot is hyperbolic if the knot complement admits a complete hyperbolic metric of finite volume. For a hyperbolic knot $K$, there is a canonical representation of the fundamental group $\pi_1(S^3 \backslash K)$ of the knot complement, called the holonomy representation of $K$, and Dunfield–Friedl–Jackson [4] conjectured that the genus and fiberedness of $K$ are determined by the twisted Alexander polynomial associated to the holonomy representation of $K$ (in what follws, we call this conjecture "Conjecture A").

In [5], for a $(-2, 3, 2n + 1)$-pretzel knot $K$ and a family of representations of $\pi_1(S^3 \backslash K)$ which contains the holonomy representation of $K$, we computed the twisted Alexander polynomials of $K$ associated to each representation in the family, and we proved that the above Conjecture A is true for $(-2, 3, 2n+1)$-pretzel knots. Moreover, in [6], we studied the twisted Alexander polynomials of all Montesinos knots with tunnel number one which contains $(-2, 3, 2n + 1)$-pretzel knots and two-bridge knots. More precisely, for a family, which contains $(-2, 3, 2n + 1)$-pretzel knots, and two-bridge knots, we computed the degree and the leading coefficient of their twisted Alexander polynomials associated to any $\mathrm{SL}(2,\mathbb{C})$-representations, and then we reduced Conjecture A to a certain condition of the holonomy representations. For other Montesinos knots with tunnel number one, in a similar way as in [5], we computed twisted Alexander polynomials associated to any $\mathrm{SL}(2,\mathbb{C})$-representations, and we proved that Conjecture A is true in this case.

Another application of the twisted Alexander polynomial is some relations to the hyperbolic volume. For a cusped hyperbolic 3-manifold, Menal-Ferrer–Porti showed that the hyperbolic volume appears in the asymptotic behavior of Reidemeister torsion [7], and Kitano [8] and Yamaguchi [9] showed some relations between the twisted Alexander polynomials of knots and the Reidemeister torsions. By using these results, Goda [10] proved that for a hyperbolic knot $K$, the hyperbolic volume $\mathrm{Vol}(S^3 \backslash K)$ of $S^3 \backslash K$ appears in the asymptotic behavior of the twisted Alexander polynomials associated to certain $\mathrm{SL}(n,\mathbb{C})$-representations $\rho_n$, where $\rho_n$ is induced from the holonomy representation of $K$. Furthermore, Park gave a generalization of the formula of the hyperbolic volume with the Reidemeister torsion, and he conjectured that the complex volume is obtained by a complexification of his results [11]. Here the complex volume $\mathrm{cv}(M)$ of a hyperbolic manifold $M$ is defined to be the complex number $\mathrm{Vol}(M) + 2\pi^2 \mathrm{cs}(M)\sqrt{-1}$ whose real part is the hyperbolic volume $\mathrm{Vol}(M)$ of $M$ and the imaginary part is a multiple of the Chern-Simons invariant $\mathrm{cs}(M)$ of $M$.

In my recent work, to obtain a complexification of Goda's formula in [10], for any hyperbolic knot $K$ of 6 crossings or fewer, we studied the asymptotic behavior of the twisted Alexander polynomials of $K$ associated to $\rho_n$, and we conjectured the equality

$$\lim_{n\to\infty} \frac{4\pi \log \Delta_{K,\rho_n}(1)}{n^2} = \mathrm{cv}(S^3 \backslash K).$$

In fact, we observed that the left hand side approaches to $\mathrm{cv}(S^3 \backslash K)$ as $n$ gets bigger.

## References


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