# 6 Research plan

The future plan of the study of quasi-$F$-splitting is divided into two parts; §1 algebraic geometric one and §2 arithmetic geometric one.

## 6.1 Quasi-$F$-split and birational geometry

Originally, $F$-splitting has been studied in various areas, especially in algebraic geometry in positive characteristic and commutative algebra. So I have a plan of a research of quasi-$F$-splitting in these two areas, especially in birational geometry.

The mininal model program (MMP) is an important theory which belongs to the birational geometry of algebraic geometry. In characteristic zero, great achievements have been established since Mori's pioneering work in eighties. In the last decades, people have made progresses on positive characteristic birational geometry, in particular, it is proved that MMP runs for projective threefolds when the characteristic of the ground field is greater than 5. The existing of the restriction on characteristic is closely related to the previously stated fact "if the characteristic $p$ is greater than 5, then all RDP's are $F$-split":

| characteristic | 3-dimensional MMP | RDP | |
|---|---|---|---|
| 0 | valids(Mori, 1988) | | |
| $\geq 7$ | valids(Hacon-Xu, 2015) | $F$-split | |
| $2, 3, 5$ | ? | not necessarily $F$-split | but quasi-$F$-split ! |

Recall that previously stated my result, all RDP's are quasi-$F$-split. Hence there is a possibility to develop 3-dimensional MMP using quasi-$F$-splitting. For this we need to establish a theory of quasi-$F$-splitting itself. The following are a list of research programs which are particularly important for this purpose. Some of these are already work in progress with Hiromu Tanaka (Tokyo University) and Jakub Witaszek (Michigan University):

- Study so called adjunction/inversion of adjunction of quasi-$F$-splitting. That is, when given a variety $X$ and its divisor $S \subset X$, study relations between quasi-$F$-splittings on $X$ and those on $S$. This will enable us to reduce a problem of higher dimensional varieties to lower dimensional ones.

- When a variety is (quasi-)$F$-splitting, then it is "Fano type" or "Calabi-Yau type". In birational geometry, "Fano type" varieties are extremely important. There is a notion of *$F$-regularity* which can capture this type of varieties in terms of Frobenius morphism. One goal of this work is define a notion of "*quasi-$F$-regularity*" and establish its properties. The existing of such a notion seems to be necessary when we apply the theory of quasi-$F$-splitting to MMP.

- It is known that $F$-split singularities are closely related to log canonical singularities, which are important singularities in birational geometry. We will study relations between quasi-$F$-split singularities and log canonical singularities.

To carry out the above programs, it is necessary to develop the theory of Witt rings, which does not exist during the study of $F$-splitting. To do this, I will study the construction of Witt rings due to D. Kaledin, which is closely related to topological cyclic homology.

## 6.2 Quasi-$F$-split and arithmetic geometry

I plan the following two which belong to arithmetic geometry.

(1) Recently, Piotr Achinger and Maciej Zdanowicz reveal that there is a close relation between $F$-splitting and the Serre-Tate theory of ordinary Calabi-Yau varieties. The goal of this work is to extend