ACHIEVEMENTS (研究成果)

Morse functions exist densely on smooth manifolds and from their singular points, appearing discretely, we have some information on the homology group and homotopy. This theory was established in the former half of the 20th century. They contributed to algebraic topological and differential topological theory of manifolds of dimensions ≥ 5 around 1950s–70s. Discoveries of 7-dimensional spheres non-diffeomorphic to the unit sphere by Milnor are due to the theory. This theory is still strong in various scenes in geometry. As a higher dimensional version, geometric and singularity theory of fold maps and more general maps have developed since Thom and Whitney's studies in 1950s. We have studied the following.

(1) **Suitable classes of fold maps.** The class of special generic maps contains the class of Morse functions with exactly two singular points on homotopy spheres and a simplest extension of the class of the functions. Hotmotopy spheres of dimensions $\neq 4$ and the 4-dimensional unit sphere are characterized as manifolds admitting such functions. Canonical projections of special generic maps are also special generic. Homotopy spheres not diffeomorphic to unit spheres admit no such maps in considerable cases. Special generic maps restrict the topologies and the differentiable structures of the manifolds in considerablecases. Such facts have been shown by Osamu Saeki and Kazuhiro Sakuma since the 1990s. Motivated by these fascinating phenomena, we are studying several classes of fold maps easy to handle systematically, starting from defining the classes. The class of round fold maps is one of fundamental classes. This is introduced in 1-1 (1-2) of "Papers". A fold map is round if it is a Morse function on a closed manifold obtained by gluing two copies of a Morse function on a compact manifold on the boundaries in a canonical way or a fold map such that the set of all singular values are embedded concentric spheres. The class contains Morse functions with exactly two singular points on homotopy spheres, canonical projections of unit spheres etc. The set of all singular points of a fold map is a smooth closed submanifold with no boundary and the restriction there is a smooth immersion of codimension 1: this makes the class of round fold maps very natural. We have also constructed such maps on 7-dimensional homotopy spheres to see phenomena as seen in studies of special generic maps. 1-3, 2, 3-7 and 3-8 of "Papers" are also related works.

(2) **Applications of (1) and others.** (Closed and simply-connected) manifolds of dimension>4 have been already classified via algebraic objects and abstract theory due to the assumption that the dimensions are sufficiently high in 1950s–70s. We are trying to understand these manifolds geometric and constructive ways via fold maps and more general maps whose codimensions are negative. This is a fundamental, natural and challenging problem. 3-1, 3-2 and 3-3 of "Papers" are related results. The spaces of all connected components of preimages of smooth maps are called Reeb spaces and for good functions such as Morse-Bott functions, graphs (Reeb graphs). Sharko proposed a natural problem asking whether "we can construct a good smooth function giving the Reeb graph isomorphic to a given graph". We add conditions on preimages in these problems and give answers to this in 1-4 and 3-6 of "Papers".