HOLOMORPHIC SECTIONS OF A HOLOMORPHIC FAMILY OF RIEMANN SURFACES INDUCED BY A CERTAIN KODAIRA SURFACE

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ABSTRACT. In this paper we will consider a holomorphic family of closed Riemann surfaces of genus two which is constructed by Riera. The goal of this paper is to estimate the number of holomorphic sections of this family.

1. INTRODUCTION

1.1. **Holomorphic family of Riemann surfaces and its sections.** Let M be a two-dimensional complex manifold and B be a Riemann surface. We assume that a proper holomorphic mapping $\pi : M \to B$ satisfies the following two conditions:

(i) The Jacobi matrix of π has rank one at every point of M.
(ii) The fiber $S_b = \pi^{-1}(b)$ over each point b of B is a closed Riemann surface of genus g_0.

We call such a triple (M, π, B) a *holomorphic family of closed Riemann surfaces* of genus g_0 over B.

A holomorphic mapping $s : B \to M$ is said to be a *holomorphic section* of a holomorphic family (M, π, B) of Riemann surfaces if $\pi \circ s$ is the identity mapping on B.

Let $\mathcal{S}$ be the set of all holomorphic sections of (M, π, B). Denote by $\sharp\mathcal{S}$ the number of all holomorphic sections of $\mathcal{S}$. Next result is called Mordell conjecture in the functional field case.

Theorem 1.1 (Manin [13],Grauert [5], Imayoshi and Shiga [8], Noguchi [14])**.** *The number of all holomorphic sections of $\mathcal{S}$ is finite.*

We remark that Shioda [17] has discussed holomorphic sections of a rational elliptic surface $(S, f, \mathbf{P}^1)$ by using and developing his theory of Mordell-Weil lattice.

Hence next it is important to estimate $\sharp\mathcal{S}$ for (M, π, B).

1.2. **Kodaira surfaces.** Kodaira constructed a holomorphic family (M, π, B) whose base surface and fiber are both compact Riemann surfaces. We briefly review its construction (c.f. Atiyah [1], Kas [10], Kodaira [12]).

Let (C, τ) be a compact Riemann surface of genus $g_0 \geq 2$ with fixed point free involution $\tau : C \to C$. Let $f : D \to C$ be a $(\mathbb{Z}/2\mathbb{Z})^{2g_0}$-unbranched covering corresponding to

$$\pi_1(C) \longrightarrow H_1(C, \mathbb{Z}) \longrightarrow H_1(C, \mathbb{Z}/2\mathbb{Z}).$$

The genus of D is $g_1 = 2^{2g_0}(g_0 - 1) + 1$.

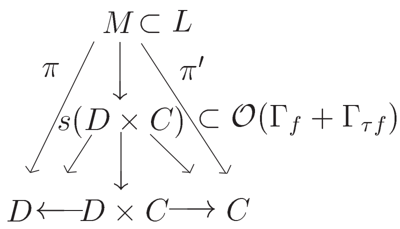
We consider the product $D \times C$ and the graphs of f and $\tau \circ f$,

$$\Gamma_f = \{(u, f(u)) \in D \times C \mid u \in D\},$$

$$\Gamma_{\tau f} = \{(u, \tau \circ f(u)) \in D \times C \mid u \in D\}.$$

As τ is fixed point free, $\Gamma_f \cap \Gamma_{\tau f} = \emptyset$ in $D \times C$. Because $\Gamma_f + \Gamma_{\tau f}$ is 2-divisible in $H_2(D \times C, \mathbb{Z})$, we can find a square root L of the holomorphic line bundle $\mathcal{O}(\Gamma_f + \Gamma_{\tau f})$, i.e., $L^{\otimes 2} \cong \mathcal{O}(\Gamma_f + \Gamma_{\tau f})$.

Let s be a section of $\mathcal{O}(\Gamma_f + \Gamma_{\tau f})$ vanishing at $\Gamma_f + \Gamma_{\tau f}$, and M be the inverse image of $s(D \times C)$ under the square mapping $L \to \mathcal{O}(\Gamma_f + \Gamma_{\tau f})$. Then the natural mapping $\pi : M \to D$ induces the following diagram.



Therefore (M, π, D) is a holomorphic family whose fiber $\pi^{-1}(u)$ is a two-sheeted branched covering of $C \cong \{u\} \times C$ in $D \times C$ branched at $(u, f(u))$ and $(u, \tau \circ f(u))$.

1.3. **Estimation of $\sharp\mathcal{S}$ for Kodaira surface (M, π, D).** For a Kodaira surface, we have an explicit estimation of $\sharp\mathcal{S}$ as follows.

First of all, a Kodaira surface has "trivial" sections s_f and $s_{\tau\circ f}$ defined by $s_f(u)$ and $s_{\tau\circ f}(u)$, where $s_f(u)$ is the branched point of $\pi^{-1}(u)$ over $(u, f(u))$ and $s_{\tau\circ f}(u)$ is the branched point of $\pi^{-1}(u)$ over $(u, \tau \circ f(u))$. Therefore

$$\sharp\mathcal{S} \geq 2.$$

Next, we estimate $\sharp\mathcal{S}$ from above by considering the canonical mapping $\mathcal{S}$ to the set $\mathrm{Hol}(D, C)$ of all holomorphic mappings from D to C,

$$\Phi : \mathcal{S} \to \mathrm{Hol}(D, C)$$

$$s \mapsto \pi' \circ s.$$

Since the involution $\tau : C \to C$ induces the covering transformation of $M \to D \times C$, Φ is 2 to 1 except for s_f and $s_{\tau \circ f}$.

Thus we have

$$\sharp \mathcal{S} = 2\sharp \Phi(\mathcal{S}) - 2.$$

We denote the set of all non-constant holomorphic mappings from D to C by $\mathrm{Hol}_{\mathrm{n.c.}}(D, C)$. Then the next claim is a key idea. (See Proposition 3.1)

Proposition 1.1. $\Phi(\mathcal{S}) \subset \mathrm{Hol}_{\mathrm{n.c.}}(D, C)$.

It is well known that $\sharp\mathrm{Hol}_{\mathrm{n.c.}}(D, C)$ is finite, for example, Tanabe [18] gave an explicit estimation of $\sharp\mathrm{Hol}_{\mathrm{n.c.}}(D, C)$,

$$\sharp\mathrm{Hol}_{\mathrm{n.c.}}(D, C) \leq (4g_1 - 3)^{2g_1} \times 6(g_1 - 1),$$

where g_1 is the genus of D. Since $g_1 = 2^{2g_0}(g_0 - 1) + 1$, we have

$$\sharp\mathrm{Hol}_{\mathrm{n.c.}}(D, C) \leq \{2^{2g_0+2}(g_0 - 1) + 1\}^{2^{2g_0+1}(g_0-1)+2} \times 3 \cdot 2^{2g_0+1}(g_0 - 1).$$

Therefore we have the following theorem.

Theorem 1.2. *The number $\sharp\mathcal{S}$ of holomorphic sections can be estimated as follows.*

$$\begin{aligned} 2 \leq \sharp\mathcal{S} &= 2\sharp\Phi(\mathcal{S}) - 2 \\ &\leq 2\sharp\mathrm{Hol}_{\mathrm{n.c.}}(D, C) - 2 \\ &\leq \{2^{2g_0+2}(g_0 - 1) + 1\}^{2^{2g_0+1}(g_0-1)+2} \times 3 \cdot 2^{2g_0+2}(g_0 - 1) - 2. \end{aligned}$$

1.4. **A Certain Kodaira Surface due to Riera.** In [15], Riera gave a holomorphic universal covering $\mathcal{D}$ of a Kodaira surface. In particular, $\mathcal{D} \subset \mathbb{C}^2$ is a Bergman domain and there exist discontinuous subgroups E and $\dot{E}$ of $\mathrm{Aut}(\mathcal{D})$ such that

$$\begin{array}{ccc} \mathcal{D} & \subset \mathbb{C}^2 & \\ \downarrow & & \\ \mathcal{D}/E & \cong M & \\ \downarrow & \downarrow & \\ \mathcal{D}/\dot{E} & \cong D \times C. & \end{array}$$

Moreover, he gave a "kind" of Kodaira surface whose base surface is a forth-punctured torus and fiber is a closed Riemann surface of genus two. This is our subject in this paper. We remark that for a Kodaira surface, the genus of the base surface must be greater than one (Kas [10], Theorem 1.1). We will estimate $\sharp\mathcal{S}$ for this surface. The detail construction will be reviewed in §2. Here we explain his idea concisely to show it is a "certain" Kodaira surface.

