(2) Abstract of results Makiko Mase

On Weierstrass semigroup of a pointed curve on a $K3$ surface (joint work with Professor Jiryo Komeda) We consider the follwing problem:

Problem: For a given numerical semigroup H, i.e., a subset of $\mathbb{N}_0 := \{0\} \cap \mathbb{N}$ such that the complement $\mathbb{N}_0 \backslash H$ is finite, can one construct a pointed curve (C, P) lying on a $K3$ surface such that the Weierstrass semigroup $H(P)$ coincides with H ?

Here, the Weierstrass semigroup $H(P)$ at a point P on a curve C is defined by

$$H(P) := \{n \in \mathbb{N}_0 \mid \exists f \in \mathbb{C}(C) \quad s.t. \quad (f)_\infty = nP\},$$

where $\mathbb{C}(C)$ is the field of rational functions on C, and $(f)_\infty$ is the polar divisor of f.

In [KM19][1], we construct pointed curves on a $K3$ surface that admit the Weierstrass semigroups

$$H = \langle 2n,\, 8n-2,\, 12n-1 \rangle, \quad \text{and} \quad H = \left\langle \begin{array}{l} 8n-8,\, 8n-6,\, 8n-4,\, 8n-2, \\ 8n,\, 16n-5,\, 16n-3,\, 16n-1 \end{array} \right\rangle$$

with $n \geq 3$.

We construct a $K3$ surface S as the minimal model of the double covering of the weighted projective plane $\mathbb{P}(1,1,4)$ branching in $B \cup \{(0:0:1)\}$, where B is the curve of degree 12 defined by

$$B : X_2(X_2 - 2X_0^4 - X_1^4)(X_2 - X_0^4 - 2X_1^4) = 0.$$

Here, the weighted projective plane $\mathbb{P}(1,1,4)$ is defined by

$$\mathbb{P}(1,1,4) := \operatorname{Proj} \mathbb{C}[x,y,z]$$

with grading in $\mathbb{C}[x,y,z]$ being determined by taking the weights $\operatorname{wt} x = \operatorname{wt} y = 1$, and $\operatorname{wt} z = 4$ to the variables. For a curve $F \subset \mathbb{P}(1,1,4)$ (resp. a point $P \in F$), denote by $\widetilde{F}$ (resp. $\widetilde{P}$) the pre-image of F (resp. of P) by the double covering.

(1) Take the Fermat curve of degree $4n$ in $\mathbb{P}(1,1,4)$: $F_n : x^{4n} + y^{4n} + z^n = 0$ with a point $P_n = (1 : \zeta : 0)$, where $\zeta^{4n} = -1$ on it. By studying intersections of F_n with the branch locus, one can show that the pre-image $\widetilde{F_n}$ of F_n is lying on the $K3$ surface S. Moreover, the Weierstrass semigroup of the pre-image $\widetilde{P_n}$ of the point P_n is given by

$$H(\widetilde{P_n}) = \langle 2n,\, 8n-2,\, 12n-1 \rangle.$$

(2) Take the following curve of degree $4n$ in $\mathbb{P}(1,1,4)$: $F_n : x^{4n-4}z + y^{4n} + z^n = 0$ with a point $P = (1:0:0)$ on it. By studying intersections of F_n with the branch locus, one can show that the pre-image $\widetilde{F_n}$ of F_n is in the $K3$ surface S. Moreover, the Weierstrass semigroup of the pre-image $\widetilde{P_n}$ of the point P_n is given by

$$H(\widetilde{P}) = \left\langle \begin{array}{l} 8n-8,\, 8n-6,\, 8n-4,\, 8n-2,\, 8n, \\ 16n-5,\, 16n-3,\, 16n-1 \end{array} \right\rangle.$$

[1] J.KOMEDA and M.MASE, Curves on weighted $K3$ surfaces of degree two with symmetric Weierstrass semigroups, Tsukuba J. Math. vol.43, No. 1, (2019), 55–70