Tight Lagrangian surfaces in $S^2 \times S^2$ [*]

Hiroshi Iriyeh and Takashi Sakai [†]

Abstract

We determine all tight Lagrangian surfaces in $S^2 \times S^2$. In particular, globally tight Lagrangian surfaces in $S^2 \times S^2$ are nothing but real forms.
Key words: Lagrangian submanifold; Killing nullity; tight map.

1 Introduction and main results

In 1991, Y.-G. Oh [10] introduced the notion of *tightness* of closed Lagrangian submanifolds in compact Hermitian symmetric spaces. Let $(\tilde{M} = G/K, \omega, J)$ be a Hermitian symmetric space of compact type and L be a closed embedded Lagrangian submanifold of $\tilde{M}$. Then L is said to be *globally tight* (resp. *tight*) if it satisfies

$$\#(L \cap g \cdot L) = SB(L, \mathbb{Z}_2)$$

for any isometry $g \in G$ (resp. close to the identity) such that L transversely intersects with $g \cdot L$. Here $SB(L, \mathbb{Z}_2)$ denotes the sum of $\mathbb{Z}_2$-Betti numbers of L.

It is known that any real forms in a compact Hermitian symmetric space G/K are (globally) tight. It is a natural problem to classify all tight Lagrangian submanifolds in G/K. Indeed, Oh [10] proved the following uniqueness theorem in $\mathbb{C}P^n$.

Theorem 1.1 (Oh). *Let L be a closed embedded tight Lagrangian submanifold in $\mathbb{C}P^n$. Then L is the standard totally geodesic $\mathbb{R}P^n$ if $n \geq 2$ or it is the standard embedding $S^1(\cong \mathbb{R}P^1)$ into $S^2(\cong \mathbb{C}P^1)$ as a latitude circle.*

And he posed the following problem:

Problem (Oh). Classify all possible tight Lagrangian submanifolds in other Hermitian symmetric spaces. Are the real forms on them the only possible tight Lagrangian submanifolds?

In this paper we give the complete solution of it in the case of $S^2 \times S^2$. Note that the following is the first result for the above problem except the case of $\mathbb{C}P^n$.

Theorem 1.2. *Let L be a closed embedded tight Lagrangian surface in $(S^2 \times S^2, \omega_0 \oplus \omega_0)$, where ω_0 denotes the standard Kähler form of $S^2(1) \cong \mathbb{C}P^1$. Then L must be one of the following cases:*
(i) the totally geodesic Lagrangian sphere

$$L = \{(x, -x) \in S^2 \times S^2 \mid x \in S^2\}.$$

(ii) a product of latitude circles $S^1(a) \subset S^2$, i.e.,

$$L = S^1(a) \times S^1(b) \subset S^2 \times S^2,$$

where $S^1(a)$ stands for the round circle with radius a $(0 < a \leq 1)$.

Corollary 1.3. *Let L be a closed embedded globally tight Lagrangian surface in $(S^2 \times S^2, \omega_0 \oplus \omega_0)$. Then L must be one of the following two cases:*
(i) the totally geodesic Lagrangian sphere

$$L = \{(x, -x) \in S^2 \times S^2 \mid x \in S^2\}.$$

(ii) the product of equators (totally geodesic Lagrangian torus)

$$L = S^1(1) \times S^1(1) \subset S^2 \times S^2.$$

As Oh[10, p. 409] pointed out, the global tightness is closely related with the Hamiltonian volume minimization problem. In fact, all globally tight Lagrangian submanifolds which are listed in Theorem 1.1 and Corollary 1.3 are Hamiltonian volume minimizing[1] (see [9, 4]).