Let $(\widehat{T}, 0)$ be a flat torus with the marked point 0 and let $\widehat{\rho} : \widehat{R} \to \widehat{T}$ be a $(\mathbb{Z}/2\mathbb{Z})^2$-unbranched covering corresponding to

$$\pi_1(\widehat{T}) \longrightarrow H_1(\widehat{T}, \mathbb{Z}) \longrightarrow H_1(\widehat{T}, \mathbb{Z}/2\mathbb{Z}).$$

We also consider the constant mapping $0 : \widehat{R} \to \widehat{T}, r \mapsto 0$. Since two graphs $\Gamma_{\widehat{\rho}}$ of $\widehat{\rho}$ and Γ_0 of 0 intersect at four points in $\widehat{R} \times \widehat{T}$, we can take $R = \widehat{R} \setminus \widehat{\rho}^{-1}(0)$ and $\rho = \widehat{\rho}|R$, and consider Γ_ρ and Γ_0 in $R \times \widehat{T}$ where Γ_ρ and Γ_0 do not intersect.

Riera constructed a two-sheeted covering $M \to R \times \widehat{T} \setminus (\Gamma_\rho + \Gamma_0)$ which induces the next diagram.

$$\begin{array}{ccccc} & \overset{\pi}{\swarrow} & M & \overset{\beta}{\searrow} & \\ & & \downarrow & & \\ R & \longleftarrow & R \times \widehat{T} & \longrightarrow & \widehat{T} \end{array}$$

Then (M, π, R) is a holomorphic family whose fiber $\pi^{-1}(r)$ is a two-sheeted branched covering of $\widehat{T} \cong \{r\} \times \widehat{T}$ in $R \times \widehat{T}$ branched at $(r, 0)$ and $(r, \rho(r))$.

1.5. **Estimation of $\sharp\mathcal{S}$ for Riera's example (M, π, R).** For the estimation of $\sharp\mathcal{S}$, we make the following strategy which is the same as in §1.2. We have "trivial" sections s_ρ and s_0 coming from ρ and $0 : R \to \widehat{T}$, hence

$$\sharp\mathcal{S} \geq 2.$$

Also we have the natural mapping

$$\Phi : \mathcal{S} \to \mathrm{Hol}(R, \widehat{T})$$
$$s \mapsto \beta \circ s$$

and the equality $\sharp\mathcal{S} = 2\sharp\Phi(\mathcal{S}) - 2$. Moreover, we will prove in §3.1 the following:

Proposition 3.1. $\Phi(\mathcal{S}) \setminus \{0\} \subset \mathrm{Hol}_{\mathrm{n.c.}}(R, \widehat{T})$.

But we can not go further because $\widehat{T}$ is not hyperbolic,

$$\sharp\mathrm{Hol}_{\mathrm{n.c.}}(R, \widehat{T}) = \infty,$$

hence the explicit estimation of $\sharp\mathcal{S}$ does not come from the idea in §1.3.

So we need another idea. First we show the following key theorem.

Theorem 3.1. *For any $g \in \Phi(\mathcal{S}) \setminus \{\rho, 0\}$, the mapping g has a holomorphic extension $\widehat{g} : \widehat{R} \to \widehat{T}$.*

As a consequence, we show in §3.1 that

Proposition 3.2. *For any $g \in \Phi(\mathcal{S}) \setminus \{\rho, 0\}$, the mapping g satisfies $\Gamma_g \cap \Gamma_\rho = \emptyset$ and $\Gamma_g \cap \Gamma_0 = \emptyset$.*

Let us denote by $\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T})$ the set of all non-constant holomorphic mappings $g : R \to \widehat{T}$ which extend to the mappings $\widehat{g} : \widehat{R} \to \widehat{T}$ and satisfy $\Gamma_g \cap \Gamma_\rho = \emptyset$ and $\Gamma_g \cap \Gamma_0 = \emptyset$.

Then Proposition 3.2 implies that $\Phi(\mathcal{S}) \subset \mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T}) \cup \{\rho, 0\}$. Now we set $\tau_1 = i$, $\tau_2 = e^{2\pi i/3}$ and put $\widehat{T}_j = \mathbb{C}_z/\Gamma_{1,\tau_j}$ $(j = 1, 2)$. The main result of this paper is as follows.

Main Theorem . *The number $\sharp\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T})$ satisfies the equality*

(a) $\sharp\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T}) = 4$, *if* $\widehat{T} \ncong \widehat{T}_1, \widehat{T}_2$.

Moreover,

(b) $\sharp\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T}_j) = 12$ *for* $j = 1, 2$.

Since $\{\rho, 0\} \subset \Phi(\mathcal{S}) \subset \mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T}) \cup \{\rho, 0\}$, we have the following:

Corollary 3.1.

(a) $2 \le \sharp\Phi(\mathcal{S}) \le 6$, *if* $\widehat{T} \ncong \widehat{T}_1, \widehat{T}_2$.

(b) $2 \le \sharp\Phi(\mathcal{S}) \le 14$, *if* $\widehat{T} \cong \widehat{T}_1$ *or* $\widehat{T} \cong \widehat{T}_2$.

Since $\sharp\mathcal{S} = 2\sharp\Phi(\mathcal{S}) - 2$, we can estimate $\sharp\mathcal{S}$ as

Corollary 3.2. *The number $\sharp\mathcal{S}$ of holomorphic sections can be estimated as follows.*

(a) $\sharp\mathcal{S} = 2, 4, \ldots, 8$ *or* 10, *if* $\widehat{T} \ncong \widehat{T}_1, \widehat{T}_2$.

(b) $\sharp\mathcal{S} = 2, 4, \ldots, 24$, *or* 26, *if* $\widehat{T} \cong \widehat{T}_1$ *or* $\widehat{T} \cong \widehat{T}_2$.

The authors thank the referee for his (or her) hearty comments and advices: The first and the third authors considered $\Phi(\mathcal{S}) = \{\rho, 0\}$ in the first version of this paper. That is, Riera's example (M, π, R) has exactly two holomorphic sections. In the referee comments, he (or she) suggested them to reconsider the complex structure on M carefully. After discussing with the second author, finally they had an idea to consider $\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T})$ and proved that $\Phi(\mathcal{S}) \subset \mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T}) \cup \{\rho, 0\}$ and $\sharp\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T}) = 1$ in general. But they could not determine whether $\Phi(\mathcal{S}) = \mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T}) \cup \{\rho, 0\}$ or not, in other words, there is "another" holomorphic section for our case, which is our next problem.

2. Construction of a Holomorphic Family due to Riera

In [15], Riera explained how to construct the holomorphic universal covering of a Kodaira surface whose fibers are branched over hyperbolic Riemann surfaces.

Since we consider a certain Kodaira surface whose fibers are branched over flat tori, we must modify his construction as follows.

2.1. Fiber as a Two-sheeted Branched Covering Surface of $\widehat{T}$.

Take a point τ in the upper half-plane $\mathbb{H}$. Let $\Gamma_{1,\tau}$ be the discrete subgroup of $\mathrm{Aut}(\mathbb{C}_w)$ generated by $w \mapsto w+1$, $w \mapsto w+\tau$. Let $\alpha_1 : \mathbb{C}_w \to \mathbb{C}_w/\Gamma_{1,\tau}$ be the canonical projection. We denote the pair $(\mathbb{C}_w/\Gamma_{1,\tau}, \alpha_1(0))$ by $(\widehat{T}, 0)$ and set $T = \widehat{T} \setminus \{0\}$.

For any point $t \in T$, we cut $\widehat{T}$ along a simple curve from 0 to t. Next we take two replicas of the torus $\widehat{T}$ with the cut and call them sheet I and sheet II. The cut on each sheet has two sides, which are labeled + side and − side. We attach the + side of the cut on I to the − side of the cut on II, and attach the − side of the cut on I to the + side of the cut on II. Now we obtain a closed Riemann surface S_t of genus two, which is the two-sheeted branched covering surface $S_t \to \hat{T}$ branched over 0 and t.

Note that the complex structure on S_t depends not only on the point t but also on the cut locus from 0 to t. Essentially there are four cuts as in Figure 1 which determine different complex structures on S_t.

