Research plan

Takanori Ayano

1. Degeneration of the inverse function of the hyperelliptic integrals.

Let $\alpha_1, \ldots, \alpha_5$ be five distinct complex numbers, V be the hyperelliptic curve of genus 2 defined by $y^2 = (x - \alpha_1) \cdots (x - \alpha_5)$, and α be a complex number such that $\alpha \neq \alpha_i$ for $i = 1, 2, 3$. Let

$$u = \int_{(\alpha_1, 0)}^{(x,y)} -\frac{x - \alpha}{2y} dx.$$

We can regard x and y as functions of u. Since $y(u)$ can be expressed in terms of $x(u)$ and $x'(u)$, by substituting this expression into the defining equation of V, we can obtain the differential equation satisfied by $x(u)$. In this research, we will derive the recurrence formulae with respect to the coefficients of the series expansion of $x(u)$ around $u = 0$. When we take $\alpha_4, \alpha_5 \to \alpha$, the curve V becomes a singular curve. This singular curve is birationally equivalent to the elliptic curve E defined by $y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$. In this research, we will give relationships between the limit of $x(u)$ under $\alpha_4, \alpha_5 \to \alpha$ and the Weierstrass elliptic function associated with E. Further, we will study whether the convergence of $x(u)$ under $\alpha_4, \alpha_5 \to \alpha$ becomes the uniform convergence. The coefficients of the series expansion of $x(u)$ around $u = 0$ are called *generalized Bernoulli-Hurwitz numbers*. The generalized Bernoulli-Hurwitz numbers appear in the definition of the universal Todd genus in algebraic topology. The inverse function of the hyperelliptic integrals is important in the Schwarz–Christoffel mapping in complex analysis. The study of the inverse function of the hyperelliptic integrals will contribute to these fields.

2. We will decompose a meromorphic function on $\mathbb{C}^2$ into a product of a holomorphic function and a meromorphic function.

Let $f(x)$ be a polynomial of degree 5, V be the hyperellptic curve of genus 2 defined by $y^2 = f(x)$, and $\sigma(u) = \sigma(u_1, u_3)$ be the sigma funciton associated with V. The function σ is a holomorphic function on $\mathbb{C}^2$. Let $\sigma_i = \frac{\partial}{\partial u_i}\sigma$ and $\wp_{i,j} = \frac{\partial}{\partial u_i \partial u_j}(-\log \sigma)$. Via the Abel-Jacobi map, the rational function x on V can be identified with the meromorphic functions $g(u) = \wp_{1,1}(2u)/2$ and $h(u) = -\sigma_3(u)/\sigma_1(u)$. Then there exists a meromorphic function k on $\mathbb{C}^2$ such that the relation $g - h = \sigma k$ holds on $\mathbb{C}^2$. In this research, we will give an explicit expression of k in terms of σ.

3. Expressions of the Abelian functions of genus 2 in terms of the elliptic functions

In this research, when a hyperelliptic curve V of genus 2 admits a morphism of degree 2 to an elliptic curve, we expressed the Abelian functions associated with V in terms of the Weierstrass elliptic functions. On the other hand, T. Shaska derived a normal form of the defining equation of a curve of genus 2 which admits a morphism of degree 3 to an elliptic curve. By applying this result, when a hyperelliptic curve V of genus 2 admits a morphism of degree 3 to an elliptic curve, we will express the Abelian functions associated with V in terms of the Weierstrass elliptic functions.