Our strategy of the proof of the main result (Theorem 1.2) is to classify all tight Lagrangian surfaces by their Killing nullities in $S^2 \times S^2$. Possible Killing nullities of Lagrangian surfaces in $S^2 \times S^2$ are $3, 4, 5$ and 6. In Section 3, we shall show that it is impossible for a tight Lagrangian surface L to have 6 or 5 as the Killing nullity using the theory of tight maps into Euclidean spaces. This part is a modification of Oh's method used in the case of $\mathbb{C}P^n$ (see [10, Theorem 4.4]). But, in our case, we essentially use the equality condition of Kuiper's inequality (see Theorem 2.7) in the case where the Killing nullity of L is 5 (Proposition 3.4).

The latter part of the paper is devoted to the determination of Lagrangian surfaces in $S^2 \times S^2$ with low Killing nullities. In Section 4, first of all, we explain basic inequality obtained by Gotoh [3], which gives a lower bound of the Killing nullity of any submanifold in compact symmetric spaces (see Theorem 4.1). In Section 5, we shall prove a sharp estimate of the lower bound in Gotoh's inequality in the case of Lagrangian surfaces in $S^2 \times S^2$ (Proposition 5.1). This formula enables us to determine all the Lagrangian surfaces with low Killing nullities. In the last section, all the Lagrangian surfaces with Killing nullities 3 or 4 are completely determined. Our argument is based on Gotoh's inequality, the above mentioned estimate and recent developments concerning Lagrangian surfaces in $S^2 \times S^2$ (see [1, 8]). In particular, Gotoh's inequality is used effectively in this context.

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[1] The Lagrangian surface (i) in Corollary 1.3 are actually homologically volume minimizing.

2 Preliminaries

Let $(\tilde{M}, \omega)$ be a closed symplectic manifold. Let L be a manifold of dimension $\frac{1}{2}\dim\tilde{M}$. In this paper all manifolds, maps, etc. are supposed to be of class C^∞. An embedding $\iota : L \to \tilde{M}$ is said to be *Lagrangian* if $\iota^*\omega = 0$. The image $\iota(L)$ is called (embedded) *Lagrangian submanifold* of $\tilde{M}$. Wherever possible, we denote $\iota(L)$ by L. In this paper, we only consider a special class of symplectic manifolds, i.e., Kähler manifolds. If J is the associated complex structure on $(\tilde{M}, \omega)$, then the metric g and ω have the relation $g(X, Y) = \omega(X, JY)$. Then $\iota : L \to \tilde{M}$ is Lagrangian if and only if

$$T_{\iota(p)}\tilde{M} = \iota_*T_pL \oplus J(\iota_*T_pL)$$

for any $p \in L$ as an orthogonal direct sum.

Let us introduce the notion of tightness of Lagrangian submanifolds. Although Oh considered the case of Hermitian symmetric spaces in [10], its definition is valid for, more generally, homogeneous Kähler manifolds.

Definition 2.1. Let $(\tilde{M}, \omega, J)$ be a homogeneous Kähler manifold and L be a Lagrangian submanifold of $\tilde{M}$. Then L is said to be *globally tight* (resp. *tight*) if

$$\#(L \cap g \cdot L) = SB(L, \mathbb{Z}_2)$$

for any holomorphic isometry g (resp. close to the identity) such that L transversely intersects with $g \cdot L$.

One of the important tools to study the tightness of Lagrangian submanifolds is the theory of tight maps into Euclidean spaces. We recall some necessary definitions and results for our discussion in the following sections.

Let M^n be a closed n-dimensional manifold.

Definition 2.2. A nondegenerate function f on M^n is said to be *tight* (or *perfect*) if it has the minimal number of critical points:

$$\#\mathrm{Crit}(f) = SB(M, \mathbb{Z}_2),$$

where $\#\mathrm{Crit}(f)$ denotes the number of critical points of f.

