Theory of Kac-Moody Symmetric Spaces.

Professor Ernst Heintze

Noted by Nobutaka Boumuki, Kaname Hashimoto

1 Finite dimensional symmetric spaces and their isotropy representations.

Abstract

We review basic properties of finite dimensional Riemannian symmetric spaces and consider their isotropy representations. We show that they are polar and that principal orbits of polar representations are isoparametric. By results of Dadok, Palais, Terng, and Thorbergsson, “symmetric spaces,” “polar representations” and “isoparametric submanifolds” of $\mathbb{R}^n$ are essentially in bijection to each other.

References:

1. E. Heintze, Toward symmetric spaces of affine Kac-Moody type, Internat. J. Geom. Methods in Modern Phys., **3**(2006), 881–888.

2. E. Heintze., and X. Liu, Homogeneity of infinite dimensional isoparametric submanifolds, Ann. of Math., **149**(1999), 149–181.

3. E. Heintze., R. S. Palais, C. L. Terng, and G. Thorbergsson, Hyperpolar actions on symmetric spaces, “Geometry, Topology and Physics for R. Bott” (S. T. Yau, ed.), Int. Press, 1995, 214–245.

4. C. L. Terng, Proper Fredholm submanifolds of Hilbert space, J. Differential Geom., **29**(1989), 9–47.

5. C. L. Terng, Polar actions on Hilbert spaces, J. Geom. Anal., **5**(1995), 129-150.

6. R. S. Palais, and C. L. Terng, Critical point theory and submanifold geometry, Lecture Notes in Math. 1353, Springer, 1988.

É. Cartan ∼1920.

$$\begin{array}{c}\text{Complex simple Lie algebras, real Lie algebras}\\ \updownarrow \\ \text{Symmetric spaces (of compact type)}\end{array}$$

X is a symmetric space, if X is a Riemannian manifold with isometry $S_p : X \longrightarrow X$ such that $S_p(p) = p$ and $(S_p)_{*p} = -\text{id}$.

Examples.

1. $\mathbb{R}^n$ $\quad S_p(x) = -x + 2p.$
2. S^n $\quad S_p(x) = -x + 2\langle x, p\rangle p.$
3. $G_k(\mathbb{R}^n)$ $\quad S_W(V) = \varphi(V) \quad$ for $V, W \in G_k(\mathbb{R}^n)$, where

$$\begin{array}{ccccc} \varphi : \mathbb{R}^n = & W \oplus W^{\perp} & \longrightarrow & W \oplus W^{\perp} \\ & \Cup & & \Cup \\ & x + y & \mapsto & x - y. \end{array}$$

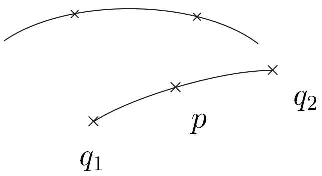
Properties:

1. $\nabla R = 0$. Because

$$\underbrace{\nabla_{S_{p*}X_1} R(S_{p*}X_2, S_{p*}X_3) S_{p*}X_4}_{\substack{\| \\ \nabla_{X_1} R(X_2,X_3)X_4 = -\nabla_{X_1} R(X_2,X_3)X_4}} = S_{p*} \nabla_{X_1} R(X_2, X_3) X_4$$

for ${}^{\forall} X_i \in T_p M$.

2. M is complete because all geodesics can always be extended.



$\Longrightarrow \quad S_p(q_1) = q_2.$

3. M is homogeneous.

$\Longrightarrow M = G/K,\ G = I_0(M),\ K = G_{p_0}$

$$\begin{array}{ccc} \| & & \\ \{gK \mid g \in G\} & \longrightarrow & M \\ gK & \mapsto & gp_0. \end{array}$$

4. $p_0 \in M$ and $\sigma : G \to G$ is defined by $g \mapsto s_{p_0} \cdot g \cdot s_{p_0}^{-1}$

$\Longrightarrow \sigma^2 = \mathrm{id}$ (involution) and $(G^\sigma)_0 \subset K \subset G^\sigma = \{g \in G \mid \sigma(g) = g\}$.

Vice versa if $M = G/K$ and K is as above (for some involution σ), then $M = G/K$ is a symmetric space with respect to any G-invariant metric (assuming K compact).

$$\begin{array}{lcl} \text{Symmetric spaces} & \overset{\longrightarrow}{\leftsquigarrow} & \text{Polar representations} \\ & \overset{\longrightarrow}{\leftsquigarrow} & \text{Isoparametric submanifolds} \end{array}$$

Let $M = G/K$, $K = G^\sigma$, and σ be the involution with $\sigma(p_0) = p_0$. K acts on $T_{p_0}M$ (isotropy representation)

$$G \xrightarrow{\pi} G/K = M$$

$$\begin{array}{cccc} \mathfrak{g} = & \mathfrak{k} & + & \mathfrak{p} \\ \sigma_* & \text{id} & & \text{-id} \end{array} \quad \xrightarrow{\pi_*} \quad T_{p_0}M \quad \& \quad \mathfrak{p} \overset{\pi_*}{\cong} T_{p_0}M.$$

Isotropy representation = Adjoint representation $\mathrm{Ad}_G(K)$ restricted to $\mathfrak{p}$.

Proposition 1.1. *The isotropy representation of a symmetric space is polar, that is, there exists a linear subspace $\Sigma \subset \mathfrak{p}$ which meets every orbit and always orthogonally. In fact, one can take Σ any maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$.*

Proof. (for the group case)
$M = G$, compact with biinvariant metric.

$$G = G \times G/\triangle, \quad \triangle = \{(g, g) \mid g \in G\} \cong G, \quad \sigma(g_1, g_2) = (g_2, g_1)$$

biinvariant metric on $\mathfrak{g}$: $\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$.

$$\begin{aligned} \mathfrak{g} + \mathfrak{g} &= \triangle + \triangle' \\ &= \underbrace{\{(X, X) \mid X \in \mathfrak{g}\}}_{\substack{\| \\ \mathfrak{k} \cong \mathfrak{g}}} + \underbrace{\{(X, -X) \mid X \in \mathfrak{g}\}}_{\substack{\| \\ \mathfrak{p} \cong \mathfrak{g}}}. \end{aligned}$$

Isotropy representation $\sim$ Adjoint representation of G on $\mathfrak{g}$.

