Abstract of future research

For the details of some notations, refer to abstract of present research.

1. **Lagrange interpolation polynomials for Laguerre-type weights**: We are studying weighted convergence condition of the Lagrange interpolation polynomial on $\mathbb{R}^+$. For a continuous function f on $\mathbb{R}^+$, we need to find the condition such that

$$\lim_{n\to\infty} \|(L^*_{n,\rho^*}(f) - f)w_\rho\|_{L^p(\mathbb{R}^+)} = 0 \tag{C}$$

for $1 < p < \infty$. Here, $L^*_{n,\rho^*}(f)$ is the Lagrange interpolation polynomial with respect to the weight $w_{\rho*} := w_{\rho+1/2p-1/4}$. We already showed (C) in the case of $p = 2$ and $1 < p < 2$. We have also shown the similarities for the cases $2 < p < \infty$ for the weight $\Phi^{*(1/2-1/p)^+}(x)w_\rho(x)$. But these conditions are very complicated and not continuous for p. Moreover, we don't know error estimates for fixed n. These problem are future tasks.
To study the theory of approximation on $\mathbb{R}^+$, we apply the theory on $\mathbb{R}$. We put $x = t^2$, then we can transform the class of weight $\tilde{\mathcal{L}}_\lambda(C^3+)$ on $\mathbb{R}^+$ into the class of weight $\mathcal{F}_\lambda(C^3+)$ on $\mathbb{R}$. To solve above problems, we have to develop the theory of the Lagrange interpolation polynomial with the weights on $\mathbb{R}$.

2. **de la Vallée Poussin mean**: At this stage, we don't know that the estimate

$$\|(f - v_n(f))w\|_{L^p(\mathbb{R})} \le CT^{1/4}(a_n)E_{p,n}(w; f). \tag{A}$$

is sharp or not. We may decrease more the power of T by technique of proof. And we also may weaken the condition of an Erdős-type weight, that is $T(a_n) \le c\,(n/a_n)^{2/3}$. We also show L^p boundedness of derivatives of the de la Vallée Poussin mean. One of these is the following: Suppose that w belongs to $\mathcal{F}_\lambda(C^4+)$ which is a smooth subclass of $\mathcal{F}(C^2+)$. If $T^{(2j+1)/4}fw \in L^p(\mathbb{R})$, then for $2 \le p \le \infty$,

$$\|v_n(f)^{(j)}w\|_{L^p(\mathbb{R})} \le C\left(\frac{n}{a_n}\right)^j \|T^{(2j+1)/4}fw\|_{L^p(\mathbb{R})} \tag{B}$$

and for $1 \le p \le 2$,

$$\|v_n(f)^{(j)}w\|_{L^p(\mathbb{R})} \le C\left(\frac{n}{a_n}\right)^j a_n^{(2-p)/2p}\|T^{(2j+1)/4}fw\|_{L^2(\mathbb{R})}$$

for all $1 \le j \le k$ and $n \in \mathbb{N}$. We use duality of L^1-norm and Riesz-Thorin interpolation theorem to prove L^p boundedness of the de la Vallée Poussin mean. But, unfortunately, we cannot use duality of L^1-norm because T remains in the proof and it is unbounded. So we could know (B) holds true or not for $1 \le p \le 2$. We would like to find the way to break through obstructions by unboundedness of T.
Moreover, convergence of the de la Vallée Poussin mean on $\mathbb{R}^+$ is the one of our future tasks. To study this problem, we use transformation $x = t^2$. The theory of polynomial approximation on $\mathbb{R}^+$ is an extension of the theory of Laguerre polynomials. It is an important field for application.