Remark 2.3. By Morse theory, for any nondegenerate function $f \in C^\infty(M^n)$ we have

$$\#\mathrm{Crit}(f) \geq SB(M, \mathbb{Z}_2).$$

Definition 2.4. A map $\phi : M^n \to \mathbb{E}^N$ from a closed manifold M^n to the N-dimensional Euclidean space $(\mathbb{E}^N, \langle \cdot, \cdot \rangle)$ with the standard inner metric $\langle \cdot, \cdot \rangle$ is said to be *tight* if the functions $z \circ \phi$ are tight for all unit vectors $z^* \in S^{N-1}$ such that $z \circ \phi$ is nondegenerate, where z is the linear function dual to z^*:

$$(z \circ \phi)(x) := \langle \phi(x), z^* \rangle \qquad (x \in M^n).$$

Remark 2.5. Sard's theorem says that the function $z \circ \phi$ is nondegenerate for almost all $z^* \in S^{N-1}$.

Definition 2.6. A map $\phi : M^n \to \mathbb{E}^N$ is said to be *substantial* (or *full*) if the image of ϕ is not contained in any hyperplane of $\mathbb{E}^N$.

The following inequality by Kuiper [5, Theorem 3A] will be used to prove the nonexistence result of tight Lagrangian surfaces in $S^2 \times S^2$ with Killing nullity 6 or 5.

Theorem 2.7 (Kuiper [5, 6], Little-Pohl [7]). *Let M^n be a closed n-dimensional manifold. If $\phi : M^n \to \mathbb{E}^N$ is a tight smooth map substantially into $\mathbb{E}^N$, then*

$$N \leq \frac{1}{2}n(n+3). \tag{2.1}$$

Moreover, the equality is only obtained if $M = \mathbb{R}P^n$, the n-dimensional real projective space and the image $\phi(M)$ is the Veronese manifold (unique up to projective transformation) of $\mathbb{E}^N$.

Note that the equality condition above was obtained by Kuiper [6] for surfaces, $n = 2$, and by Little and Pohl [7] for n-manifolds in general. We will use the equality condition for the case of surfaces essentially in Section 3.

At the end of this section, we review the definition of the Killing nullity. Let $\tilde{M}$ be a Riemannian manifold and M be a submanifold in $\tilde{M}$. Let $\mathfrak{i}(\tilde{M})$ be the Lie algebra consisting of all Killing vector fields of $\tilde{M}$. Consider the following vector space

$$\mathfrak{i}(\tilde{M})^{NM} := \{Z^{NM} \in \Gamma(NM) \mid Z \in \mathfrak{i}(\tilde{M})\},$$

where NM denotes the normal bundle of M and Z^{NM} indicates the normal component of a vector field Z of $\tilde{M}$. The dimension of $\mathfrak{i}(\tilde{M})^{NM}$ is called the *Killing nullity* of M and denoted by $\mathrm{nul}_K(M)$. For any $p \in M$, we consider a linear map

$$\Phi_p : \mathfrak{i}(\tilde{M})^{NM} \longrightarrow N_pM \oplus \mathrm{Hom}(T_pM, N_pM)$$

defined by

$$\Phi_p(Z^{NM}) := (Z_p^{NM}, (\nabla^{NM}Z^{NM})_p),$$

where ∇^{NM} denotes the normal connection of the normal bundle NM. By definition of Φ_p, we have

$$\mathrm{nul}_K(M) \geq \dim \mathrm{Im}\Phi_p.$$

As we explain in Section 4, this estimate can be described in terms of Lie algebra in the case where $\tilde{M}$ is a compact Riemannian symmetric space.

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3   Nonexistence of tight Lagrangian surfaces with large Killing nullities

Let G be the identity component of the full isometry group of $S^2 \times S^2$, that is, $G = SO(3) \times SO(3)$. Then the isotropy group K at $o = (p_1, p_2)$ in $S^2 \times S^2$ is isomorphic to $SO(2) \times SO(2)$, and $S^2 \times S^2$ is expressed as a coset space G/K. Assume that G is equipped with an invariant metric normalized so that G/K becomes isometric to the product of unit spheres.