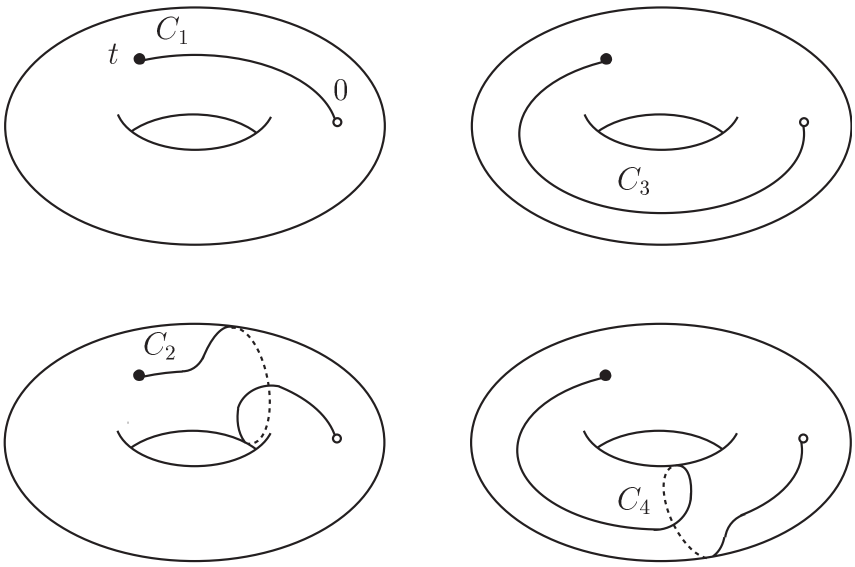


Figure 1. Four cuts on T

Hence we can not construct a family whose fibers are S_t over T. We construct a holomorphic family with fibers S_t. Let $\Gamma_{2,2\tau}$ be the discrete subgroup of $\mathrm{Aut}(\mathbb{C}_z)$ generated by $z \mapsto z+2$, $z \mapsto z+2\tau$. Let $\alpha_2 : \mathbb{C}_z \to \mathbb{C}_z/\Gamma_{2,2\tau}$ be the canonical projection and denote the pair $(\mathbb{C}_z/\Gamma_{2,2\tau}, \alpha_2(0))$ by $(\widehat{R}, 0)$.

Define $\widetilde{\rho} : \mathbb{C}_z \to \mathbb{C}_w$ by $\widetilde{\rho}(z) = z$. Then $\widetilde{\rho}$ induces a $(\mathbb{Z}/2\mathbb{Z})^2$-unbranched covering $\widehat{\rho} : \widehat{R} \to \widehat{T}$ which corresponds to

$$1 \longrightarrow \widehat{\rho}(\pi_1(\widehat{R})) \longrightarrow \pi_1(\widehat{T}) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow 1.$$

Set $R = \widehat{R} \setminus \widehat{\rho}^{-1}(0)$ and $\rho = \widehat{\rho}|R$. For any point $r \in R$, we take a simple curve $\widetilde{C}$ from 0 to r such that $\widehat{\rho}(\widetilde{C})$ is a cut from 0 to $\widehat{\rho}(r)$. By using this cut, we construct a two-sheeted covering $S_r := S_{\rho(r)} \to \widehat{T}$. Now S_r is uniquely determined by $r \in R$ not depending on the cut $\widetilde{C}$. Hence we have a family whose fibers are S_r over R as a set.

Next we introduce a complex structure in this family.

2.2. **Quasi-conformal Deformation.** We fix a point $r_0 \in R$ and a simple arc from 0 to r_0 in R. The image of this under ρ is a curve C on $\widehat{T}$ from 0 to $\rho(r_0)$. Cutting $\widehat{T}$ along C, we have a closed Riemann surface S_{r_0} of genus two. We realize this two-sheeted branched covering $S_{r_0} \to \widehat{T}$ in terms of Fuchsian groups as follows.

We choose a Fuchsian group $\dot{G} \subset PSL(2,\mathbb{R})$ which satisfies the following conditions:

- (i) there exist two elliptic elements $\dot{g}_1$ and $\dot{g}_2$ in $\dot{G}$ such that each $g_j (j = 1, 2)$ has the fixed point z_j in $\mathbb{H}$,
- (ii) $\mathbb{H}/\dot{G}$ is biholomorphically equivalent to $\widehat{T}$,
- (iii) The canonical projection $\mathbb{H} \to \mathbb{H}/\dot{G}$ sends z_1 and z_2 to 0 and $\rho(r_0)$ under a biholomorphical mapping from $\mathbb{H}/\dot{G}$ to $\widehat{T}$, respectively.

Then we can find an index 2 normal subgroup G_1 of $\dot{G}$ such that $\mathbb{H}/G_1 \to \mathbb{H}/\dot{G}$ realizes $S_{r_0} \to \widehat{T}$. From the definition of α_2, $\widetilde{\rho} : \mathbb{C}_z \to \mathbb{C}_w$ defined by $\widetilde{\rho}(z) = z$ is a lift of $\widehat{\rho} : \widehat{R} \to \widehat{T}$ to the universal coverings $\mathbb{C}_z$ of $\widehat{R}$ and $\mathbb{C}_w$ of $\widehat{T}$, and let $\widetilde{r_0}$ be a point $r_0 = \alpha_2(\widetilde{r_0})$.

Let $V : \mathbb{H} \to \mathbb{C}_w$ be the mapping with $V(z_1) = 0$ which makes the next diagram commutative. Then V becomes a two-sheeted branched covering with $V(\dot{G}z_1) = \Gamma_{1,\tau}0$ and $V(\dot{G}z_2) = \Gamma_{1,\tau}\widetilde{\rho}(\widetilde{r_0})$, where $\dot{G}z_j$ is the orbit under $\dot{G}$ of z_j, and $\Gamma_{1,\tau}\widetilde{\rho}(\widetilde{r_0})$ and $\Gamma_{1,\tau}0$ are the orbits under $\Gamma_{1,\tau}$ of $\widetilde{\rho}(\widetilde{r_0})$ for $\widetilde{r_0} \in \mathbb{C}_z$ and 0, respectively.

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{V} & \mathbb{C}_w \\ \downarrow & & \downarrow \\ \mathbb{H}/\dot{G} & \longrightarrow & \widehat{T} \end{array}$$

We construct for $z \in \mathbb{C}_z$, a quasi-conformal mapping $\omega_z : \mathbb{C}_w \to \mathbb{C}_w$ satisfying the following conditions:

(i) $\omega_z(\widetilde{\rho}(\widetilde{r}_0)) = \widetilde{\rho}(z)$,
(ii) $\omega_z \circ g \circ \omega_z^{-1} = g$ for all $g \in \Gamma_{1,\tau}$,

In order to construct such a quasi-conformal mapping ω_z, we make the following observations:

First, let $\gamma(t), 0 \leq t \leq 1$ be a path from $\widetilde{\rho}(\widetilde{r}_0)$ to $\widetilde{\rho}(z)$ in $\mathbb{C}_w$ which contains no points of $L(1,\tau) = \{m + n\tau \in \mathbb{C} \mid m, n \in \mathbb{Z}\}$. For each t, there exists a Dirichlet fundamental region D_t for $\Gamma_{1,\tau}$ centered at $\gamma(t)$. Choose an Euclidean disk B_t centered at $\gamma(t)$ sufficiently small that the closure $\overline{B}_t$ is contained in D_t and has no points of $L(1,\tau)$. Moreover we take a finite covering of γ, say $B_{t_1}, \ldots, B_{t_{n+1}}$, such that $\gamma(t_1) = \widetilde{\rho}(\widetilde{r}_0)$ and $\gamma(t_{n+1}) = \widetilde{\rho}(z)$ and $\gamma(t_{j+1}) \in B_{t_{j+1}}$.

Next, we set

$$\omega_j(\zeta) = \begin{cases} \dfrac{\zeta + \gamma(t_{j+1}) - 2\gamma(t_j)}{1 + \frac{1}{r_j^2}(\gamma(t_{j+1}) - \gamma(t_j))(\overline{\zeta} - \overline{\gamma(t_j)})} + \gamma(t_j), & \text{on } B_{t_j} \\ \zeta, & \text{on } \overline{D_{t_j} \setminus B_{t_j}}. \end{cases}$$

where r_j is the radius of B_{t_j}. Moreover put $\omega_j = g \circ \omega_j \circ g^{-1}$ on $g(D_{t_j})$ for all $g \in \Gamma_{1,\tau}$.