(i) Let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal abelian subalgebra.
Fix $X \in \mathfrak{g}$.

$$T_X(\mathrm{Ad}(G)X) = [\mathfrak{g}, X].$$

Because

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{g_t} X = \left.\frac{d}{dt}\right|_{t=0} g_t X g_t^{-1} = YX - XY = [X, Y],$$

where $\left.\frac{d}{dt}\right|_{t=0} g_t = Y$ with $g_0 = e$.

$$\nu_X(\mathrm{Ad}(G)X) = \mathfrak{c}_X \qquad \text{centralizer.}$$

Because

$$Y \perp [\mathfrak{g}, X] \Longleftrightarrow 0 = \langle [\mathfrak{g}, X], Y \rangle = \langle \mathfrak{g}, [X, Y] \rangle.$$

(ii) X is regular ($\mathrm{Ad}G(X)$ is principal)

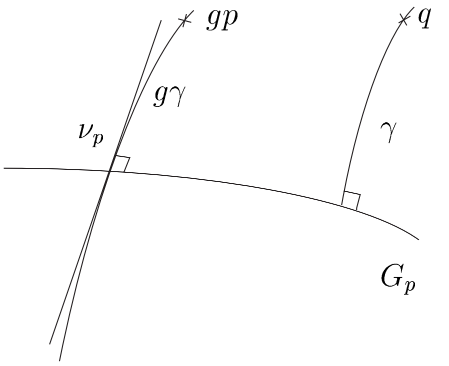
($\Longrightarrow G_X$ acts trivially on ν_X)

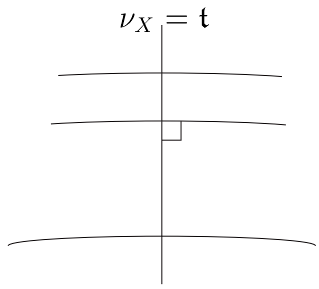
$\Longrightarrow [\mathfrak{c}_X, \nu_X] = 0 \Longrightarrow \nu_X$ is abelian.
Let $\mathfrak{t}$ be a maximal abelian subalgebra containing ν_X.

$$\Longrightarrow \nu_X \subset \mathfrak{t} \underset{\text{(i)}}{\subset} \nu_X$$

$$\Longrightarrow \nu_X = \mathfrak{t}.$$

(iii) If G is compact and acts isometric $\Longrightarrow \exp(\nu_p(G(p)))$ meets all orbits.





$\Longrightarrow \exists g \in G$ s.t. $g\bar{q} = p$.

□

Corollary 1.1. *Maximal tori in compact connected Lie groups are conjugate.*

Proof. (i) Let the action of G on M (any Riemannian manifold) be polar. Σ is a section $\Longrightarrow \{g\Sigma \mid g \in G\}$ is the set of sections (Σ' is another section and $p \in \Sigma$ is regular $\Longrightarrow \exists g \in G$ s.t. $gp \in \Sigma' \cup g\Sigma$. gp is regular, and hence a section through gp is unique because it is equal to $\exp \nu_{gp}(G_{gp})$).

(ii) $G \supset \underset{\text{max. torus}}{T} \longleftrightarrow \mathfrak{t} \subset \underset{\text{max. abelian}}{\mathfrak{g}}$

$$\begin{aligned} T_1 &\longleftrightarrow \mathfrak{t}_1 \\ T_2 &\longleftrightarrow \mathfrak{t}_2 \end{aligned}$$

Proposition 1.2. $\exists g \in G$ *s.t.* $\mathfrak{t}_2 = (\mathrm{Ad} g)\mathfrak{t}_1 \Longrightarrow T_2 = gT_1g^{-1}$.

□

$$\begin{aligned} &V_0 = \left\{A \in M(n \times n, \mathbb{R}) \;\mid\; A^t = A, \mathrm{tr} A = 0\right\} \\ &\cup \\ &\Sigma = \{A \in V_0 \;\mid\; A \text{ diag}\} \end{aligned}$$

$SO(n)$ acts on V_0 by conjugate.
$SU(n)/SO(n) \qquad \mathfrak{su}(n) = \mathfrak{so}(n) + \sqrt{-1}V_0$
$\sigma A = \bar{A}$.

Dadok (1985) Any irreducible polar representation is orbit equivalent to the isotropy representation of a symmetric space.

$$\left.\begin{matrix} G_1 \longrightarrow O(V_1) \\ G_2 \longrightarrow O(V_2) \end{matrix}\right\} \text{ are orbit equivalent, if there is a linear isometry}$$

$$\varphi : V_1 \longrightarrow V_2 \text{ s.t. } \varphi(G_1 v_1) = G_2(\varphi v_1) \text{ for } {}^\forall v_1 \in V_1.$$

Proof. (J. Eschenberg-E. Heintze) (cohomog. $\neq 2$)
Let K be a group which acts polarly on V and irreducibly:
($K = \{g \in O(V) \mid g$ leaves the K-orbits invariant$\}$).

$$\underset{\substack{\uparrow \\ \mathrm{Lie}(K)}}{\mathfrak{g} := \mathfrak{k} + V} \qquad\qquad \text{(as vector space).}$$

bracket For $A, B \in \mathfrak{k}$ and $X, Y \in V$,

$[A, B]$ is the given one,
$[A, X] := A(X)$,
$[X, Y] \in \mathfrak{k}$ is the element with $\langle A, [X, Y]\rangle_{\mathfrak{k}} = \langle A(X), Y\rangle_V$.

↗ the given invariant inner product

Here, $\langle A, B\rangle_{\mathfrak{k}} := -B_{\mathfrak{k}}(A, B) - \mathrm{tr}(A \cdot B : V \longrightarrow V)$.

Problem: Prove Jacobi identity

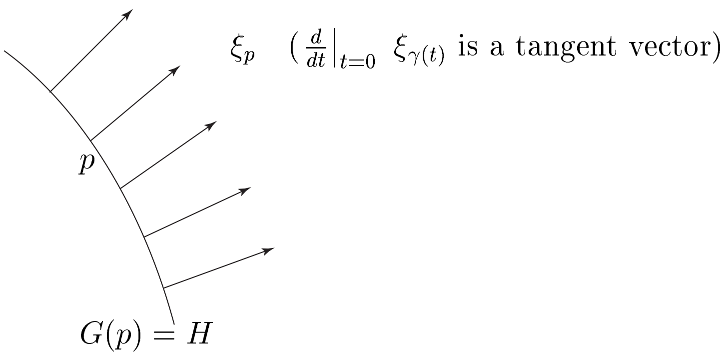
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \qquad\qquad \text{(not so simple !)}$$

□

Geometric Implications of Polarity:

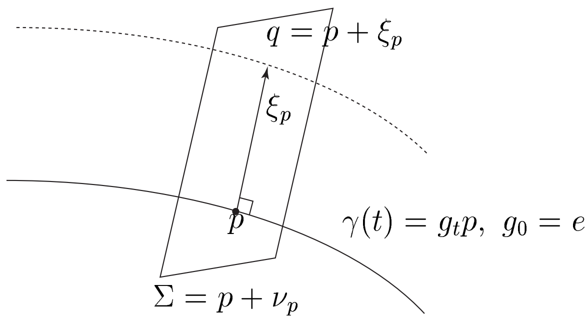
Proposition 1.3. *Suppose that $G \subset I(\mathbb{R}^n)$ acts polarly on $\mathbb{R}^n$. Let $G(p)$ be a principal orbit and $\xi_p \in \nu_p(G(p))$. Then ξ_p extends to a global normal field by the group action and thus parallel in $\nu(G_p)$.*

Proof.



