Future research plans

I want to construct an invariant of G_2-manifolds (and even a wider class of 7-manifolds) by G_2-dDT connections. In other words, I aim to construct nontrivial invariants such as a Gromov-Witten invariant (resp. Donaldson invariant) that "counts" pseudoholomorphic curves (resp. ASD connections). Indeed, it is one of central problems in of $G_2, \mathrm{Spin}(7)$ geometry to construct an enumerative invariant by using calibrated submanifolds or $G_2, \mathrm{Spin}(7)$-instantons, which are considered to be similar to $G_2, \mathrm{Spin}(7)$-dDT connections.

I expect to obtain more interesting results in the case of G_2-dDT connections. The reason is: (i) (More precisely,) G_2-dDT connections correspond to "graphical" calibrated submanifolds, and the singular set seems to be more tractable. (ii) From [15, 19], the moduli space of G_2-dDT connections seems to have better properties than calibrated submanifolds and G_2-instantons. (iii) Significant research has been done on dHYM connections, with deeper results than calibrated submanifolds.

For the construction of invariants, it is necessary to compactify the moduli space and investigate its properties in detail. To that end, I will first work on the following **compactness theorem for minimal connections**.

From my research so far, it seems to work well if basic ideas are taken from the submanifold side, and techniques for the proof are taken from the connection (gauge theory) side. Therefore, I would like to take the mirror "volume" V from the submanifold side and show the compactness theorem using the method on the connection (gauge theory) side.

(Here, critical points of V are called **minimal connections**. The **compactness theorem** is a statement such that a sequence of minimal connections with uniform bounded "volume" subconverges away from an "energy concentration set" S and "bubbles" occur on S. (We can consider this problem for general Riemannian manifolds, and the G_2 structures etc. are not necessary.) I am currently working on this problem with Daniel Fadel and Gonçalo Oliveira, who are familiar with $G_2, \mathrm{Spin}(7)$ -instantons.)

In (higher dimensional) gauge theory, Uhlenbeck, Price, Nakajima, Tian et al. give the compactness theorem of Yang-Mills connections. Then the "energy concentration set" S is studied in detail and related to calibrated geometry for ASD, HYM, $G_2, \mathrm{Spin}(7)$-instantons, which are special classes of Yang-Mills connections. From [19], G_2-dDT connections minimize V, and hence they are minimal connections. (That is, the situation is similar to the gauge theory described above.) Recently, I have shown that the minimality condition is given by an equation similar to the Yang-Mills connection. Thus we expect that an argument similar to the case of Yang-Mills connections will work.

However, there are some problems. To prove this, it is necessary to show that the "energy density" (integrating factor) v of V satisfies **Bochner type inequality**. (That is, v should be a subsolution of an elliptic operator of divergence form.) Since v is much more complicated than the energy density of the Yang-Mills functional, it will be more difficult to prove such an inequality. However, we recently showed a similar formula to the Weitzenböck formula and found that terms with the highest derivative would be handled well. The low-order derivative terms still seems to be complicated, but I expect that they can be estimated well with a little more technical ingenuity.

Another problem is that $\underline{V \text{ is not scaling-invariant}}$ (in some sense). This property is important for the study of S. However, there is a fact that V is "equivalent" to the sum of scaling-invariant functionals. I think that this problem can be solved by applying a similar method to the gauge theory for each term and combining the results nicely.

After that, I try to prove key claims such as ϵ regularity theorem using the Bochner type inequality. (Assuming these, it is already known that the Hausdorff dimension of S is less than or equal to 3. That is, from a dimensional point of view, similar results of Tian et al. are expected.)