A simple calculation shows that $\omega_j : \mathbb{C}_w \to \mathbb{C}_w$ is a quasi-conformal mapping with the Beltrami coefficient

$$\tau_j(\zeta) = \begin{cases} -\frac{1}{r_j^2}(\gamma(t_{j+1}) - \gamma(t_j))(\omega_j(\zeta) - \gamma(t_j)), & \text{on } B_{t_j} \\ 0, & \text{on } \overline{D_{t_j} \setminus B_{t_j}}. \end{cases}$$

We remark that $|\gamma(t_{j+1}) - \gamma(t_j)| < r_j$ and $|\omega_j(\zeta) - \gamma(t_j)| < r_j$ imply $\|\tau_j\|_\infty < 1$. Moreover since $\gamma(t_{n+1}) = \widetilde{\rho}(z)$, τ_j depends holomorphically on z in B_{t_j}.

Finally, we set $\omega_z = \omega_n \circ \omega_{n-1} \circ \cdots \circ \omega_1$. By the construction of each ω_j, we see that ω_z satisfies the conditions (i), and (ii). Hence we have the desired quasi-conformal mapping ω_z.

2.3. **Construction of $\mathcal{D}$.** For $z \in \mathbb{C}_z$, we put

$$\mu_z(\zeta) = \tau_z(V(\zeta))\frac{\overline{V'(\zeta)}}{V'(\zeta)},$$

then μ_z is the Beltrami coefficient for $\dot{G}$. We define W_{μ_z} as a unique quasi-conformal mapping of $\mathbb{H}$ which has the complex dilatation μ_z and leaves $0, 1$, and ∞ fixed, respectively. Set

$$\widehat{\mu}_z(\zeta) = \begin{cases} \mu_z(\zeta), & \zeta \in \mathbb{H} \\ 0, & \zeta \in \mathbb{C} \setminus \mathbb{H} \end{cases} \tag{2.1}$$

Then there exists a unique quasi-conformal mapping W^{μ_z} of $\widehat{\mathbb{C}}$ which has the complex dilatation $\widehat{\mu}_z$ and leaves $0, 1$, and ∞ fixed, respectively. Now put $D(\mu_z) = W^{\mu_z}(\mathbb{H})$. Then we have the following commutative diagrams:

$$\begin{array}{ccccccc} \mathbb{H} & \xrightarrow{W_{\mu_z}} & \mathbb{H} & \quad & \mathbb{H} & \xrightarrow{W^{\mu_z}} & D(\mu_z) \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle V_z} & & {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle V^z} \\ \mathbb{C} & \xrightarrow{\omega_z} & \mathbb{C} & & \mathbb{C} & \xrightarrow{\omega_z} & \mathbb{C} \end{array}$$

where $V_z = \omega_z \circ V \circ (W_{\mu_z})^{-1}$ and $V^z = \omega_z \circ V \circ (W^{\mu_z})^{-1}$ are branched coverings branched over the orbits $\Gamma_{1,\tau} w$ and $\Gamma_{1,\tau} 0$.

Since μ_z depends holomorphically on z, it is known that W^{μ_z} also depends holomorphically on z. Thus we set

$$\mathcal{D} = \{(z, \zeta) \mid z \in \mathbb{H}, \zeta \in D(\mu_z)\}.$$

Then $\mathcal{D}$ becomes a domain in $\mathbb{C}^2$, so called a Bergman domain.

2.4. **Construction of E.** Next we construct a subgroup E of automorphisms of $\mathcal{D}$ which acts properly discontinuously without fixed points.

Let H be the covering transformation group of a four punctured torus R, that is $R = \mathbb{H}/H$. Denote by $\mathrm{mod}(G_1)$ the set of all equivalence classes $\langle \omega \rangle$ of quasi-conformal mapping $\omega : \mathbb{H} \to \mathbb{H}$ with $\omega G_1 \omega^{-1} = G_1$, where two quasi-conformal mappings ω_1 and ω_2 are said to be equivalent if $\omega_1 = \omega_2$ on $\mathbb{R}$. Then there exists a homomorphism $\delta : H \to \mathrm{mod}(G_1)$ such that

$$W_{\mu_{h(z)}} = \alpha \circ W_{\mu_z} \circ \delta(h)^{-1} \qquad (z \in \mathbb{H}, h \in H) \tag{2.2}$$

where $\alpha \in \mathrm{Aut}(\mathbb{H})$ is chosen so that $\alpha \circ W_{\mu_z} \circ \delta(h)^{-1}$ fixes each of $0, 1$, and ∞.

It should be remarked that we have a homomorphism $\theta_2 : H \to \mathrm{Aut}(G_1)$ given by $\theta_2(h)(g) = \delta(h) \circ g \circ \delta(h)^{-1}$. By using this homomorphism, we define E to be the semidirect product of H and G_1. In order to define the action of E on $\mathcal{D}$, we make the following observations:

First, we need the following result.

Proposition 2.1 (Bers [2], Lemma 3.1). *Let* $[\mu] \in T(G)$ *and* $\langle\omega\rangle \in \mathrm{mod}(G)$. *Define a quasi-conformal mapping* W_ν *by the formula*

$$W_\nu = \alpha \circ W_\mu \circ \omega^{-1},$$

where $\alpha \in \mathrm{Aut}(\mathbb{H})$ *such that* $\alpha \circ W_\mu \circ \omega^{-1}$ *fixes each of* $0, 1,$ *and* ∞. *Then the mapping* $\zeta \mapsto \widehat{\zeta}$ *given by*

$$\widehat{\zeta} = W^\nu \circ \omega \circ (W^\mu)^{-1}(\zeta)$$

is a conformal bijection from $D(\mu)$ *onto* $D(\nu)$.

Moreover if $[\mu]$ *varies holomorphically according to a parameter, so does* $\widehat{\zeta}$ *for a fixed value of* ζ.

By (2.2) and Theorem 2.1, the mapping

$$\widehat{\zeta} = W^{\mu_{h(z)}} \circ \delta(h) \circ (W^{\mu_z})^{-1}(\zeta)$$

is a conformal bijection from $D(\mu_z)$ onto $D(\mu_{h(z)})$. It follows from the second part of Theorem 2.1 that $\widehat{\zeta}$ depends holomorphically on z.

Thus we define the action of E on $\mathcal{D}$ by

$$\begin{aligned}(h, g_1)(z, \zeta) &= (h(z), W^{\mu_{h(z)}} \circ g_1 \circ (W^{\mu_{h(z)}})^{-1}(\widehat{\zeta})) \\ &= (h(z), W^{\mu_{h(z)}} \circ g_1 \circ \delta(h) \circ (W^{\mu_z})^{-1}(\zeta)),\end{aligned}$$

where $(z, \zeta) \in \mathcal{D}$ and $(h, g_1) \in H \ltimes G_1$. We can check this is a group action.

Let $F(G_1)$ be the Bers fiber space over the Teichmüller space $T(G_1)$ defined by $F(G_1) = \{([\mu_z], \zeta) \mid [\mu_z] \in T(G_1), \zeta \in D(\mu_z)\}$. Every element $\langle\omega\rangle$ of $\mathrm{mod}(G_1)$ acts on $F(G_1)$ by

$$([\mu_z], \zeta) \mapsto ([\nu_z], W^{\nu_z} \circ \omega \circ (W^{\mu_z})^{-1}(\zeta)).$$

We set

$$A = \{(z, ([\mu_z], \zeta)) \mid z \in \mathbb{H},\ ([\mu_z], \zeta) \in F(G_1)\}.$$

Then $\mathcal{D}$ is identified with A under the mapping

$$(z, \zeta) \mapsto (z, ([\mu_z], \zeta)),$$

and the action of E on $A \cong \mathcal{D}$ can be written as

$$(h, g_1)(z, ([\mu_z], \zeta)) = (h(z), g_1 \circ \delta(h)([\mu_z], \zeta)),$$

where $g_1 \circ \delta(h)$ is an element of $\mathrm{mod}(G_1)$.

Theorem 2.1 (Bers [2], Theorem 7). *If* $\dim_{\mathbb{C}} T(G) < \infty$, *then* $\mathrm{mod}(G)$ *acts properly discontinuously on* $F(G)$.

Hence E acts properly discontinuously on $\mathcal{D}$ as $\dim_{\mathbb{C}} T(G_1) = 3$. Moreover the action of E on $\mathcal{D}$ is fixed point free since H and G_1 are fixed point free.

2.5. **Holomorphic Family** (M, π, R)**.** The quotient space $\mathcal{D}/E$ becomes a 2-dimensional complex manifold. We set $M = \mathcal{D}/E$.