G_p acts trivially on $\nu_p = \nu_p(G(p))$

$\implies \xi_{gp} = g_* \xi_p$ defines a global normal field.



$$\left.\begin{array}{l} \left.\dfrac{d}{dt}\right|_{t=0} g_t p \perp \nu_p, \\ \left.\dfrac{d}{dt}\right|_{t=0} g_p(p + \xi_p) \perp \nu_p \end{array}\right\} \implies \left.\frac{d}{dt}\right|_{t=0} g_t \xi_p \perp \nu_p.$$

□

Corollary 1.2. *Principal orbits of a polar representation (linear or affine) are isoparametric, that is, νM is flat and A_ξ has constant eigenvalues for any parallel ξ.*

Proof. νM is flat by the Proposition. If ξ is parallel in νM, then

$$\xi_{gp} = g_* \xi_p \Longrightarrow A_{\xi_{gp}} = A_{g_* \xi_p} = g_* A_{\xi_p} g_*^{-1}.$$

$\square$

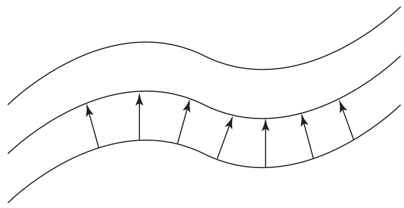
É. Cartan: Isoparametric hypersurfaces of $\mathbb{R}^n$, S^n, $\mathbb{H}^n = X$.

$$M = f^{-1}(t), \quad f : X \longrightarrow \mathbb{R}$$

f is isoparametric, if $\parallel \mathrm{grad} f \parallel$ and $\triangle f$ are constant along level surfaces.

$\nwarrow$ $\nearrow$
parameter

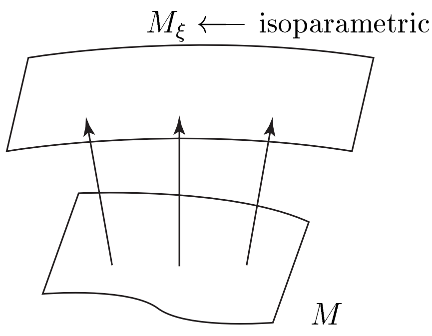
$\parallel \mathrm{grad} f \parallel = a(f)$
$\triangle f = b(f)$



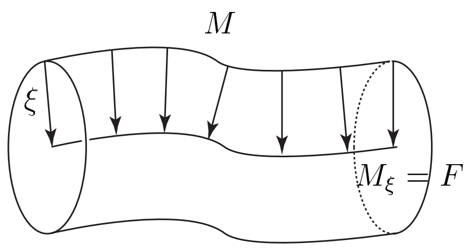
$\Downarrow$

level surfaces are parallel.
level surfaces have constant mean curvatures.
($\Longrightarrow$ principal curvatures are constant).

- Isoparametric submanifolds of $\mathbb{R}^n$ have a rich geometry.
  Let $M \subset \mathbb{R}^n$ be isoparametric and ξ a parallel vector field.
  Then $M_\xi = \{p + \xi_p \mid p \in M\}$ is a parallel submanifold.



$M \rightsquigarrow$ isoparametric foliation
(singular foliation)



F = focal submanifold

Ricci equation: $[A_{\xi_1}, A_{\xi_2}] = 0$ for $\forall \xi_1, \xi_2 \in \nu_p M$.

$$TM = E_1 \oplus \cdots \oplus E_g,\ A_\xi|_{E_i} = \langle n_i, \xi\rangle \cdot \mathrm{id} = \lambda_i(\xi)\cdot \mathrm{id},$$

where $n_1, \cdots, n_g$ are normal fields (curvature normals).

Let $\alpha = 2^{\mathrm{nd}}$ be the fundamental form.

$$\langle \alpha(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle \Longrightarrow \alpha(X_i, Y) = \langle X_i, Y\rangle n_i$$

$$\boxed{\nabla_{X_i}\alpha(Y_j, Z_k) = \langle \nabla_{X_i} Y_j, Z_k\rangle (n_j - n_k)}$$

Codazzi : This is symmetric in i, j, k.

For $X_i \in E_i$, $Y_j \in E_j$, $Z_k \in E_k$ with $i = j \neq k$,

$$\begin{aligned}
&\langle \nabla_{X_i} Y_j, Z_k\rangle (n_j - n_k) = \langle \nabla_{Z_k} X_j, X_i\rangle \underbrace{(n_j - n_i)}_{\to 0} \\
&\Longrightarrow \nabla_{X_i} Y_j \perp E_k \quad {}^\forall k \neq i, j \\
&\Longrightarrow \nabla_{X_i} Y_i \in E_i \\
&\Longrightarrow E_i \text{ is integrable and its leaves are totally geodesic}
\end{aligned}$$

(actually an around sphere).

2 Affine Kac-Moody groups and corresponding infinite dimensional symmetric spaces

Abstract

A compact Lie group with biinvariant metric is one of the simplest examples of a finite dimensional symmetric space. Probably the most direct generalization to infinite dimensions is an affine Kac-Moody group. We describe affine Kac-Moody algebras and their corresponding groups and show that they can be viewed as symmetric spaces. More generally we consider symmetric spaces G/K where G is an affine Kac-Moody group. We show that their isotropy representations are polar and are closely related to Terng's $P(G,H)$-actions on Hilbert space by gauge transformations.

