2. Research Projects

The research project can be divided into three parts.

2.A. Study on Iitaka's Conjecture In this research, we study Iitaka's conjecture, which is mentioned in 1.A-i. As mentioned above, inequality (I) does not hold in positive characteristic, so we also treat a weaker inequality

$$\kappa(X) \geq \kappa(Y) + \kappa'(\overline{G}). \tag{I'}$$

Here, $\overline{G}$ is the geometric generic fiber, $\kappa'(\overline{G})$ is the Kodaira dimension of $\overline{G}$ in the sense of Luo [6], which is the same as the classical definition ($\kappa'(\overline{G}) = \kappa(\overline{G})$) if $\overline{G}$ is non-singular. In characteristic 0, we have $\kappa'(\overline{G}) = \kappa(G)$, and in positive characteristic, we only have $\kappa'(\overline{G}) \leq \kappa(G)$. Also, in characteristic 0, Viehweg [8] conjectured that if $\kappa(Y) \geq 0$, then

$$\kappa(X) \geq \mathrm{Max}\,(\kappa(Y), \mathrm{Var}(f)) + \kappa(G), \tag{V}$$

where $\mathrm{Var}(f)$ denotes the variant of f. This inequality is stronger than inequality (I'). The goal of this research is to clarify when inequalities (I), (I') and (V) hold respectively. In particular, we mainly focus on the following three cases.

2.A-i. Case when Y is of maximal Albanese dimension The Albanese morphism α of a non-singular projective variety X induces an algebraic fiber space whose total (resp. base) space is X (resp. a variety of maximal Albanese dimension), if α is separable over its image. Therefore, one can expect that the results obtained by this research will promote the study of the class of non-singular projective varieties with non-trivial Albanese morphism. Also, in this case, Iitaka's conjecture was solved affirmatively by Cao–Păun [2], but their proof relies an analytic method. From the viewpoint of giving new knowledge for algebraic solution, the solution in positive characteristic is also meaningful.

2.A-ii. Case when the geometric generic fiber$\overline{G}$ is of general type ($\dim \overline{G} = \kappa(\overline{G})$) This case was proved by Kollár [5] in characteristic 0. In the research until now, we have proved Iitaka's conjecture by combining the positivity theorem and some special assumption on Y, but in this research, we aim to prove the conjecture when Y is arbitrary non-singular projective variety.

2.A-iii. Case when the base space Y is of general type ($\dim Y = \kappa(Y)$) This case is reduced to the study on the weak positivity of $f_*\omega^m_{X/Y}$ (2.B) by Viehweg's argument [8].

2.B. Study on the positivity theorem We continue the study on the positivity of the direct images of (relative) pluricanonical bundles mentioned in 1.A-i, and the purpose of this research is to establish the positivity theorem in a larger class of algebraic fiber spaces. The positivity theorem is an important theorem that is applied to not only Iitaka's conjecture but also moduli problem. A concrete goal of this research is to remove the global assumption on the geometric generic fiber $\overline{G}$ and the generic fiber G in the statements of the known results. In view of this research until now ([E7, Theorem 1.1], [E5, Theorem 1.4]), we can expect that if $\overline{G}$ has only mild singularities, then the direct images of sufficiently large powers of (relative) canonical bundle have kinds of positivity. Also, we investigate the positivity of subsheaves of $f_*\omega^m_{X/Y}$ and $f_*\omega^m_X$. For applications, it is enough to show the existence of non-trivial weakly positive subsheaves.

2.C. Study of canonical bundle formula The canonical bundle formula has been generalized in various form and applied in many situation, since it was established by Kodaira [4] in the study of elliptic surfaces. In this research, we consider the following form of canonical bundle formula due to Ambro [1]:

Question: Assume that there is a $\mathbb{Q}$-divisor L on Y such that $K_{X/Y} \sim_{\mathbb{Q}} f^*L$. Then is there an effective $\mathbb{Q}$-divisor Δ_Y on Y such that $L \sim_{\mathbb{Q}} \Delta_Y$ and that the pair (Y, Δ_Y) has only "mild" singularities?

Here, $\sim_{\mathbb{Q}}$ denotes $\mathbb{Q}$-linear equivalence. If this question is solved affirmatively, then for example, positivity theorems proved in this research until now ([E7, Theorem 1.1], [E10, Theorem 1.3]) Iitaka's inequalities ([E7, Theorem 1.4], [E10, Theorem 1.5]), and techniques used in their proof can be applied to the class of algebraic fiber spaces whose ω_X is relatively semi-ample. By this research until now, it was proved that if $\overline{G}$ has only F-pure singularities, then a $\mathbb{Q}$-divisor L on Y satisfying $K_{X/Y} \sim_{\mathbb{Q}} f^*L$ is pseudo-effective ([E5, Theorem 1.4]). Furthermore, it was also shown that if $\overline{G}$ is globally F-split, then $\kappa(L) \geq 0$, which is stronger than the pseudo-effectivity ([E7, Theorem 3.17]). In this research, at first, we let the intermediate goal be to prove $\kappa(L) \geq 0$ with removing the assumption that "$\overline{G}$ is globally F-split". Next, we investigate the singularity (Y, Δ_Y), where Δ_Y is an effective $\mathbb{Q}$-divisor on Y such that $\Delta_Y \sim_{\mathbb{Q}} L$, and try to solve the above question.

Reference

[E1]⋯[E10] are referred from 「(3) 論文リスト」, using the same numbering. The following are other references. [1] F. Ambro. The moduli b-divisor of an lc-trivial fibration. *Compos. Math.*, 141(2):385–403, 2005. [2] J. Cao and M. Păun. Kodaira dimension of algebraic fiber spaces over Abelian varieties. *Invent. Math.*, 207(1):345–387, 2017. [3] Y. Kawamata. Characterization of abelian varieties. *Compos. Math.*, 43(2):253–276, 1981. [4] K. Kodaira. On compact analytic surfaces: Ⅱ. *Ann. of Math.*, 77(3):563–626, 1963. [5] J. Kollar. Subadditivity of the Kodaira dimension: fibers of general type. *Adv. Stud. in Pure Math.*, 10:361–398, 1987. [6] Z. Luo. Kodaira dimension of algebraic function fields. *Amer. J. Math.*, 109(4):669–693, 1987. [7] M. Popa and C. Schnell. On direct images of pluricanonical bundles. *Algebra Number Theory*, 8(9):2273–2295, 2014. [8] E. Viehweg. Weak positivity and the additivity of the Kodaira dimension for certain fiber spaces. In Algebraic Varieties and Analytic Varieties, pages 329–353. Kinokuniya, North-Holland, 1983.