The group $\dot{E} = H \ltimes \dot{G}$ also acts on $\mathcal{D}$ and the quotient space $\mathcal{D}/\dot{E}$ is biholomorphically equivalent to $R \times \widehat{T}$. Therefore we have a two-sheeted branched covering $\Pi : M \to R \times \widehat{T}$ branched over two graphs Γ_0 and Γ_ρ.

We define π to be the composite $P_R \circ \Pi$ of the covering mapping Π and the projection $P_R : R \times \widehat{T} \to R$, and β to be $P_{\widehat{T}} \circ \Pi$, where $P_{\widehat{T}} : R \times \widehat{T} \to \widehat{T}$. Then the triple (M, π, R) is a holomorphic family such that for any point $r \in R$, $\beta|S_r : S_r = \pi^{-1}(r) \to \widehat{T}$ is a two-sheeted branched covering.

3. Proof of Main Theorem

Let us recall $\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T})$ is the set of all holomorphic mappings $g : R \to \widehat{T}$ which extend to the mappings $\widehat{g} : \widehat{R} \to \widehat{T}$ and satisfy $\Gamma_g \cap \Gamma_\rho = \emptyset$ and $\Gamma_g \cap \Gamma_0 = \emptyset$. Set $\tau_1 = i$, $\tau_2 = e^{2\pi i/3}$ and put $\widehat{T}_j = \mathbb{C}_z/\Gamma_{1,\tau_j}, j = 1, 2$.

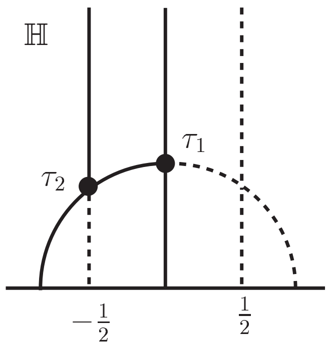


Figure 2

Main Theorem . *The number* $\sharp\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T})$ *satisfies the equality*

(a) $\sharp\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T}) = 4$, *if* $\widehat{T} \ncong \widehat{T}_1, \widehat{T}_2$.

Moreover,

(b) $\sharp\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T}_j) = 12$ *for* $j = 1, 2$.

Since $\{\rho, 0\} \subset \Phi(\mathcal{S}) \subset \mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T}) \cup \{\rho, 0\}$, we have the following:

Corollary 3.1. (a) $2 \leq \sharp\Phi(\mathcal{S}) \leq 6$, *if* $\widehat{T} \ncong \widehat{T}_1, \widehat{T}_2$.

(b) $2 \leq \sharp\Phi(\mathcal{S}) \leq 14$, *if* $\widehat{T} \cong \widehat{T}_1$ *or* $\widehat{T} \cong \widehat{T}_2$.

Since $\sharp\mathcal{S} = 2\sharp\Phi(\mathcal{S}) - 2$, we can estimate $\sharp\mathcal{S}$ as

Corollary 3.2. *The number $\sharp\mathcal{S}$ of holomorphic sections can be estimated as follows.*

(a) $\sharp\mathcal{S} = 2, 4, \ldots, 8,$ *or* 10, *if* $\widehat{T} \not\cong \widehat{T}_1, \widehat{T}_2$.

(b) $\sharp\mathcal{S} = 2, 4, \ldots, 24,$ *or* 26, *if* $\widehat{T} \cong \widehat{T}_1$ *or* $\widehat{T} \cong \widehat{T}_2$.

3.1. **Key Theorem.**

Proposition 3.1. $\Phi(\mathcal{S}) \setminus \{0\} \subset \mathrm{Hol}_{\mathrm{n.c.}}(R, \widehat{T})$.

Proof of Proposition 3.1. Assume there exists a constant mapping $g \in \Phi(\mathcal{S}) \setminus \{0\}$ which is written as $g(r) = c$, where c is not equal to 0. Since $\rho : R \to T$ is surjective, there exists a point r_0 such that $\rho(r_0) = c$, hence $\widehat{\rho}(r_0) = c$. Since $\widetilde{\rho}(z) = z$ is a lift of $\widehat{\rho}$, we can find $z_0 \in \mathbb{C}_z \setminus L(1, \tau)$ such that $\alpha_2(z_0) = r_0$ and

$$z_0 = c. \tag{3.1}$$

For sufficiently small $\epsilon > 0$, $\Delta(z_0, \epsilon) = \{z \in \mathbb{C}_z \mid |z - z_0| < \epsilon\}$ and $\Delta(c, \epsilon) = \{w \in \mathbb{C}_w \mid |w - c| < \epsilon\}$ can be taken as local charts at $r_0 \in R$ and $c \in \widehat{T}$, respectively. Then the graph $\Gamma_g = \{(r, c) \mid r \in R\}$ in $R \times \widehat{T}$ can be locally written as

$$w = c$$

in $\Delta(z_0, \epsilon) \times \Delta(c, \epsilon)$. Thus M is locally represented as

$$u^2 = w - c$$

in $\mathbb{C}_u \times \Delta(z_0, \epsilon) \times \Delta(c, \epsilon)$ (see Wavrik [19], Theorem in Appendix). Take $\epsilon' > 0$ with $\epsilon' < \epsilon$, and set $z = z_0 + \epsilon' e^{i\theta}$. By using (3.1), we have

$$\begin{aligned} u^2 &= w - c \\ &= z_0 + \epsilon' e^{i\theta} - c \\ &= \epsilon' e^{i\theta}. \end{aligned}$$

When θ goes from 0 to 2π, $u = u(\theta)$ becomes two-valued which means that $s = s(\theta)$ is two-valued. We have a contradiction. ■

Theorem 3.1. *For any $g \in \Phi(\mathcal{S}) \setminus \{\rho, 0\}$, the mapping g has a holomorphic extension $\widehat{g} : \widehat{R} \to \widehat{T}$.*

Proof of Theorem 3.1. First, we use the following theorem about the canonical extension of holomorphic families:

Theorem 3.2 (Imayoshi [6],Theorem 4 and Theorem 5)**.** *Let $(N, \pi, \Delta - \{0\})$ be a holomorphic family of compact Riemann surfaces of genus g over the punctured disk. If the homotopical monodromy is of infinite order, then $(N, \pi, \Delta - \{0\})$ can be canonically completed in the holomorphic family $(\widehat{N}, \widehat{\pi}, \Delta)$ with a singular fiber over the origin, where*

$\widehat{N}$ is a two-dimensional normal complex space. Moreover any holomorphic section $s : \Delta - \{0\} \to N$ has a unique holomorphic extension $\widehat{s} : \Delta \to \widehat{N}$.

To use this result, we need to show the following claim.

Claim 1. *For any puncture p of R, the homotopical monodromy $\mathcal{M}_p$ of (M, π, R) around p is of infinite order.*

Proof. First, we consider the case where p is 0. Fix a point r_0 in a neighborhood of 0 in R and fix r_0. When a point r moves from r_0, and turns around 0 once, and comes back to r_0, the cut between 0 and $\rho(r_0)$ on T as in Figure 3 also turns around 0 once. Thus the curve ℓ on T as in Figure 3 changes to ℓ'. When the point r moves as above, by the construction of the fiber S_{r_0}, the curve $\widetilde{\ell}$ on S_{r_0} as in Figure 4 changes to $\widetilde{\ell'}$.

Hence the monodromy $\mathcal{M}_0$ is the twice product of a negative Dehn twist about the simple closed curve C_1, where C_1 is a separating curve as in Figure 5. Therefore $\mathcal{M}_0$ is of infinite order.