$\mathfrak{g}$	: finite dim. complex, semi-simple Lie alg..
$\cup$	
$\mathfrak{h}$	: Cartan subalg.—that is, $\mathfrak{h}$ is abelian & ${}^\forall X \in \mathfrak{h}$, $\mathrm{ad}\, X : \mathfrak{g} \to \mathfrak{g}$ diagonal..
$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\triangle} \mathfrak{g}_\alpha$	: root-space decomposition w.r.t. $\triangle \subset \mathfrak{h}^*$.
$\alpha_1, \cdots, \alpha_n \in \triangle$	: basis

Remark that ${}^\forall\alpha \in \triangle$ can be described as follows: $\alpha = \sum_{i=1}^{n} m_i\alpha_i$, either $m_i \geq 0$ or $m_i \leq 0$, $m_i \in \mathbb{Z}$.

$A = (a_{ij})$ Cartan Matrix:

$$a_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \in \mathbb{Z}.$$

Cartan matrix $\overset{1:1}{\longleftrightarrow}$ Dynkin diagram
where $\alpha_1, \cdots, \alpha_n$
α_i, α_j are connected by $a_{ij} \cdot a_{ji}$ edges

$$0 \leq a_{ij} \cdot a_{ji} = 4\frac{(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_i)\cdot(\alpha_j, \alpha_j)} \leq 4.$$

A_n ○—○⋯○—○ $A = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$

B_3 ○—○⇒○ $A = \begin{pmatrix} 2 & -1 & \\ -1 & 2 & -2 \\ & -1 & 2 \end{pmatrix}$

$\vdots$

G_2 ○⇛○ $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

${}^\exists$ RECIPE (due to Serre) to construct $\mathfrak{g}$ from Cartan matrix A ($n \times n$-matrix): H_i, E_i, F_i, $1 \leq i \leq n$

$$\begin{array}{lll} [H_i, H_j] = 0, & [H_i, E_j] = a_{ji}E_j, & [H_i, F_j] = -a_{ji}F_j, \\ [E_i, F_i] = 2H_i, & (\mathrm{ad}\, E_i)^{1-a_{ji}}(E_j) = 0, & (\mathrm{ad}\, F_i)^{1-a_{ji}}(F_j) = 0. \end{array}$$

Properties of Cartan matrix:

(C1) $a_{ii} = 2$, $a_{ij} \in \mathbb{Z}$.

(C2) $a_{ij} = 0 \Longleftrightarrow a_{ji} = 0$.

(C3) $a_{ij} \leq 0$ for ${}^\forall i \neq j$.

(C4) $\det (a_{ij})_{i,j\in J'} > 0$ for $\forall J' \subset J$.

Kac-Moody: If (C4) is dropped, then A is called a generalized Cartan matrix. The corresponding Lie algebra is called a Kac-Moody algebra.
Borderline case: replace (C4) by

(C4′) $\det (a_{ij})_{i,j\in J'} > 0$ for $\forall J' \subsetneqq J$
$\det A = 0$

→ affine Kac-Moody alg. (∞-dim., but very close to finite dim. ones).

$n = 2$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$A_1 \times A_2 \qquad A_2 \qquad B_2 \qquad G_2$$

affine $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

An affine Cartan matrix.

Realization of corresponding Kac-Moody algebra.

$\mathfrak{g}$: complex, simple, $\sigma \in \operatorname{Aut} \mathfrak{g}$, $\sigma^l = \mathrm{id}$.

$$L_{\mathrm{alg}}(\mathfrak{g}, \sigma) := \{u : \mathbb{R} \to \mathfrak{g} \mid u(t + 2\pi) = \sigma u(t)\}.$$

$$u(t) = \sum_{|n|\leq N} u_n e^{\sqrt{-1}nt/l}, \ u_n \in \mathfrak{g}.$$
$$(\widetilde{u}(t) = u(lt))$$

$[\,\cdot\,,\,\cdot\,]_0$ pointwise

$\widehat{L}_{\mathrm{alg}}(\mathfrak{g}, \sigma) := L_{\mathrm{alg}}(\mathfrak{g}, \sigma) + \mathbb{C} \cdot c + \mathbb{R} \cdot d.$

$[u, v] := [u, v]_0 + (u', v) \cdot c$, where $(u, v) := \int_0^{2\pi} (u(t), v(t))_0 dt$
Killing ⤴

$[x, c] = 0$ for $\forall x$, $\quad [d, u] = u'$

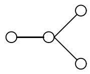
$\Longrightarrow \widehat{L}_{\mathrm{alg}}(\mathfrak{g}, \sigma)$ is an affine Kac-Moody algebra.

Conversely, any affine Kac-Moody algebra is isomorphic to some $\widehat{L}_{\mathrm{alg}}(\mathfrak{g}, \sigma)$.

$L(\mathfrak{g},\sigma) = \{u : \mathbb{R} \to \mathfrak{g} \ \mid \ u(t+2\pi) = \sigma u(t)\}$, $\sigma \in \operatorname{Aut}\mathfrak{g}$.
$\widehat{L}(\mathfrak{g},\sigma) = L(\mathfrak{g},\sigma) + \mathbb{C}\cdot c + \mathbb{R}\cdot d.$

$$\begin{aligned} L_{\text{alg}}(\mathfrak{g},\sigma) \cong L_{\text{alg}}(\mathfrak{g},\widetilde{\sigma}) &\Longleftrightarrow \widehat{L}_{\text{alg}}(\mathfrak{g},\sigma) \cong \widehat{L}_{\text{alg}}(\mathfrak{g},\widetilde{\sigma}) \\ &\Longleftrightarrow [\sigma], [\widetilde{\sigma}] \in \operatorname{Aut}\mathfrak{g}/\operatorname{Int}\mathfrak{g} \text{ are} \\ &\qquad \text{conjugate (have the same order)} \end{aligned}$$

$\operatorname{Aut}\mathfrak{g}/\operatorname{Int}\mathfrak{g} = 1,\ \mathbb{Z}_2,\ \mathbb{Z}_3.$



D_4 ($\mathfrak{so}(8)$).

$k =$ order of $[\sigma]$

$$\mathfrak{g}^{(k)} := \widehat{L}_{\text{alg}}(\mathfrak{g},\sigma) \quad 1 \le k \le 3.$$

$$\begin{array}{l} A_n^{(1)},\ B_n^{(1)},\ C_n^{(1)},\ A_n^{(2)},\ D_n^{(2)},\ E_6^{(2)},\ D_4^{(3)} \\ \updownarrow \\ \widehat{L}(\mathfrak{su}(n+1), \text{id}) \end{array}$$

$\mathfrak{g}$: compact simple Lie algebra, $\sigma \in \operatorname{Aut}\mathfrak{g}$.

$$\underbrace{\widehat{L}(\mathfrak{g},\sigma)}_{\text{"compact real form"}} \subset \widehat{L}(\mathfrak{g}_{\mathbb{C}},\sigma_{\mathbb{C}})$$

Groups associated to $\widehat{L}(\mathfrak{g},\sigma)$.

