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Transition process of factor graphs of Sturmian words

The **Sturmian words (sequences)**, defined as aperiodic infinite words of minimal complexity, share many interesting properties and can be defined differently.

In this study, we elucidated the properties of Sturmian words using the **factor graph** introduced by Rauzy (1982) [2]. We focus on the transition process of the factor graph for the length of factors and examine in detail the case where the shape of the graph changes when the order of the factor graph increases by one. We have improved our outlook by using the Farey sequence and considering the entire real number set, including rational numbers as well as irrational numbers. Furthermore, a comprehensive understanding of mechanical words was obtained by examining the properties of those using geometric mappings introduced by Yasutomi (1998)[3] and Akiyama (2021)[1].

In this study, we focused on the fact that the division of the central path, the 0-side path, and the 1-side path in the factor graph changes continuously when α is fixed and colored the three divisions correspondingly.

If α coincides with any fraction $\dfrac{p}{q}$ in the Farey sequence $Farey_{n+1}$, then the factor graph is a loop of q vertices and the line segment $[0,1) \times \{1-\alpha\}$ is equally divided into q. Otherwise, the factor graph is determined by the Farey pair surrounding α in $Farey_{n+1}$.

If $\dfrac{p_0}{q_0} < \dfrac{p_1}{q_1}$ is a Farey pair surrounding α, then both line segments corresponding to common factors of $S_{\frac{p_0}{q_0}}$ and $S_{\frac{p_1}{q_1}}$ are both bottoms of a map trapezoid and the factors corresponding to the points in the trapezoid belong to the central path of the factor graph $G_n(s_\alpha)$. The line segment corresponding to a factor with only $S_{\frac{p_0}{q_0}}$ is the base of a triangle with another vertex at a point on $[0,1) \times \left\{1-\frac{p_1}{q_1}\right\}$ on the map, and the factor corresponding to a point in the triangle belongs to the 0-side path of the graph. The line segment corresponding to the factor with only $S_{\frac{p_1}{q_1}}$ is the base of a triangle whose other vertex is $[0,1) \times \left\{1-\frac{p_0}{q_0}\right\}$ on the map, and the factor corresponding to a point in the triangle belongs to 1-side path of the factor graph.

From the above, we can see how the shape of the factor graph of order n changes with the value of α when each convex cell obtained by this partition is colored to distinguish the central path, the 0-side path, and the 1-side path, respectively.

The main results are as follows.

- The symmetry of factor graphs.
- The factor graph of a mechanical word is determined by its slope α, the order n of the factor graph, and the Farey pairs surrounding α in the Farey sequence $Farey_{n+1}$.
- Given three nonnegative integers $k \geq 1,\ l \geq 0,\ m \geq 0\ (l+m>0)$, a necessary and sufficient condition for the existence of a factor graph with k, l and m vertices on the central path, 0-side path and 1-side path, respectively.
- When the mechanical word $S_{\alpha,\rho}$ has a bispecial factor of length n, the sum of possible lengths of values of $\alpha \in [0,1]$ and its asymptotic formula, the sum of the areas of the possible values of $(\rho,\alpha) \in [0,1) \times [0,1]$ and its asymptotic formula, as well as the area of each of the three paths of the factor graph in the geometric mapping and its asymptotic formula.

References

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[3] S. Yasutomi. *The continued fraction expansion of α with $\mu(\alpha)=3$*, Acta Arith. **84** (1998): no. 4, 337-374.