（英訳文）

Although we have a tentative settlement of the existence problem of supercritical deformed Hermitian–Yang–Mills (dHYM) equation [1], it is still unknown what happens when there are no solutions. Also we need to extend the dHYM equation to high rank vector bundles and study its solvability in order to reveal relations with the Bridgeland stability for derived category of coherent sheaves $D^b\mathrm{Coh}(X)$. From the above, I would like to propose the following research projects.

Research (A) Construction of optimal subvarieties

According to the Nakai–Moishezon type criterion [1], we have a comprehensive understanding of the solvability of the supercritical dHYM equation in terms of holomorphic intersection numbers including subvarieties $Y \subset X$. In other words, there is at least one subvariety $Y \subset X$ which destabilizes X when X admits no solutions. Then a question is that "is there optimal choice of such a subvariety in some sense?" This problem can be regarded as an analogy (subvariety version) of finding Harder–Narasimhan filtrations on slope unstable vector bundles. An important clue is the equivalence between the solvability of hypercritical dHYM equation and the coercivity of the Calabi–Yau functional (a convex functional which has dHYM metrics as critical points) due to a recent work by Chu–Lee. Since the coercivity captures the asymptotic behavior of the Calabi–Yau functional near the boundary, we can formulate the optimal element in the sense that it minimizes the slope of the functional at infinity. More concretely, for any given subvariety $Y \subset X$, we consider a model curve (a ray on the space of potentials which forms singularities along Y at infinity), and try to obtain an algebro-geometric description of the slope of the Calabi–Yau functional along it at infinity. As an analogy of this, we often compute non-Archimedean functionals for test configurations constructed as the deformation to the normal cone of subvarieties $Y \subset X$ in the context of K-stability, which will help us to solve the problem. Also when X has complex dimension 2, the problem can be reduced to the usual Nakai–Moishezon criterion. In this case, the dHYM equation depends quadratically on the reference metric, thus computations will be more explicit and simple.

Research (B) Existence problem of dHYM metrics on holomorphic vector bundles

A vector bundle version of the dHYM equation (referred as dHYM equation for simplicity) was introduced by Collins–Yau in 2018. However, there are no non-trivial examples until a recent work by Dervan–McCarthy–Sektnan, in which they constructed solutions as a small deformation from the Hermitian–Einstein equation. Motivated by this, I introduced the notion of J-equation on holomorphic vector bundles and investigate basic properties as well as example of them [15]. In particular, I constructed solutions to the dHYM equation with Large Lagrangian angles as a small deformation from solutions to the J-equation, whereas all examples constructed by Dervan–McCarthy–Sektnan had small Lagrangian angles. In [15], I obtained examples only in the case when X has complex dimension 2 or E has rank 2 due to technical difficulties. One of the reasons is that we have to deal with high order terms of the second fundamental form when considering subbundles of E. To overcome this problem, I try to introduce the notion of geodesics on the space of Hermitian metrics, and study the behavior of functionals as well as second fundamental forms along them. Then I will formulate the stability notion for the J-equation in terms of coherent subsheaves and subvarieties. In [15], I have already constructed the stability in a certain asymptotic situation, or for vortex bundles, which will help us to extend the result to general case.