The vector space of all Killing vector fields on $S^2 \times S^2$ is denoted by $\mathfrak{i}(G/K)$, which is isomorphic to the Lie algebra $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ of G. Let $\iota : L \to S^2 \times S^2$ be a Lagrangian embedding of a closed surface L. Let us consider the Killing nullity of L: $\mathrm{nul}_K(L) = \dim_{\mathbb{R}} \mathfrak{i}(G/K)^{NL}$. Since the dimension of G is 6, we have $\mathrm{nul}_K(L) \le 6$.

Proposition 3.1. *If $\mathrm{nul}_K(L) = 6$, then the closed embedded Lagrangian surface L in $S^2 \times S^2$ cannot be tight in the sense of Definition 2.1.*

Proof. Assume that $\mathrm{nul}_K(L) = 6$. It suffices to show the following:

Claim 1. There exists some $g \in G$ which is arbitrarily close to the identity such that L transversely intersects with $g \cdot L$ and

$$\#(L \cap g \cdot L) > SB(L, \mathbb{Z}_2).$$

It is equivalent to

Claim 2. There exists some $W \in \mathfrak{i}(G/K)$ such that all zeros of $W^{NL} \in \Gamma(NL)$ are nondegenerate and

$$\#\mathrm{Zero}(W^{NL}) > SB(L, \mathbb{Z}_2).$$

Indeed, for $W \in \mathfrak{i}(G/K)$ as in Claim 2, $\exp(tW) \in G$ will be an isometry satisfying the property of Claim 1 for sufficiently small t. Hence, we shall prove Claim 2.

Choose $W_1, W_2, \ldots, W_6 \in \mathfrak{i}(G/K)$ such that $\{W_1^{NL}, W_2^{NL}, \ldots, W_6^{NL}\}$ form a basis of $\mathfrak{i}(G/K)^{NL}$. Since the isometry group action of G on G/K is Hamiltonian, there exists a function $f_\xi \in C^\infty(G/K)$ corresponding to any element $\xi \in \mathfrak{i}(G/K) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ such that

$$df_\xi = \omega(\xi, \cdot).$$

Therefore, for each $W_i \in \mathfrak{i}(G/K)$ ($i = 1, 2, \ldots, 6$), there exists a function $f_i \in C^\infty(G/K)$ such that $df_i = \omega(W_i, \cdot)$.

Let us define $\phi_i := \iota^* f_i \in C^\infty(L)$ ($i = 1, 2, \ldots, 6$) and introduce a smooth map $\phi : L \to \mathbb{E}^6$ defined by

$$\phi(p) := (\phi_1(p), \phi_2(p), \ldots, \phi_6(p)) \quad (p \in L).$$

We shall show that $\phi : L \to \mathbb{E}^6$ is substantial. Assume that ϕ is not substantial, i.e., there exists a hyperplane $H \subset \mathbb{E}^6$ such that $\phi(L) \subset H$. This condition is equivalent to the one that there exists a nonzero vector $z^* = (b_1, b_2, \ldots, b_6) \in \mathbb{R}^6$ such that

$$(z \circ \phi)(p) = \langle z^*, \phi(p) \rangle = \mathrm{const}.$$

for all $p \in L$. Hence the differential of $z \circ \phi$ vanishes identically and we have

$$
\begin{aligned}
0 = d(z \circ \phi) &= d(b_1\phi_1 + b_2\phi_2 + \cdots + b_6\phi_6) \\
&= b_1 d(\iota^* f_1) + b_2 d(\iota^* f_2) + \cdots + b_6 d(\iota^* f_6) \\
&= \iota^*(b_1 df_1 + b_2 df_2 + \cdots + b_6 df_6) \\
&= \iota^*(b_1 \omega(W_1, \cdot) + b_2 \omega(W_2, \cdot) + \cdots + b_6 \omega(W_6, \cdot)) \\
&= \omega(b_1 W_1 + b_2 W_2 + \cdots + b_6 W_6, \iota_*(\cdot)).
\end{aligned}
$$

Putting $V := b_1 W_1 + b_2 W_2 + \cdots + b_6 W_6 \in \mathfrak{i}(G/K)$, then we obtain

$$
0 = \omega(V^{NL}, \iota_*(\cdot)) = g(JV^{NL}, \iota_*(\cdot))
$$

and it yields $V^{NL} = b_1 W_1^{NL} + b_2 W_2^{NL} + \cdots + b_6 W_6^{NL} = 0$. Since $W_1^{NL}, W_2^{NL}, \ldots, W_6^{NL}$ are linearly independent, we have $z^* = (b_1, b_2, \ldots, b_6) = 0$. This is a contradiction. Therefore, $\phi : L \to \mathbb{E}^6$ is substantial.