G: compact, simply connected Lie group with Lie algebra $\mathfrak{g}$, $\sigma : G \longrightarrow G$.

$$L(G,\sigma) = \{g : \mathbb{R} \to G \ \mid \ g(t+2\pi) = \sigma g(t), g \in C^\infty\}$$
pointwise multiplication

Pressly-Segal: Loop Groups

$$\underbrace{\widetilde{L}(\mathfrak{g},\sigma)}_{\text{derived algebra}} := L(\mathfrak{g},\sigma) + \mathbb{R}c \overset{\text{ideal}}{\lhd} \widehat{L}(\mathfrak{g},\sigma)$$

$$[u,v] = [u,v]_0 + (u',v)\cdot c$$

$\widehat{L}(\mathfrak{g},\sigma)$ = semi-direct product of $\widetilde{L}(\mathfrak{g},\sigma)$ and $\mathbb{R}$.

<u>AIM</u> : Find the group corresponding to $\widetilde{L}(\mathfrak{g},\sigma)$.
$\widetilde{L}(\mathfrak{g},\sigma)$ = central extension of $L(\mathfrak{g},\sigma)$;

$$0 \to \mathbb{R}\cdot c \to \widetilde{L}(\mathfrak{g},\sigma) \to L(\mathfrak{g},\sigma) \to 0.$$

In general, $\widetilde{\mathfrak{a}}$ is a 1-dim. central extension of a Lie algebra $\mathfrak{a}$, if

$$0 \to \mathbb{R} \stackrel{\alpha}{\to} \widetilde{\mathfrak{a}} \to \mathfrak{a} \to 0$$

is exact and $\alpha(\mathbb{R}) \subset$ center of $\widetilde{\mathfrak{a}}$.
$\Longrightarrow \widetilde{\mathfrak{a}} = \mathfrak{a} + \mathbb{R}$ (as vector spaces): for $x, y \in \mathfrak{a}$

$$[x,y] = [x,y]_0 + \omega(x,y)\cdot c \ \text{ necessarily.}$$

Here, $\omega : \mathfrak{a}\times\mathfrak{a} \longrightarrow \mathbb{R}$ with

(i) $\omega(x,y) = -\omega(y,x)$

(ii) $\omega([x,y]_0,z) + \omega([y,z]_0,x) + \omega([z,x]_0,y) = 0$

$$\text{central extension} \longleftrightarrow \text{cocycle.}$$

In our case,

$$\omega_\lambda(u,v) = \lambda(u',v) \quad \left(= \int_0^{2\pi} (u'(t), v(t))_0 dt\right).$$

This ω_λ is a cocycle on $L(\mathfrak{g},\sigma)$. It extends by left translation to a 2-form ω_λ on $L(G,\sigma)$. This is closed (by (i) and (ii)).

$$\begin{array}{ccc} \lambda[\omega] = [\omega_\lambda] \in H^2(L(G,\sigma),\mathbb{R}) & \supset & H^2(L(G,\sigma),\mathbb{Z}) \\ \wr\!\| & & \wr\!\| \\ \mathbb{R} & & \mathbb{Z} \end{array}$$

$$\Longrightarrow {}^\exists\lambda_0 \text{ s.t. } [\omega_\lambda] \in H^2(L(G,\sigma),\mathbb{Z}) \text{ iff } \lambda = k\lambda_0 \text{ for some } k \in \mathbb{Z}.$$

To $[\omega_{k_{\lambda_0}}]$ corresponds on S^1-bundle $\widetilde{L}(G,\sigma)_k \stackrel{S^1}{\longrightarrow} L(G,\sigma)$.

$$k = 1 \qquad \widetilde{L}(G,\sigma) := \widetilde{L}(G,\sigma)_1 \stackrel{S^1}{\longrightarrow} L(G,\sigma) \qquad \text{(simply connected).}$$

$$1 \to S^1 \to \widetilde{L}(G,\sigma) \to L(G,\sigma) \to 1$$

central extension.

$$\widehat{L}(G,\sigma) = \widetilde{L}(G,\sigma) \times S^1 \quad \text{(semi-direct)}$$

affine Kac-Moody group.

$$\widetilde{L}(\mathfrak{g},\sigma) \quad = \underset{u,v}{\overset{\cup}{L(\mathfrak{g},\sigma)}} + \mathbb{R}\cdot c + \mathbb{R}\cdot d,$$

↖ compact, simple $\sigma \in \mathrm{Aut}(\mathfrak{g})$

$$\widehat{L}(G,\sigma) \xrightarrow{T^2} L(G,\sigma)$$

biinvariant metric on $\widehat{L}(\mathfrak{g},\sigma)$:

$$(u,v) = \int_0^{2\pi} (u(t),v(t))_0 dt, \quad c,d \perp L(G,\sigma)$$

Killing ↗

L^2-inner product

Lorentz metric

$$(c,c) = (d,d) = 0$$
$$(c,d) = 1$$
$$(c+d,c+d) = 2$$
$$(c-d,c-d) = -2.$$

biinvariance: e.g.

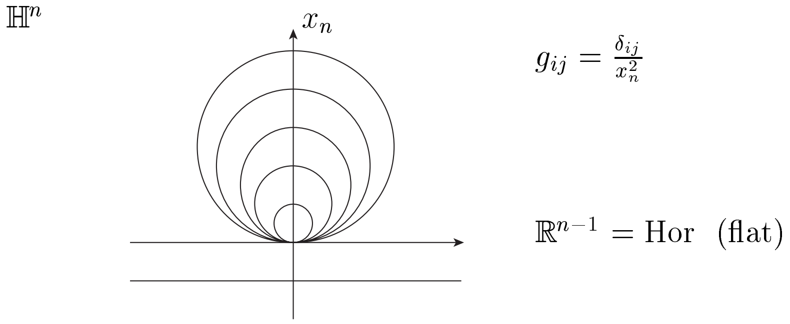
$$\begin{aligned}(d,[u,v]) &= (d,[u,v]_0 + (u',v)c) \\ &= (u',v), \\ ([d,u],v) &= (u',v).\end{aligned}$$

