(2) Research Results Until Now and Research Projects

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1. Research results Until Now

1.A. Studies on algebraic fiber spaces ([E5,6,7,9,10]) In this research, we studied on algebraic fiber spaces, that is, separable surjective morphisms between non-singular projective varieties defined over an algebraically closed field k with irreducible general fibers.

1.A-i. On Iitaka's conjecture and weak positivity theorem ([E6,7,10]) Iitaka conjectured that, when the characteristic of k is 0, the Kodaira dimension κ of the total space X, the generic fiber G and the base space Y of an algebraic fiber space $f : X \to Y$ satisfy the subadditivity

$$\kappa(X) \geq \kappa(Y) + \kappa(G). \tag{I}$$

The Kodaira dimension is a key birational invariant in the classification theory of varieties. The conclusions of Iitaka's conjecture include important results such as the birational characterization of abelian varieties due to Kawamata [3]. A clue to study the conjecture is the direct images $f_*\omega^m_{X/Y}$ of pluricanonical bundles. Viehweg [8] showed that $f_*\omega^m_{X/Y}$ is weakly positive for each m (weak positivity theorem), and used it to give a partial solution to Iitaka's conjecture. Recently, Popa–Schnell [7] gave a partial solution to a relative version of Fujita's conjecture, and as a consequence, gave a new proof of the weak positivity theorem. In this research, we extended respectively the weak positivity theorem and Popa–Schnell's result to **positive characteristic**, that is, the case where the base field k has positive characteristic ([E7, Thm 1.1], [E10, Thm 1.3]), and used them to give partial results to Iitaka's conjecture ([E7, Thm 1.4],[E10, Thm 1.5]). In particular, in joint work with Lei Zhang, we proved that Iitaka's conjecture holds true when the characteristic is at least 7 and the dimension of X is three ([E6, Thm 1.3]). Also, by joint work with Paolo Cascini, János Kollár, Lei Zhang, we constructed counterexamples to inequality (I) in positive characteristic ([E1]).

1.A-ii. On positivity of anti-canonical divisors ([E5]) The positivity of anti-canonical divisor is an important notion that is closely related to the geometric property of a variety. Given an algebraic fiber space $f : X \to Y$, we can study the positivity of $-K_Y$ from the positivity of $-K_X$ and the property of fibers. At first, we consider the case when f is a smooth morphism. Kollár–Miyaoka–Mori proved that in arbitrary characteristic, if X is Fano ($-K_X$ is ample), then so is Y. By an argument similar to the one used to prove the above result, the same statement as above holds for the nefness of anti-canonical divisors. Fujino–Gongyo showed that in characteristic 0, if X is weak Fano ($-K_X$ is nef and big), then so is Y. In the case when f is not necessarily smooth, in characteristic 0, it is proved that if $-K_X$ is nef and big (resp. nef), then $-K_Y$ is big (resp. pseudo-effective). In this research, we extended all of the above results in characteristic 0 to **positive characteristic** ([E5, Thms 1.1,1.3 and 4.1]), and furthermore, generalized Kollár–Miyaoka–Mori's reuslt to semi-stable morphisms in arbitrary characteristic ([E5, Thm 5.8]).

1.A-iii. On positivity of relative anti-canonical divisors([E9]) Kollár–Miyaoka–Mori proved that the relative anti-canonical divisor $-K_{X/Y}$ of an algebraic fiber space $f : X \to Y$ is not ample. Using the notion of augmented base locus $mathbbB_+$, we can express the above result as $\mathbb{B}_+(-K_{X/Y}) \neq \emptyset$. In this research, in characteristic 0, we investigated how large large $\mathbb{B}_+(-K_{X/Y})$ is, and showed that it dominates Y, in particular $\dim \mathbb{B}_+(-K_{X/Y}) \geq \dim Y$. Note that the same statements hold in positive characteristic by a proof similar to the one proved the above results.

1.B. On rational connectedness of varieties ([E2]) Let the base field be an algebraically closed field of characteristic 0. Campana and Kollár–Miyaoka–Mori studied a Fano variety and proved that it is rationally connected, that is, given a Fano variety, for every two points are contained in one rational curve on the variety. This property is important. Hacon–M^{c}Kernan generalized the above result to weak Fano varieties, and left a further generalization as a question:
Question: If the anti-canonical divisor $-K_X$ of a non-singular projective variety X is nef, then does a family consists of "rationally connected subvariety in X of dimension at least $\kappa(-K_X)$" cover a dense open subsets of X?

In this research, we dealt with this question in joint work with Yoshinori Gongyo, and answered affirmatively by studying rationally connected fibrations, using the weak positivity theorem [E2, Thm 1.1].

1.C. On characterizations of abelian varieties ([E3,4]) Let X be a non-singular projective variety defined over an algebraically closed field of characteristic $p > 0$. We studied necessary and sufficient conditions for X to be an abelian variety.

1.C-i. Characterization by the structure of $F^m_*\mathcal{O}_X$ ([E4]) From here on, we assume that K_X is pseudo-effective. We consider the following condition on X:
Condition $(*)_m$: $F^m_*\mathcal{O}_X$ is decomposed into a direct sum of line bundles.

If X is an ordinary abelian variety, then $(*)_m$ holds for every m. The converse statement of this was studied by Sannai–Tanaka (Hiromu), and they proved that X is an ordinary abelian variety if $(*)_m$ holds for "infinitely many" m. In this research, we studied the case when $(*)_{m_0}$ holds for one m_0, for the purpose of removing "infinitely many" from Sannai–Tanaka's result. This is joint work with Akiyoshi Sannai. As a result, we proved that if "$p \geq 3$" or "$p = 2$ and $m_0 \geq 2$", then X is an ordinary abelian variety [E4, Theorem 1.2].

1.C-ii. Characterization by Albanese morphism ([E3]) In this research, in positive characteristic, we gave a sufficient condition for the Albanese morphism to be an algebraic fiber space [E3, Theorems 1.1–1.3]. As its corollary, we showed that ordinary abelian varieties are characterized as non-singular projective F-split varieties X with $b_1(X) = 2\dim X$. Also, as another application, we solved the abundance conjecture in a special case [E8].