Research plan (April 2022 ∼)

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The study of elliptic operators in the framework of global setting brought deep understandings of the manifold theory and analysis. In my coming research activity, I am going to study global phenomena of mainly sub-elliptic operators including elliptic cases.

Laplace-Beltrami operator and Dirac operator are defined by means of geometric structures on manifolds, the first one is a second order elliptic differential operator and the second one is a first order elliptic differential operator, respectively. In the study here sub-Riemannian structures and associated sub-Laplacians are the main subjects. A sub-Laplacian is a sub-elliptic second order differential operator and is defined based on a sub-Riemannian structure.

This geometric structure (= sub-Riemannian structure) requires that there exists a bracket generating sub-bundle in the tangent bundle.

The opposite structure, foliation structure, was studied since many years ago. On the other hand until recently it was not so much studied of such structure from geometric and global analytic point of view, so that I am expecting that to pursue the research of this subject has an enough meaning even by comparing with Riemannian manifolds theory and analysis on them.

Contact manifolds and nilpotent Lie groups are basic examples of manifolds carrying a sub-Riemannian structure and it happens often that the total space of a Riemannian submersion has both structures, foliation and sub-Riemannian. Hence there are ample examples of such manifolds to be studied.

Although we can define a transversely elliptic operator on foliated manifolds, there are no natural differential operator associated to the foliation structure. On the other hand there is an intrinsically defined second order differential operator (we call it a sub-Laplacian) on sub-Riemannian manifolds reflecting the sub-Riemannian structure. Hence, again there must be enough meaning to study the sub-Laplacian in contrast with the Laplacian in the Riemannian case.

Since this operator satisfies the "*sub-elliptic estimate*" (= sub-ellipticity) proved by Hörmander, the basic structure of the spectrum is similar to elliptic operator cases. However there is non trivial characteristic variety so that from the geometric point of view it is not enough to treat them in the topological framework(K-theory). This fact may include difficulty in the study, and at the same time we can expect the possibility of new phenomena other than obtained by the property of ellipticity. So, I am going to continue the research on these topics under the direction to find possible new phenomena which will not be included in the elliptic operator theory, and together including the problems whether named classical manifolds have this structure and analytically similar properties of this type operator with the elliptic cases, like Weyl low or an explicit construction of the heat kernel.

Although in these academic years 2020 and 2022 March my activities were forced to be restricted by the covid-19 pandemic situation and so I could not invite nor visit my coworkers in Europe, I expect I can continue joint research work with them in the coming years.

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"From elliptic operators to sub-elliptic operators"

I am planing the research on the following explicit problems (a) $\sim$ (d) **:**

(a) Nilpotent Lie groups are examples of sub-Riemannian manifolds carrying a "good" sub-Riemannian structure (i.e., equi-regular sub-Riemannian structure). Among them I was studying a class of algebras (and groups) attached to Clifford algebras (= pseudo H-type algebras and groups) in these years. As a next step, I am going to classify "*integral lattices*". Also by comparing the spectral zeta functions of Laplacian and sub-Laplacian on their compact nilmanifolds I will determine their zeta regularized determinants by making clear how they are expressed in terms of classical special functions and seek a possibility of their functional equation.

In particular, I am going to study the inverse spectral problem of the sub-Laplacians from a point of a classical famous problem relating with the Riemann zeta function which must be interesting since the residues of the spectral zeta function of the sub-Laplacian of pseudo H-type nil-manifolds relate with the values of Riemann zeta function at integral points.

(b) I had being expect that "Weinstein's eigenvalue theorem" must be valid also for sub-Laplacians under similar assumptions(= Maslov quantization condition), and I established a complete proof of such the theorem. However important is to find concrete examples of such Lagrangian submanifolds satisfying the condition in this case.

So I will try to find various examples of such Lagrangian submanifolds, especially examples other than tori, since these are well-known as realized as common hyper surfaces in the completely integrable geodesic (or bi-characteristic) flow cases.

At the same time I started a study of Radon transformation from the point of the Fourier integral operator theory. It turned out that this problem highly related with the problem (d) below.

(c) Since the last year, I had started a problem on the construction of the Green kernel of a sub-Laplacian on a manifold with conic singularity by the method of symbolic calculus, however under corona pandemic situation it was not so easy to discuss with the colaborator of this problem with the enough times, so that there were no progress on this subject in the year 2021. In this year I am expecting to be possible to make some progress.

(d) Although Lie groups have always invariant sub-Riemannian structures, it is not clear whether their symmetric spaces have such a structure always or not. One problem on this I will try possibly within this year is to construct explicitly such a structure or classify them for a general cases or a particular symmetric space.