Similarly, for another puncture p of R with $p \neq 0$, we see that monodromy $\mathcal{M}_p$ is the twice product of a negative Dehn twist about the simple closed curve C_2, where C_2 is a non-separating curve as in Figure 5. Therefore $\mathcal{M}_p$ is of infinite order. ∎

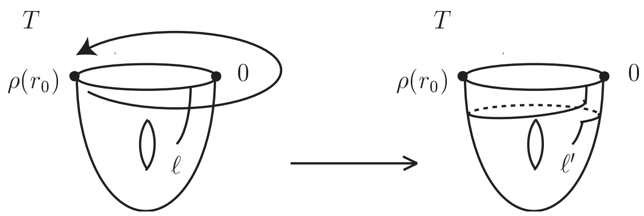


FIGURE 3

By means of Theorem 3.2, we see that our family (M, π, R) can be canonically completed in the degenerated family $(\widehat{M}, \widehat{\pi}, \widehat{R})$, where $\widehat{M}$ is a compact two dimensional normal complex space. Moreover every holomorphic section $s : R \to M$ has a unique holomorphic extension $\widehat{s} : \widehat{R} \to \widehat{M}$. Let $\widehat{s}_0 : \widehat{R} \to \widehat{M}$ be the holomorphic extension of the zero section s_0. Since $\widehat{R}$ is compact, two tori $\widehat{s}(\widehat{R})$ and $\widehat{s}_0(\widehat{R})$ intersects each other at most finitely many times on $\widehat{M}$. Then the set $S = g^{-1}(0)$ is a finite subset of R, hence the restriction of g to $R \setminus S$ induces the holomorphic mapping $R \setminus S \to \widehat{T} \setminus \{0\}$ between hyperbolic Riemann surfaces. Now we recall a generalization of the "big" Picard Theorem:

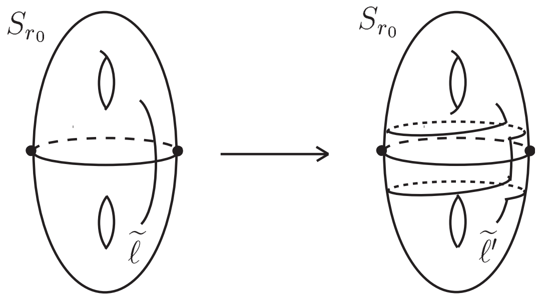


FIGURE 4

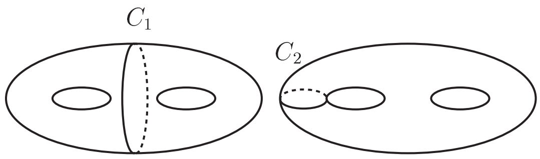


FIGURE 5

Theorem 3.3 (Royden [16]). *Let f be a holomorphic mapping of the punctured disk Δ^* into a hyperbolic Riemann surface W. Then either f extends to a holomorphic mapping of the disk Δ into W or else W is contained in a Riemann surface $W^* = W \cup \{p\}$, so that f extends to a holomorphic mapping of Δ into W^*.*

From this result, the mapping $R \setminus S \to \widehat{T} \setminus \{0\}$ extends uniquely to a holomorphic mapping $\widehat{g} : \widehat{R} \to \widehat{T}$. ■

Proposition 3.2. *For any $g \in \Phi(\mathcal{S}) \setminus \{\rho, 0\}$, the mapping g satisfies $\Gamma_g \cap \Gamma_\rho = \emptyset$ and $\Gamma_g \cap \Gamma_0 = \emptyset$.*

Proof of Proposition 3.2. Every element g in $\Phi(\mathcal{S}) \setminus \{\rho, 0\}$ is extended to a holomorphic mapping $\widehat{g}$ from $\widehat{R}$ to $\widehat{T}$ by Theorem 3.1. We remark that $\widehat{g}$ becomes an unbranched covering from $\widehat{R}$ onto $\widehat{T}$ by Riemann-Hurwitz formula. Let $\widetilde{g} : \mathbb{C}_z \to \mathbb{C}_w$ be a lift of $\widehat{g}$ to the universal coverings of $\widehat{R}$ and $\widehat{T}$ which satisfies $\alpha_1 \circ \widetilde{g} = \widehat{g} \circ \alpha_2$. Since $\widehat{g}$ is non-constant, $\widetilde{g}$ must be an automorphism of $\mathbb{C}$, hence $\widetilde{g}$ is written as

$$\widetilde{g}(z) = Az + B,$$

where A and B are complex numbers and $A \neq 0$. It should be remarked that $\widetilde{g}$ is not unique, because we may replace $\widetilde{g}$ by $\gamma_1 \circ \widetilde{g} \circ \gamma_2$, where $\gamma_1 \in \Gamma_{1,\tau}$ and $\gamma_2 \in \Gamma_{2,2\tau}$.

$$\begin{array}{ccc} \mathbb{C}_z & \xrightarrow{\widetilde{g}} & \mathbb{C}_w \\ {\scriptstyle \alpha_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ \widehat{R} & \xrightarrow{\widehat{g}} & \widehat{T} \\ \cup & \nearrow_{g} & \\ R & & \end{array}$$

For three graphs Γ_g, Γ_0 and Γ_ρ in $R \times \widehat{T}$, we consider the following two cases:

Case (1) $\Gamma_g \cap \Gamma_0 \neq \emptyset$.

Case (2) $\Gamma_g \cap \Gamma_\rho \neq \emptyset$.

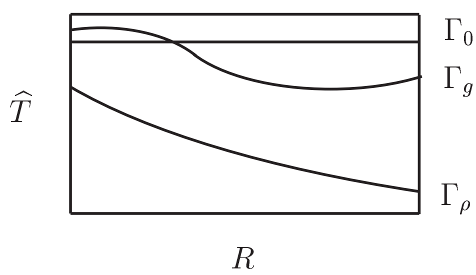


FIGURE 6. Case (1)

Case (1) In this case, there exists a point $r_0 \in R$ such that $g(r_0) = 0$, hence $\widehat{g}(r_0) = 0$. Then we can find $z_0 \in \mathbb{C}_z \setminus L(1,\tau)$ such that $\alpha_2(z_0) = r_0$ and

$$Az_0 + B = 0. \tag{3.2}$$

For sufficiently small $\epsilon > 0$, $\Delta(z_0, \epsilon) = \{z \in \mathbb{C}_z \mid |z - z_0| < \epsilon\}$ and $\Delta(0, \epsilon) = \{w \in \mathbb{C}_w \mid |w| < \epsilon\}$ can be taken as local charts at $r_0 \in R$ and $0 \in \widehat{T}$, respectively. Then the graph $\Gamma_0 = \{(r, 0) \mid r \in R\}$ in $R \times \widehat{T}$ can be locally written as

$$w = 0$$

in $\Delta(z_0, \epsilon) \times \Delta(0, \epsilon)$. Thus M is locally represented as

$$u^2 = w$$

in $\mathbb{C}_u \times \Delta(z_0,\epsilon)\times\Delta(0,\epsilon)$. Take $\epsilon' > 0$ with $\epsilon' < \epsilon$, and set $z = z_0 + \epsilon' e^{i\theta}$. By using (3.2), we have

$$\begin{aligned} u^2 &= Az + B \\ &= A(z_0 + \epsilon' e^{i\theta}) + B \\ &= A\epsilon' e^{i\theta}. \end{aligned}$$

By the same argument as in the proof of Proposition 3.1, we have a contradiction.

Case (2) In this case, there exists a point $r_0 \in R$ such that $g(r_0) = \rho(r_0)$, hence $\widehat{g}(r_0) = \widehat{\rho}(r_0)$. Since $\widetilde{\rho}(z) = z$ is a lift of $\widehat{\rho}$, we can find $z_0 \in \mathbb{C}_z \setminus L(1,\tau)$ such that $\alpha_2(z_0) = r_0$ and

$$Az_0 + B = z_0. \tag{3.3}$$

For sufficiently small $\epsilon > 0$, $\Delta(z_0,\epsilon)$ and $\Delta(w_0,\epsilon)$ can be taken as local charts at $r_0 \in R$ and $\rho(r_0) \in \widehat{T}$, respectively.

Then $\Gamma_\rho = \{(r,\rho(r)) \mid r \in R\}$ in $R \times \widehat{T}$ can be locally written as

$$w = z$$

in $\Delta(z_0,\epsilon)\times\Delta(w_0,\epsilon)$. Thus M is locally represented as

$$u^2 = w - z$$

in $\mathbb{C}_u \times \Delta(w_0,\epsilon)\times\Delta(z_0,\epsilon)$. Take $\epsilon' > 0$ with $\epsilon' < \epsilon$, set $z = z_0 + \epsilon' e^{i\theta}$. By using (3.3), we have

$$\begin{aligned} u^2 &= Az + B - z \\ &= A(z_0 + \epsilon' e^{i\theta}) + B - (z_0 + \epsilon' e^{i\theta}) \\ &= (A-1)\epsilon' e^{i\theta}. \end{aligned}$$

By the same argument as in the proof of Proposition 3.1, we have a contradiction. Thus we have the assertion. ■

3.2. **Proof of Main Theorem.** From now on, we assume τ is in the domain F in $\mathbb{C}$ defined by the following four conditions : (i) $\mathrm{Im}\tau > 0$ (ii) $-1/2 \le \mathrm{Re}\tau < 1/2$, (iii) $|\tau| \ge 1$, (iv) $\mathrm{Re}\tau \le 0$ if $|\tau| = 1$, since any flat torus is biholomorphically equivalent to $\mathbb{C}/\Gamma_{1,\tau}$ for some $\tau \in F$.

