Research Result

Let $\beta > 1$. Hereafter, we write the integer part of y is $\lfloor y \rfloor$ and the fractional part is $\{y\}$. Define $T: [0,1] \to [0,1)$ by $T(x) = \{\beta x\}$. Then we have the expansion of $x \in [0,1]$, that is,

$$x = c_1\beta^{-1} + c_2\beta^{-2} + \cdots + c_n\beta^{-n} + \cdots = \sum_{n=1}^{\infty} c_n\beta^{-n}$$

where $c_n = \lfloor \beta T^{n-1}(x) \rfloor$. We call such expansion β-expansion and write $d_\beta(x) = c_1c_2\cdots$. We say that $d_\beta(x)$ is finite if its tail is only zero. Denote by $\mathrm{Fin}(\beta)$ the set of $x \in [0,1)$ such that $d_\beta(x)$ is finite.

It is known that any positive integer has a finite decimal expansion. As a generalization of this finiteness property, Frougny and Solomyak introduced the following conditions:

(PF) $\mathbb{Z}_+[\beta^{-1}] \cap [0,1) \subset \mathrm{Fin}(\beta)$

(F) $\mathbb{Z}[\beta^{-1}] \cap [0,1) \subset \mathrm{Fin}(\beta)$

(1) **Finite $\boldsymbol{\beta}$-expansion and Odometers** ([1] in the List of papers)

For $x \in (0,1]$, let $d_\beta^*(x) = d_\beta(x-0)$ and $\{x\}^* = \lim_{y\uparrow x}\{x\}$. The set M is defined by $M = d_\beta([0,1)) \cup d_\beta^*((0,1])$. Let the function $v: M \to [0,1]$ be given by $v(w_1w_2\cdots) = \sum_{n=1}^{\infty} w_n\beta^{-n}$. For real number γ, define $H_\gamma: M \to M$ by

$$H_\gamma(\boldsymbol{w}) = \begin{cases} d_\beta(\{\gamma + v(\boldsymbol{w})\}) & \mathit{if}\ \boldsymbol{w}\ \mathit{is\ finite} \\ d_\beta^*(\{\gamma + v(\boldsymbol{w})\}^*) & \mathit{if}\ \boldsymbol{w}\ \mathit{is\ not\ finite} \end{cases}$$

We call $H_{\beta^{-1}}$ odometer associated with β-numeration system. Then we proved the followings: β has property (F) if and only if $H_{\beta^{-1}}$ is surjective, and β has property (PF) if and only if $H_{\beta^{-1}}$ is injective. Furthermore, when β is an algebraic integer, we can represent a procedure of carry operation in $H_{\beta^{-1}}$ by a transducer. As a result, we also proved that β has property (F) if and only if $H_{\beta^{-1}}$ is computable. This is a joint work with M. Yoshida.

(2) **Some class of cubic Pisot numbers with finiteness property** ([2] in the List of papers)

Akiyama characterized cubic Pisot units with property (F). Also, he found cubic Pisot numbers with property (F) by using a set of witnesses in joint work with Brunotte and others. However, in general, a set of witnesses is very large set. So it is difficult to determine that β has property (F) by hand computing. By using transducer constructed [1], we proved a generalization of Akiyama's cubic Pisot units theorem by hand computing. Moreover, in this proof, we found a class of cubic Pisot numbers with property (F) by using the set smaller than set of witnesses. As a result, we found a new class of cubic Pisot numbers with property (F). This is joint work with M. Yoshida.