Future research plan

We will continue study from algebra-geometric viewpoint. Specifically, we will study the wall-crossing formula by Takuro Mochizuki, quiver varieties, and weighted projective spaces, and apply them to integrable systems.

Moduli spaces M(r,n) of framed coherent sheaf on the projective plane $\mathbf{P}^2$ are constructed as Jordan quiver varieties. The partition functions defined by integrals over this moduli spaces

$$Z = \sum_{n=0}^{\infty} q^n \int_{M(r,n)} \varphi$$

are called the Nekrasov partition functions.

Here, integrals are defined by counting fixed points of the torus action. The description of this torus fixed point is different depending on stability conditions used to define quiver varieties. Using this, Nakajima-Yoshioka derived blow-up formula from wall-crossing formula developed by Takuro Mochizuki. The applicant also obtained several functional equations by the similar method. To extend these results further, I plan the following research.

Plan A (moduli of quiver representations): Based on the computational techniques developed so far, we will summarize formulas obtained from wall-crossing phenomenon of certain quiver representations. As an example, we get combinatorial description of integrals over quiver varieties, Laumon spaces, flag manifolds, etc. Furthermore, in joint research with Ikuji Terashima, we study relationship between framed moduli of quiver representations and quantum group.

Plan B (Gauge theory): Nakajima-Yoshioka [NY] derived a blow-up formula by comparing the projective plane $\mathbf{P}^2$ with its one-point blow-up, that is, the (-1) curve. Using this formula, they derived relations between the Donaldson invariants and the Seiberg-Witten invariants predicted by Witten for algebraic surfaces.

In this study, we compare the *(-n)* curves and the projective plane $\mathbf{P}^2$ for all positive integers *n*, and try to prove the (-n) blow-up formula. Recently, Donaldson invariants are defined using higher rank vector bundles. We will consider the applications of our study to such Donaldson invariants.

Plan C (Integrable System): Bershtein-Shchechkin [BS] derived (-2) blow-up formula by representation theory. As an application, they show that the Painlevé τ function is equal to infinite sum of Nekrasov partition functions. We aim to give another proof to this by geometric methods.

In addition, we introduce *K*-theoretic integrals over moduli spaces, and show functional equations by extending results by the applicant. We will also work about conjecture for the type VI discrete Painlevé τ function proposed by Jimbo-Nagoya-Sakai [JNS].

Plan D (Derived Algebraic Geometry): Based on calculation of integrals over the Grassmannian, we will formulate wall-crossing formula in derived algebra-geometric setting. We will examine possibility of calculations of integrals in terms of derived algebraic geometry. We hope that it will be applied to the study of Macdonald polynomials in the future.

References

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[NY] H. Nakajima and K. Yoshioka, Instanton counting on blowup. I. 4-dimensional pure gauge theory, Invent. Math. 162 (2005), no. 2, 313-355