Research Plan

Relation of Pisot finiteness properties

Let $\beta > 1$. Frougny and Solomyak introduced the following conditions:

$$(\mathrm{F}_1)\quad \mathbb{Z} \subset \mathrm{Fin}(\beta)$$

$$(\mathrm{PF})\quad \mathbb{Z}_+[\beta^{-1}] \subset \mathrm{Fin}(\beta)$$

$$(\mathrm{F})\quad \mathbb{Z}[\beta^{-1}] \subset \mathrm{Fin}(\beta)$$

where $\mathrm{Fin}(\beta)$ is the set of real number x such that $|x|$ has finite β-expansion. We know some interesting results on these finiteness properties. The following table shows previous research results:

	Number class	Structure of $\mathrm{Fin}(\beta)$	Sufficiency for (F)	Sufficiency for (PF)
(F_1)	Pisot	?	?	?
(PF)	Pisot	Closed under addition	$d_\beta(1)$ is finite	—
(F)	Pisot	Ring	—	—

(F_1) is not well-known as well as I know. So, I'll study relations between (F_1) and other *Pisot finiteness* properties. Also, I am interested in an algebraic structure of $\mathrm{Fin}(\beta)$ under (F_1).

Decidability of $(\mathbf{F}_1)$

Let $\beta > 1$ be an algebraic integer with minimal polynomial $x^d - a_{d-1}x^{d-1} - \cdots - a_1 x - a_0$ and define τ_β, which is a transformation on $\mathbb{Z}^{d-1}$, by

$$\tau_\beta(l_1, l_2, \cdots, l_{d-1}) \coloneqq \left(l_2, \cdots, l_{d-1}, -\left\lfloor l_1 a_0 \beta^{-1} + l_2(a_1\beta^{-1} + a_0\beta^{-2}) + \cdots + l_{d-1}(a_{d-2}\beta^{-1} + \cdots + a_0\beta^{-d+1})\right\rfloor\right).$$

Then τ_β is a kind of generalization of β-transformation T when β is an algebraic integer with degree d. Define $\tau_\beta^*(\boldsymbol{l}) = -\tau_\beta(-\boldsymbol{l})$ and

$$Q_\beta = \left\{\boldsymbol{l} = (l_1, l_2, \cdots, l_{d-1}) \in \mathbb{Z}^{d-1} \middle| \exists\{\boldsymbol{l}_n\}_{n=1}^N \ s.t.\ \boldsymbol{l}_N = \boldsymbol{l},\ \ \boldsymbol{l}_{n+1} \in \left\{\tau_\beta(\boldsymbol{l}_n), \tau_\beta^*(\boldsymbol{l}_n)\right\} \ and\ \boldsymbol{l}_1 = (0, \cdots, 0, 1)\right\}.$$

Then if β is a Pisot number, then Q_β is a finite set. Recently, I proved that for each $\boldsymbol{l} \in Q_\beta$, there is $n \geq 0$ such that $\tau_\beta^n(\boldsymbol{l}) = \boldsymbol{0}$ if and only if β has property (F), and there is $n \geq 0$ such that $\left(\tau_\beta^*\right)^n(\boldsymbol{l}) = \boldsymbol{0}$ if and only if β has property (PF). So, I expect the same statement for (F_1). Questions in the following table show my research aims.

	Structure of $\mathrm{Fin}(\beta)$	Sufficiency for (F)	Sufficiency for (PF)	Decidability on Q_β
(F_1)	?	?	?	?
(PF)	Closed under addition	$d_\beta(1)$ is finite	—	$\forall \boldsymbol{l} \in Q_\beta, \exists n; \left(\tau_\beta^*\right)^n(\boldsymbol{l}) = \boldsymbol{0}$
(F)	Ring	—	—	$\forall \boldsymbol{l} \in Q_\beta, \exists n; \tau_\beta^n(\boldsymbol{l}) = \boldsymbol{0}$