<u>Adjoint action:</u>

$$\left\{x \in \widehat{L}(\mathfrak{g},\sigma) \;\middle|\; (x,x) = -1\right\}$$

two sheeted hyperboloid is invariant under $\mathrm{Ad}\widehat{L}(G,\sigma)$ and

$$\begin{aligned}\mathrm{Hor} &:= \left\{x \in \widehat{L}(\mathfrak{g},\sigma) \;\middle|\; (x,x) = -1, (x,c) = 1\right\} \\ &= \left\{u + d - \frac{||u||^2+1}{2}c \;\middle|\; u \in L(\mathfrak{g},\sigma)\right\} \\ &\longleftrightarrow u \in L(\mathfrak{g},\sigma).\end{aligned}$$



Proposition 2.1. *The adjoint action of* $\widetilde{L}(G,\sigma)$ $(\subset \widehat{L}(G,\sigma))$ *restricted to* Hor *induces on* $L(\mathfrak{g},\sigma)$ *the following affine action:*
Let $\pi : \widetilde{L}(G,\sigma) \to L(G,\sigma)$ *and* $\pi(\widetilde{g}) = g$

$$\Longrightarrow Ad(\widetilde{g})u = gug^{-1} - g'g^{-1} \qquad (\mathfrak{g}, G \subset \mathrm{End}(\mathbb{R}^n)).$$

Proof. We compute modulo $\mathbb{R}\cdot c$, i.e. we work in $\widehat{L}(\mathfrak{g},\sigma)/\mathbb{R}\cdot c = L(\mathfrak{g},\sigma)+\mathbb{R}\cdot d$. Let $\mathfrak{g}, G \subset \mathrm{End}(\mathbb{R}^n)$ and $V = \{v : \mathbb{R} \to \mathbb{R}^n \mid v \in C^\infty\}$. Then $L(\mathfrak{g},\sigma) + \mathbb{R}\cdot d$ embeds into $\mathrm{End}(V)$: for $u \in L(\mathfrak{g},\sigma)$ and $v \in V$

$$\begin{cases} (u(v))(t) = u(t)v(t), \\ (dv)(t) = v'(t). \end{cases}$$

$\Longrightarrow \mathrm{Ad}\,\tilde{g}(u) = gug^{-1}$, $\mathrm{Ad}\,\tilde{g}(d) = gdg^{-1}$,

$$gdg^{-1}(v) = g(g^{-1}v)' = -gg^{-1}g'g^{-1}v + gg^{-1}v' = (-g'g^{-1} + d)v.$$

Hence, $\mathrm{Ad}\,\tilde{g}(d) = d - g'g^{-1}$.

$$\mathrm{Ad}\,\tilde{g}\left(u + d - \tfrac{||u||^2+1}{2}c\right) = \underbrace{gug^{-1} - g'g^{-1}} + d + \lambda c$$

$\updownarrow$ $\in$ Hor $\qquad\qquad \updownarrow$

$u \in L(\mathfrak{g},\sigma)$ $\qquad$ $gug^{-1} - \underbrace{g'g^{-1}}_{\text{constant}} \in L(\mathfrak{g},\sigma)$

AIM This action is polar.
But first take completion of $L(\mathfrak{g},\sigma)$ and $L(G,\sigma)$. $L^2([0,2\pi],\mathfrak{g})$ is the completion of $L(\mathfrak{g},\sigma)$ w.r.t. $(\cdot,\cdot)$.

$$L_{H^1}(G,\sigma) := \{g : \mathbb{R} \to G \mid g(t+2\pi) = \sigma g(t), g \in H^1\ (g' \in L^2)\}.$$

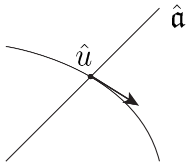
The action $\mathrm{Ad}\,\tilde{g}$ of $L(G,\sigma)$ on $L(\mathfrak{g},\sigma)$ extends to the action of $L_{H^1}(G,\sigma)$ on $L^2([0,2\pi],\mathfrak{g})$.

Proposition 2.2. *This action is polar. In fact, if $\mathfrak{a} \subset \mathfrak{g}^\sigma$ is maximal abelian $\Rightarrow \hat{\mathfrak{a}}$ (consent curve in $\mathfrak{a}$) is section.*

Proof. a) $\hat{\mathfrak{a}}$ meets orbits always orthogonally. Let $u \in \mathfrak{a}$ and $\hat{u}$ be the corresponding consent curve.

$$\frac{d}{ds}\bigg|_{s=0} \left(g_s u g_s^{-1} - g_s' g_s^{-1}\right) = Xu - uX - X'(t) = [X,u] - X',$$

where $g_s(t) \in L_{H^1}(G,\sigma)$ with $g_0(t) = e$ and $\frac{d}{ds} g_s(t) = X(t)$. For $^\forall \hat{v} \in \hat{\mathfrak{a}}$, $([X,u] - X', \hat{v}) = (X, [\hat{u},\hat{v}]) - (X',\hat{v}) = -(X(2\pi) - X(0), v) = 0$.



C. L. Terng: b) $\hat{\mathfrak{a}}$ meets every orbit ($\sigma = \mathrm{id}$, $\mathfrak{a} = A \subset \mathfrak{g}$: maximal torus). Let $u \in L^2([0,2\pi],\mathfrak{g})$ with $u = -g'g^{-1}$ and $g(0) = e$
$\Longrightarrow g(2\pi) = X_0 \exp(-2\pi a) X_0^{-1}$ for some $a \in \mathfrak{a} = A$ and $X_0 \in G$.
Let $X(t) := g(t) X_0 \exp ta \Longrightarrow X \hat{a} X^{-1} - X' X^{-1} = u$. □

□

3 Classification and geometry of Kac-Moody symmetric spaces

Abstract

We show that these spaces are classified by pairs of automorphisms of order less or equal two of finite dimensional simple Lie algebras (up to a certain equivalence relation) and indicate that several basic geometric properties of finite dimensional symmetric spaces carry over to this new situation. We conjecture that these spaces, polar actions on Hilbert spaces and isoparametric submanifolds of Hilbert spaces are essentially also in bijection to each other. Finally we address some of the unsolved problems in this theory.

Let $\widehat{L}(G,\sigma)$ be an affine Kac-Moody group with (Lorentz) biinvariant metric. In Group Case, $S_{\hat{e}}(\hat{g}) := \hat{g}^{-1}$ is the symmetry w.r.t. $\hat{e} \in \widehat{L}(G,\sigma)$, and Isotropy representation induces an affine polar action on a Hilbert space. Let $\widehat{G} := \widehat{L}(G,\sigma)$ and $\widehat{K} := \widehat{G}^{\hat{\rho}}$, where $\hat{\rho}$ is an involution of $\widehat{G}$. Today, we investigate the isotropy representation of $\widehat{G}/\widehat{K}$.