We recall

$$L(1,\tau) = \{m + n\tau \in \mathbb{C} \mid m,n \in \mathbb{Z}\}$$

and call an element of $L(1,\tau)$ a lattice point, and set

$$L(2,2\tau) = \{2m + 2n\tau \in \mathbb{C} \mid m,n \in \mathbb{Z}\}.$$

Every element g of $\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T})$ has a holomorphic extension $\widehat{g} : \widehat{R} \to \widehat{T}$. A lift $\widetilde{g}$ be of $\widehat{g}$ is written as

$$\widetilde{g}(z) = Az + B,$$

where A and B are complex numbers and $A \neq 0$.

We need two lemmas.

Lemma 3.1. *$A \neq 1$.*

Proof of Lemma 3.1. Suppose $A = 1$. If $B = 0$ modulo $A \cdot \Gamma_{2,2\tau} + \Gamma_{1,\tau} = \Gamma_{1,\tau}$, then $\widetilde{g}$ is a lift of ρ, while ρ is not an element of $\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T})$, a contradiction. Hence B is not equal to 0 modulo $\Gamma_{1,\tau}$. Put $z_0 = -B$ then we have $\alpha_2(z_0) \in R$ and $g(\alpha_2(z_0)) = 0$, since $\alpha_1 \circ \widetilde{g} = \widehat{g} \circ \alpha_2$. Therefore the graphs of g and 0 in $R \times \widehat{T}$ intersects each other, which contradicts the assumption that g is contained in $\mathrm{Hol}_{\mathrm{dis}}(R, \widehat{T})$. ∎

From now on, we may assume that $A \neq 1$.

Lemma 3.2. *$\widetilde{g}$ can be written as $\widetilde{g}(z) = A(z + \omega)$ where $\omega = 0, 1, \tau$ and $1 + \tau$.*

Proof of Lemma 3.2. Take the point $z_0 = -B/(A-1)$. Then $\tilde{g}(z_0) = z_0$. If $z_0 \in \mathbf{C} \setminus \Gamma_{1,\tau}$, we see that $\Gamma_g \cap \Gamma_\rho \neq \emptyset$, a contradiction. Hence $z_0 \in \Gamma_{1,\tau}$. Then there exist integers m, n such that $z_0 = -B/(A-1) = -m - n\tau$. The result follows. ∎

To determine A, we may assume $\widetilde{g}(z) = Az$. Since $\widetilde{g}(L(2, 2\tau)) \subset L(1, \tau)$, we have

$$2A = p + q\tau, \tag{3.4}$$

$$2A\tau = u + v\tau, \tag{3.5}$$

where p, q, u, and v are integers. The Euclidean areas of $\widehat{R}$ and $\widehat{T}$, and $\deg(\widehat{g}) \leq 4$ implies that

$$1 \leq pv - qu \leq 4. \tag{3.6}$$

and

$$|2A| = |p + q\tau| \leq 2. \tag{3.7}$$

By (3.4) and (3.5), we get

$$q\tau^2 + (p - v)\tau - u = 0. \tag{3.8}$$

Since the assumption $\tau \in F$ implies that the discriminant is negative, we have

$$(p + v)^2 < 4(pv - qu). \tag{3.9}$$

The root τ of (3.8) with $\mathrm{Im}(\tau) > 0$ is given by

$$\tau = \begin{cases} \dfrac{v - p + \sqrt{4(pv - qu) - (p+v)^2}\ i}{2q}, & \text{if } q > 0, \\ \dfrac{v - p - \sqrt{4(pv - qu) - (p+v)^2}\ i}{2q}, & \text{if } q < 0. \end{cases} \tag{3.10}$$

First by the assumption $\tau \in F$ and (3.7), we see that the possibilities of p and q are follows.

(i) If $q = 0$, then $p = \pm 1, \pm 2$.
(ii) If $q = 1$, then $p = 0, \pm 1, \pm 2$.
(iii) If $q = 2$, then $p = 0, \pm 1, \pm 2$.

When $q = 0$, from (3.8) and $\tau \in F$, we have $(p, q, u, v) = (\pm 1, 0, 0, \pm 1)$, $(\pm 2, 0, 0, \pm 2)$.

When $q \neq 0$, for each (p, q), we get v satisfying $-1/2 \leq \mathrm{Re}(\tau) < 1/2$. Next for each (p, q, v) we obtain u with (3.6). Finally, finding (p, q, u, v) in these p, q, u, v such that τ represented in (3.10) is an element of F, we have the list of $p, q, u, v, \tau, 2A$ and a fixed point of $\widetilde{g}$ in the following Table 1 and 2.

In these Tables, when some lift $\widetilde{g}$ has a fixed point which is not contained in $L(1, \tau)$, we see that Γ_g intersects Γ_ρ, a contradiction.

Next when $(p, q, u, v) = (1, -1, 1, 2), (1, -1, 2, 2), (2, 1, -1, 1), (2, 1, -2, 1)$, we see that Γ_g intersects Γ_0, a contradiction. Finally when $(p, q, u, v) = (2, 0, 0, 2)$, $\tilde{g}$ is a lift of ρ, a contradiction. Consequently, we have the following

(a) $\sharp\mathrm{Hol}_{\mathrm{dis}}(R, \hat{T}) = 4$, if $\tau \neq i, e^{2\pi i/3}$.
(b) $\sharp\mathrm{Hol}_{\mathrm{dis}}(R, \hat{T}) = 3 \times 4 = 12$, if $\tau = i$ or $e^{2\pi i/3}$.

Thefefore we have the assertion. ∎

p	q	u	v	τ	$2A = p + q\tau$	fixed point
0	1	−1	0	i	i	$(4+2i)/5$
0	1	−2	0	$\sqrt{2}i$	$\sqrt{2}i$	$(2+\sqrt{2}i)/3$
0	1	−3	0	$\sqrt{3}i$	$\sqrt{3}i$	$(2+\sqrt{3}i)/7$
0	1	−4	0	$2i$	$2i$	$(1+i)/2$
0	1	−1	−1	$e^{2\pi i/3}$	$e^{2\pi i/3}$	$(5+\sqrt{3}i)/7$
0	1	−2	−1	$(-1+\sqrt{7}i)/2$	$(-1+\sqrt{7}i)/2$	$(5+\sqrt{7}i)/8$
0	1	−3	−1	$(-1+\sqrt{11}i)/2$	$(-1+\sqrt{11}i)/2$	$(5+\sqrt{11}i)/9$
0	1	−4	−1	$(-1+\sqrt{15}i)/2$	$(-1+\sqrt{15}i)/2$	$(5+\sqrt{15}i)/10$
0	−1	1	0	i	$-i$	$2(1+2i)/5$
0	−1	2	0	$\sqrt{2}i$	$-\sqrt{2}i$	$2(1+\sqrt{2}i)/3$
0	−1	3	0	$\sqrt{3}i$	$-\sqrt{3}i$	$2(3+2\sqrt{3}i)/7$
0	−1	4	0	$2i$	$-2i$	$(1+\sqrt{3}i)/2$
0	−1	1	1	$e^{2\pi i/3}$	$-e^{2\pi i/3}$	$(3-\sqrt{3}i)/3$
0	−1	2	1	$(-1+\sqrt{7}i)/2$	$(1-\sqrt{7}i)/2$	$(5+\sqrt{7}i)/4$
0	−1	3	1	$(-1+\sqrt{11}i)/2$	$(1-\sqrt{11}i)/2$	$(3-\sqrt{11}i)/5$
0	−1	4	1	$(-1+\sqrt{15}i)/2$	$(1-\sqrt{15}i)/2$	$(3-\sqrt{15}i)/6$
0	2	−2	0	i	$2i$	$(1+i)/2$
0	2	−2	−1	$(-1+\sqrt{15}i)/4$	$(-1+\sqrt{15}i)/2$	$(5+\sqrt{15}i)/10$
0	2	−2	−2	$e^{2\pi i/3}$	$2e^{2\pi i/3}$	$\sqrt{3}i/3$
0	−2	2	0	i	$-2i$	$(1+i)/2$
0	−2	2	1	$(-1+\sqrt{15}i)/4$	$(1-\sqrt{15}i)/2$	$(3-\sqrt{15}i)/6$
0	−2	2	2	$e^{2\pi i/3}$	$-2e^{2\pi i/3}$	lattice point
1	0	0	1	*any*	1	lattice point
1	1	−1	1	$e^{2\pi i/3}$	$1+e^{2\pi i/3}$	$(3+\sqrt{3}i)/3$
1	1	−2	0	$(-1+\sqrt{7}i)/2$	$(1+\sqrt{7}i)/2$	$(3+\sqrt{7}i)/4$
1	1	−3	0	$(-1+\sqrt{11}i)/2$	$(1+\sqrt{11}i)/2$	$(5+\sqrt{11}i)/5$
1	1	−4	0	$(-1+\sqrt{15}i)/2$	$(1+\sqrt{15}i)/2$	$(3+\sqrt{15}i)/6$
1	1	−1	1	i	$1+i$	lattice point
1	1	−2	1	$\sqrt{2}i$	$1+\sqrt{2}i$	$(1+\sqrt{2}i)/3$
1	1	−3	1	$\sqrt{3}i$	$1+\sqrt{3}i$	$(1+\sqrt{3}i)/2$
1	−1	1	1	i	$1-i$	lattice point
1	−1	2	1	$\sqrt{2}i$	$1-\sqrt{2}i$	$2(1-\sqrt{2}i)/3$
1	−1	3	1	$\sqrt{3}i$	$1-\sqrt{3}i$	$(1-\sqrt{3}i)/2$
1	−1	1	2	$e^{2\pi i/3}$	$1-e^{2\pi i/3}$	lattice point
1	−1	2	2	$(-1+\sqrt{7}i)/2$	$(3-\sqrt{7}i)/2$	lattice point
1	2	−2	−1	$e^{2\pi i/3}$	$1+2e^{2\pi i/3}$	$2(2+\sqrt{3}i)/7$
1	2	−2	0	$(-1+\sqrt{15}i)/4$	$(1+\sqrt{15}i)/2$	$(3+\sqrt{15}i)/6$

