RESEARCH PLAN

I will study the **Prasad conjecture for local theta correspondences**, which describes the local theta correspondence in terms of Langlands parameters. Since the local theta correspondence is compatible with some representation theoretic operations, the conjecture is a powerful tool to study representation theoretic properties via Langlands parameters, which I am interested in. In this research, I will particularly study quaternionic dual pairs. A difference from the case of non-quaternionic dual pair is the necessity of the use of the theory of **rigid inner forms** of Kaletha. And we need to construct a new formulation in this case. Then, I will also study the restriction problem in terms of Langlands parameters as an application of the Prasad conjecture.

(1)Prasad Conjecture. Let F be a local field of characteristic 0, let (G, H) be a reductive dual pair of equal or almost equal rank, and let $\psi\colon F \to \mathbb{C}^1$ be a non-trivial additive character. If a tempered irreducible representation π of $G(F)$ satisfies $\sigma = \theta_\psi(\pi, H)$ is non-zero, then their L-parameters satisfy a certain relation. The converse is also true. Thus, the local theta correspondence defines a bijection of unions of L-packets. Now, according to the theory of the rigid inner form of Kaletha, we can associate a certain rigid inner form z with an embedding $\iota_z\colon \Pi_\phi(G(F)) \to \mathrm{Irr}(\mathcal{S}_\phi^+)$. Here, $\mathcal{S}_\phi^+$ is a finite group called the **S-group**. In the case (G, H) is a non-quaternionic dual pair, Prasad conjectured that the local theta correspondence preserves the structure of L-packets which is described by the characters of the S-group [Pra93][Pra00] (**Prasad conjecture**). This conjecture has been proved by Gan-Ichino [GI16] and [Ato18]. Hence, my research aims to extend the Prasad conjecture to quaternionic dual pairs. But in this case, we need to construct a new formulation because the choice of z is not canonical.

What I plan to do. Then, I explain the goal I want to attain. I will construct the set $\mathcal{P}(G)$ and the surjection

$$\mathcal{P}(G) \to \mathrm{Irr}_{\mathrm{temp}}(G(F))\colon u \mapsto \pi(u)$$

properly. Then, it suffices to construct the graph

$$\Xi(\theta_\psi) \subset \mathcal{P}(G) \times \mathcal{P}(H)$$

explicitly such that

- if $(u, u') \in \Xi(\theta_\psi)$ then $\theta_\psi(\pi(u), H) = \pi(u)$;
- if $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(F))$ satisfies $\theta_\psi(\pi, H) \neq 0$, there exists $(u, u') \in \Xi(\theta_\psi)$ so that $\pi(u) = \pi$ and $\pi(u') = \theta_\psi(\pi, H)$.

When $F = \mathbb{R}$, there is progress for this problem. And I plan to complete the formulation and to prove it for some special cases of the non-Archimedean cases by March 2022. Under some conditions such as the ranks of G and H are sufficiently small, Arthur's multiplicity formula will be available. However, to consider general cases, we must establish the **local intertwining relation**, and verify the compatibility of the local intertwining operators and the local theta correspondence as in [Ato18]. To do this, I will focus on the center $\mathcal{Z}_\phi^+$ of the S-groups instead of the S-groups themselves since their characters are still enough to classify L-packets of quaternionic unitary groups ([Art13, Chapter 9]).

(2)Restrection Problem. Let F be a local field, let G and H be connected reductive groups over F. We suppose that $H \subset G$. Then, we consider the problem to determine $\dim \mathrm{Hom}_{H(F)}(\pi, \sigma)$ for $\pi \in \mathrm{Irr}(G(F))$ and $\sigma \in \mathrm{Irr}(H(F))$. If they satisfy the condition so that the **see-saw identity** holds, we can translate the problem to that on other groups. For example, it is possible to solve the following problem as an application of the see-saw identity and the Prasad conjecture. Let D be the division quaternion algebra over F, let G be the unitary group of the 2-dimensional Hermitian space over D, and let H be the subgroup $D^1 \times D^1$ of G. Then, we consider the problem to determine $\dim \mathrm{Hom}_{H(F)}(\pi, 1 \boxtimes \sigma)$ for $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(F))$ and $\sigma \in \mathrm{Irr}(D^1)$. By see-saw identity, this problem is equivalent to a restriction problem for $\Delta G' \subset G' \times G'$. Here, G' denotes the unitary group of a 2-dimensional skew-Hermitian space over D, and $\Delta G'$ denotes the diagonal subgroup of $G' \times G'$. But by the accidental isomorphism, we can solve this by the diagonal restriction problem for D^1 and $\mathrm{SL}_2(F)$. I will complete the above discussions and seek its generalization.

References

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