Theorem 3.1. (i) *Let $\hat{\phi}$ be an autom. of $\widehat{L}(\mathfrak{g},\sigma)$. Then, $^{\exists}\varepsilon \in \{\pm 1\}$, $^{\exists}\nu \in \mathbb{R}$, $^{\exists}u_\phi \in L(\mathfrak{g},\sigma)$ and an isom. $^{\exists}\phi : L(\mathfrak{g},\sigma) \to L(\mathfrak{g},\sigma)$, s.t.*

$$\begin{cases} \hat{\phi}(c) = \varepsilon c, \\ \hat{\phi}(d) = \varepsilon d + u_\phi + \nu c, \\ \hat{\phi}(u) = \phi(u) - \varepsilon(u_\phi, u)c. \end{cases}$$

Here, 1^{st} (resp. 2^{nd}) kind if $\varepsilon = +1$ (resp. -1).

(ii) *There exist $\phi_t \in \mathrm{Aut}(\mathfrak{g})$ and $t_0 \in \mathbb{R}$, s.t. $(\phi u)(t) = \phi_t(u(\varepsilon t + t_0))$ for $^{\forall}t \in \mathbb{R}$, $t \mapsto \phi_t$ is C^∞, and $\phi_{t+2\pi} = \sigma\phi_t\sigma^{-\varepsilon}$ ($\Leftrightarrow$ $(\phi u)(t+2\pi) = \sigma(\phi u(t))$).*

(iii) $\mathrm{ad}(u_\phi(t)) = -\phi_t'\phi_t^{-1}$ ($\in \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{Der}\,\mathfrak{g}$).

Remark 3.1. Let us comment on correspondence: $\hat{\phi} \longrightarrow (\phi,\nu)$.

If $\hat{\phi}$ is of finite order, then $\nu = -\varepsilon||u_\phi||^2/2$. Hence, $\hat{\phi} \Longleftrightarrow \phi$ in this case.

(iv) *Conversely: Any smooth curve ϕ_t in $\mathrm{Aut}(\mathfrak{g})$, $\varepsilon \in \{\pm 1\}$, $t_0, \nu \in \mathbb{R}$ with $\phi_{t+2\pi} = \sigma\phi_t\sigma^{-\varepsilon}$ determines an autom. of $\widehat{L}(\mathfrak{g},\sigma)$.*

Simplest Case: $\phi_t \equiv \phi_0 \in \mathrm{Aut}(\mathfrak{g})$ ($\Leftrightarrow u_\phi = 0$)

$$\Longrightarrow \quad \begin{aligned} &\hat{\phi}(c) = \varepsilon c, \\ &\hat{\phi}(d) = \varepsilon d + \nu c \quad \leftarrow \nu = 0 \text{ if } \hat{\phi} \text{ is of finite order.} \\ &\hat{\phi}(u) = \phi u, \quad \phi u(t) = \phi_0(u(\varepsilon t + t_0)). \end{aligned}$$

Theorem 3.2. (i) *Not every* $\hat{\phi}$ *is conjugate on* $\widehat{L}(\mathfrak{g},\sigma)$ *to one with* $\phi_t \equiv \phi_0$ *constant* (*Even not true for involutions*).

(ii) *For any autom.* $\hat{\phi}$ *of finite order, there exists* $\tilde{\sigma} \in \mathrm{Aut}(\mathfrak{g})$ *and an isom.* $\hat{\psi} : \widehat{L}(\mathfrak{g},\sigma) \to \widehat{L}(\mathfrak{g},\tilde{\sigma})$ *s.t.* $\hat{\chi} := \hat{\psi}\hat{\phi}\hat{\psi}^{-1}$ *has* $\chi_t \equiv \chi_0$ ($\hat{\phi}, \hat{\chi}$ *quasi conjugate*).

Let $\hat{\rho} : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ be an involution, where $\hat{\mathfrak{g}} := \widehat{L}(\mathfrak{g},\sigma)$. Then, one may assume that $\hat{\rho}(c) = \varepsilon c$, $\hat{\rho}(d) = \varepsilon d$ and $\hat{\rho}u(t) = \rho_0(u(\varepsilon t + t_0))$. Let us denote the canonical decomposition by $\hat{\mathfrak{g}} = \hat{\mathfrak{k}} + \hat{\mathfrak{p}}$.

CASE $\varepsilon = 1$: $c, d \in \hat{\mathfrak{k}}$ and $\widehat{L}(G,\sigma)/\widehat{L}(G,\sigma)^{\hat{\rho}} \simeq L(G,\sigma)/L(G,\sigma)^{\rho}$.

CASE $\varepsilon = -1$: $c, d \in \hat{\mathfrak{p}}$.

$\hat{\rho}(c) = -c$,

$\hat{\rho}(d) = -d$,

$\hat{\rho}u(t) = \rho_0(u(-t))$ $\quad$ ($t_0 = 0$ after a certain conjug.).

$\rho_0^2 = \mathrm{id}$ and $\rho_0 = \sigma\rho_0\sigma$. Let $\rho_+ := \rho_0$ and $\rho_- := \rho_0\sigma^{-1} \Longrightarrow \rho_\pm^2 = \mathrm{id}$.

Theorem 3.3. *Let* $\rho_\pm \in \mathrm{Aut}(\mathfrak{g})$ *with* $\rho_\pm^2 = \mathrm{id}$, *and let* $\sigma := \rho_-\rho_+$. *Then,* $\hat{\rho}$ *with* $\hat{\rho}(c) = -c$, $\hat{\rho}(d) = -d$ *and* $\hat{\rho}u(t) = \rho_+(u(-t))$ *defines an involution on* $\widehat{L}(\mathfrak{g},\sigma)$. *Up to conjugation, these are all involutions of the second kind on the various* $\widehat{L}(\mathfrak{g},\sigma)$ ($\mathfrak{g}$ *:fixed,* σ *:varies*).

Remark 3.2. Conjugacy classes of involutions of the second kind correspond to equivalence classes of pair (ρ_+, ρ_-) with $\rho_\pm \in \mathrm{Aut}(\mathfrak{g})$ and $\rho_\pm^2 = \mathrm{id}$.