TABLE 1. $p = 0, 1$

p	q	u	v	τ	$2A$	fixed point
-1	0	0	-1	any	-1	$2(1+\tau)/3$
-1	1	-1	-2	$e^{2\pi i/3}$	$-1+e^{2\pi i/3}$	$(7+\sqrt{3}i)/13$
-1	1	-2	-2	$(-1+\sqrt{7}i)/2$	$(-3+\sqrt{7}i)/2$	$(7+\sqrt{7}i)/14$
-1	1	-1	-1	i	$-1+i$	$(3+i)/5$
-1	1	-2	-1	$\sqrt{2}i$	$-1+\sqrt{2}i$	$2(3+\sqrt{2}i)/11$
-1	1	-3	-1	$\sqrt{3}i$	$-1+\sqrt{3}i$	$(3+\sqrt{3}i)/6$
-1	-1	1	-1	i	$-1-i$	$2(2+i)/5$
-1	-1	2	-1	$\sqrt{2}i$	$-1-\sqrt{2}i$	$2(2+3\sqrt{2}i)/11$
-1	-1	3	-1	$\sqrt{3}i$	$-1-\sqrt{3}i$	$(1+\sqrt{3}i)/2$
-1	-1	1	0	$e^{2\pi i/3}$	$-1-e^{2\pi i/3}$	$(5-\sqrt{3}i)/7$
-1	-1	2	0	$(-1+\sqrt{7}i)/2$	$-(1+\sqrt{7}i)/2$	$(5-\sqrt{7}i)/8$
-1	-1	3	0	$(-1+\sqrt{11}i)/2$	$-(1+\sqrt{11}i)/2$	$(5-\sqrt{11}i)/9$
-1	-1	4	0	$(-1+\sqrt{15}i)/2$	$-(1+\sqrt{15}i)/2$	$(5-\sqrt{15}i)/10$
-1	-2	2	0	$(-1+\sqrt{15}i)/4$	$-(1+\sqrt{15}i)/2$	$(5-\sqrt{15}i)/10$
-1	-2	2	1	$e^{2\pi i/3}$	$-1-2e^{2\pi i/3}$	$2(2-\sqrt{3}i)/7$
2	0	0	2	any	2	lattice point
2	1	-1	1	$e^{2\pi i/3}$	$2+e^{2\pi i/3}$	lattice point
2	1	-2	1	$(-1+\sqrt{7}i)/2$	$(3+\sqrt{7}i)/2$	lattice point
2	2	-2	0	$e^{2\pi i/3}$	$2+2e^{2\pi i/3}$	lattice point
-2	0	0	-2	any	-2	$1/2$
-2	-1	1	-1	$e^{2\pi i/3}$	$-2-e^{2\pi i/3}$	$(7-\sqrt{3}i)/13$
-2	-1	2	-1	$(-1+\sqrt{7}i)/2$	$-(3+\sqrt{7}i)/2$	$(7-\sqrt{7}i)/14$
-2	-2	2	0	$e^{2\pi i/3}$	$-2-2e^{2\pi i/3}$	$(3+\sqrt{3}i)/6$

TABLE 2. $p = -1, \pm 2$

References

[1] M. F. Atiyah, *The signature of fibre-bundles*, in Global Analysis, Papers in Honor of K. Kodaira, Univ. Tokyo Press, Tokyo, (1969), pp.73-84.

[2] L. Bers, *Fiber spaces over Teichmüller spaces*, Acta. Math. **130** (1973), 83-126.

[3] C. H. Clemens, A Scrapbook of Complex Curve Theory. Plenum Press, New York and London, 1980.

[4] W. Fulton, Algebraic Topology, Graduate Texts in Mathematics **153**, Springer-Verlag, Tokyo and New York, 1995.

[5] H. Graurert, *Mordells Vermutung über Punkte auf algebraischen Kurven und Funktionenkörper*, Publ.Math. I.H.E.S. **25** (1965), 131-149.

[6] Y. Imayoshi, *Holomorphic families of Riemann surfaces and Teichmüller Spaces,* in Riemann surfaces and Related Topics, 1978 Stony Brook Conference, Ann. Math. Studies, No 97, pp. 277-300, edited by I. Kra and B. Maskit, Princeton University Press, Princeton, New Jersey, 1981.

[7] Y. Imayoshi and T. Nogi, *A remark on holomorphic sections of certain holomorphic families of Riemann surfaces*, "Complex analysis and applications", proceedings of the 13th International conference on Finite or Infinite Dimensional Complex Analysis and Applications, Shantou University, Chaina, pp.8-12 August 2005 / editors Yuefei Wang, Shengjian Wu, Hasi Wulan, Lo Yang, 101-108.

[8] Y. Imayoshi, and H. Shiga, *A finiteness theorem of a holomorphic family of Riemann surfaces*, In Holomorphic Functions and Moduli II, pp.207-219, Springer-Verlag; MSRI publications, volume 10, 1988.

[9] Y. Imayoshi, and M. Taniguchi, An Introduction to Teichmüller Spaces, Springer-Verlag, Tokyo and New York, 1992.

[10] A. Kas, *On deformations of a certain type of irregular algebraic surface*, Amer. J. Math. **90** (1968), 789-804.

[11] S. Kobayashi, Hyperbolic Manifolds and Holomorphic Mappings, An Introduction, Second Edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, 2005.

[12] K. Kodaira, *A certain type of irregular algebraic surfaces*, J. Analyse Math. **19** (1967), 207-215.

[13] J. Manin, *Rational points of algebraic curves over function fields*, Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1395-1440 (Russian). *Translations II AMS* **50** (1966), 189-234.

[14] J. Noguchi, *Hyperbolic fibre spaces and Mordell's conjecture over function fields*, Publ. RIMS, Kyoto Univ. **21** (1985), 27-46.

[15] G. Riera, *Semi-direct products of fuchsian groups and uniformization*, Duke Math. J **44** (1977), 291-304.

[16] H. L. Royden, *The Picard theorem for Riemann surfaces*, Proc. Amer. Math. Soc. **90** (1984), no. 4, 571-574.

[17] T. Shioda, *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Paul. **39** (1990), No. 2, 211-240.

[18] M. Tanabe, *A bound for the theorem of de Franchis*, Proc. Amer. Math. Soc. **127** (1999), No. 8, 2289-2295.

[19] J. J. Wavrik *Deformations of Banach [branched] coverings of complex manifolds*, Amer. J. Math. **90** (1968), 926-960.

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