By Theorem 2.7, the smooth map ϕ cannot be tight. Hence, there is some $z^* = (a_1, a_2, \ldots, a_6) \in S^5(1) \subset \mathbb{E}^6$ such that

$$
z \circ \phi = a_1\phi_1 + a_2\phi_2 + \cdots + a_6\phi_6 \in C^\infty(L)
$$

is nondegenerate but not tight, that is,

$$
\#\mathrm{Crit}(z \circ \phi) > SB(L, \mathbb{Z}_2).
$$

The differential of $z \circ \phi$ is calculated as

$$
d(z \circ \phi) = \omega(a_1 W_1 + a_2 W_2 + \cdots + a_6 W_6, \iota_*(\cdot)).
$$

Here, if we put

$$
W := a_1 W_1 + a_2 W_2 + \cdots + a_6 W_6 \in \mathfrak{i}(G/K), \tag{3.1}
$$

then the critical points of $z \circ \phi$ coincide with the zeros of $W^{NL} \in \mathfrak{i}(G/K)^{NL}$. The Killing vector field W in (3.1) satisfies the requirement of Claim 2:

$$
\#\mathrm{Zero}(W^{NL}) = \#\mathrm{Crit}(z \circ \phi) > SB(L, \mathbb{Z}_2).
$$

$\square$

Remark 3.2. The above proposition can be generalized to the case of complex hyperquadrics $Q_n(\mathbb{C})$. It is expressed as a coset space $G/K = SO(n+2)/SO(2) \times SO(n)$. Let L be a closed Lagrangian submanifold in G/K. Since the dimension of the Lie algebra of $G = SO(n+2)$ is $(n+2)(n+1)/2$, we have $\mathrm{nul}_K(L) \le (n+2)(n+1)/2$. The same argument of Proposition 3.1 implies that *if* $\mathrm{nul}_K(L) = (n+2)(n+1)/2$, *then the closed embedded Lagrangian submanifold L in $Q_n(\mathbb{C})$ cannot be tight in the sense of Definition 2.1.*

Before we proceed to the case where $\mathrm{nul}_K(L) = 5$, let us mention a topological restriction for embedded Lagrangian surfaces of $S^2 \times S^2$.

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Lemma 3.3. *Let $\iota : L \to S^2 \times S^2$ be a Lagrangian embedding of a closed surface. Then the Euler characteristic $\chi(L)$ of L is even.*

Proof. Let $\iota : L \to S^2 \times S^2$ be a Lagrangian embedding. Then $\iota_*[L]$ defines an element of 2-dimensional integral homology class $H_2(S^2 \times S^2, \mathbb{Z})$. Since the homology class is generated by $S := [S^2 \times \{\mathrm{pt}\}]$ and $T := [\{\mathrm{pt}\} \times S^2]$, the element $\iota_*[L]$ is represented as

$$\iota_*[L] = mS + nT$$

for some $m, n \in \mathbb{Z}$. The self-intersection index of the cycle $\iota_*[L] \in H_2(S^2 \times S^2, \mathbb{Z})$ is calculated as

$$\iota_*[L] \cdot \iota_*[L] = (mS + nT) \cdot (mS + nT) = 2mn.$$

This fact and Arnold's formula (see [2, p. 200])

$$\iota_*[L] \cdot \iota_*[L] = \chi(L)$$

(in the nonorientable case the equality is modulo 2) implies that $\chi(L)$ is even. □

Let us consider the case where $\mathrm{nul}_K(L) = 5$.