$(\rho_+, \rho_-) \sim (\rho_-, \rho_+)$;

$(\rho_+, \rho_-) \sim (\alpha\rho_+\alpha^{-1}, \beta\rho_-\beta^{-1})$ for $^\forall\alpha, \beta \in \mathrm{Aut}(\mathfrak{g})$ with $\alpha^{-1}\beta \in \mathrm{Int}(\mathfrak{g})$.

(Corollary) All symm. sp. of K-M type thus correspond to pairs $(G/K_+, G/K_-)$ of symm. sp., where $K_\pm := G^{\rho_\pm}$.

$\rho_\pm$: given, $\varepsilon = -1$.

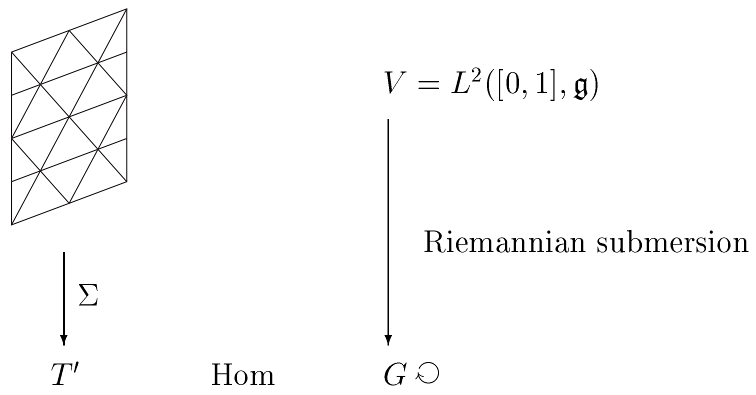
Define σ by $\sigma := \rho_+\rho_+$, and suppose that $\hat{\rho}$ satisfies $\hat{\rho}u(t) = \rho_+(u(-t))$.

$$\widehat{L}(G,\sigma)^{\hat{\rho}} \simeq L(G,\sigma)^{\rho} = \{g : \mathbb{R} \to G \,|\, g(t + 2\pi) = \sigma g(t),\ \rho_+ g(-t) = g(t)\}.$$

$$L_{H^1}(G,\sigma)^{\rho} \simeq \{g : [0,\pi] \to G \,|\, g(0) \in G^{\rho_+},\ g(\pi) \in G^{\rho_-}\} \simeq P(G, K_+ \times K_-).$$

Here $K_\pm := G^{\rho_\pm}$ and $P(G,H) := \{g : [0,1] \to G \,|\, g \in H^1,\ (g(0), g(1)) \in H\}$. Terng: The $P(G, K_+ \times K_-)$ action on Hilbert sp. is polar and is induced from the isotropy representation.

Theorem 3.4. *Any symm. sp. of affine Kac-Moody type induces an affine polar action on a Hilbert space. The principal orbits are isoparametric submanifolds.*



Geometry of these infinite dim. symm. sp.

Problem: e.g.

$L(G, \sigma)$ is only a Fréchet manifold modelled on $L(\mathfrak{g}, \sigma)$, which is a Fréchet vector sp..

The topology causes from a family of norms: $||u||_n := \sum_{k=1}^{n} ||u^{(k)}||_\infty$, where $u^{(k)}$ denotes the k-th derivative.

Inverse function theorem does not hold for Fréchet manifolds.

R. Hamilton: Tame Fréchet manifolds.

Proposition 3.1 (B. Popescu). *Let $\widehat{G} := \widehat{L}(G, \sigma)$ and $\widehat{K} := \widehat{G}^{\hat{\rho}}$. Then, $\widehat{G}$ and $\widehat{G}/\widehat{K}$ are tame Fréchet manifolds, and $\widehat{G} \to \widehat{G}/\widehat{K}$ is a submersion.*

$\widehat{L}(G, \sigma)$ has a biinvariant metric and $L(G, \sigma)$ has a biinvariant metric ($= L^2$ metric). But the tangent sp. ($= L(\mathfrak{g}, \sigma)$) is not complete w.r.t. weak metric.

Theorem 3.5 (B. Popescu).

(i) *$\widehat{G}$ and $\widehat{G}/\widehat{K}$ are symm. sp., and $\widehat{G} \to \widehat{G}/\widehat{K}$ is (pseudo) Riem. submersion.*

(ii) *The Levi-Civita connection exists, and its curvature and geodesics are computed as in the finite dim. case, e.g.*

  (a) $R(X, Y)Z = -[[X, Y], Z]$ *for* $^\forall X, Y, Z \in \hat{\mathfrak{p}}$ ($\hat{\mathfrak{g}} = \hat{\mathfrak{k}} + \hat{\mathfrak{p}}$).

  (b) $\langle R(X, Y)Y, X \rangle = \langle [X, Y], [X, Y] \rangle \geq 0$.

(iii) $\nabla R = 0$.

Let $X = G/K$ be a finite dim. symm. sp.. Then, a submanifold $F \subset X$ is called flat, if F is totally geodesic and flat.

$$\operatorname{rank} X := \max\{\dim F \mid F \text{ is flat}\}.$$

Theorem 3.6 (B. Popescu)**.** *In $\widehat{G}$ and $\widehat{G}/\widehat{K}$, there exist maximal flat, totally geod. submfds of finite dim.* ($^{\exists}$*also infinite dim. flat totally geod. submflds*)*. These finite dim. max. flat are all conjugate to each other* ($\operatorname{rank}\widehat{L}(G,\sigma) = \operatorname{rank} G + 2$)*. They are compact.*

Problem: Duality

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$	$\longleftrightarrow$	$\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p}$
G/K		G^*/K (dual symm. sp.)
compact		non-compact
sec. curv. ≥ 0		sec. curv. ≤ 0
		Hadamard-Cartan
		(G^*/K is diffeom. to $\mathbb{R}^n$)
e.g.		
$G = (G \times G)/\triangle$	$\longleftrightarrow$	$G_{\mathbb{C}}/G$
$\triangle \simeq G$		

Mostow Rigidity Theorem: Let $X := G^*/K$ and $\tilde{X} := \tilde{G}^*/\tilde{K}$ ($X, \tilde{X}$ contain no 2-dim. factor), and let X/Γ and $\tilde{X}/\tilde{\Gamma}$ be compact quotients, where $\Gamma \subset I(X)$. Then, $\Gamma \simeq \tilde{\Gamma} \Longrightarrow X/\Gamma$ and $\tilde{X}/\tilde{\Gamma}$ are isometric to each other. (Proof in the higher rank case: Tits building).