Proposition 3.4. *If $\mathrm{nul}_K(L) = 5$, then the closed embedded Lagrangian surface L in $S^2 \times S^2$ cannot be tight in the sense of Definition 2.1.*

Proof. Let $\iota : L \to S^2 \times S^2$ be a Lagrangian embedding of a closed surface L. Suppose that L is tight, i.e., it satisfies

$$\#(L \cap g \cdot L) = SB(L, \mathbb{Z}_2)$$

for any isometry $g \in G$ close to the identity such that L transversely intersects with $g \cdot L$. This condition implies that

$$\#\mathrm{Zero}(W^{NL}) = SB(L, \mathbb{Z}_2) \tag{3.2}$$

for any $W \in \mathfrak{i}(G/K)$ such that all zeros of $W^{NL} \in \mathfrak{i}(G/K)^{NL}$ are nondegenerate.

Since $\mathrm{nul}_K(L) = 5$, we can choose $W_1, \ldots, W_5 \in \mathfrak{i}(G/K)$ such that $\{W_1^{NL}, \ldots, W_5^{NL}\}$ form a basis of $\mathfrak{i}(G/K)^{NL}$. For any $W_i \in \mathfrak{i}(G/K)$ $(i = 1, \ldots, 5)$, there exists a function $f_i \in C^\infty(G/K)$ such that $df_i = \omega(W_i, \cdot)$.

Define $\phi_i := \iota^* f_i \in C^\infty(L)$ $(i = 1, \ldots, 5)$ and consider a smooth map $\phi : L \to \mathbb{E}^5$ defined by

$$\phi(p) := (\phi_1(p), \ldots, \phi_5(p)) \quad (p \in L).$$

As in the proof of Proposition 3.1, we see that ϕ is a substantial map.

For any $z^* = (a_1, \ldots, a_5) \in S^4(1) \subset \mathbb{E}^5$ such that $z \circ \phi = a_1\phi_1 + \cdots + a_5\phi_5$ is nondegenerate, we obtain

$$d(z \circ \phi) = \omega(a_1 W_1 + \cdots + a_5 W_5, \iota_*(\cdot)).$$

Putting $W := a_1 W_1 + \cdots + a_5 W_5 \in \mathfrak{i}(G/K)$, then we have
$$d(z \circ \phi)(p) = \omega \left(W^{NL}(p), \iota_{*p}(\cdot) \right) \qquad (3.3)$$
for any $p \in L$. By equations (3.2) and (3.3), we obtain
$$\#\mathrm{Crit}(z \circ \phi) = \#\mathrm{Zero}(W^{NL}) = SB(L, \mathbb{Z}_2)$$
for any $z^* = (a_1, \ldots, a_5) \in S^4(1) \subset \mathbb{E}^5$ such that $z \circ \phi$ is nondegenerate. Hence, $\phi : L \to \mathbb{E}^5$ is a tight substantial map of a closed surface.

Theorem 2.7 implies that ϕ satisfies the equality of (2.1). Hence, we have $L = \mathbb{R}P^2$ and ϕ is the Veronese embedding. But Lemma 3.3 shows that $\mathbb{R}P^2$ cannot be realized as a Lagrangian embedding into $S^2 \times S^2$, since $\chi(\mathbb{R}P^2) = 1$. $\qquad \square$

4 Symmetric spaces and Gotoh's inequality

In this section, we explain Gotoh's inequality which gives a lower bound of the Killing nullity of any submanifold in compact symmetric spaces.

Let (G, H) be a Riemannian symmetric pair and $\mathfrak{g} = \mathfrak{h} + \tilde{\mathfrak{m}}$ its canonical decomposition, where $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively, and $\tilde{\mathfrak{m}}$ is naturally identified with the tangent space $T_o(G/H)$ of the origin $o = H$ in the symmetric space G/H. Let M be a compact submanifold in G/H. We may assume that M contains the origin $o = H$. Then $\tilde{\mathfrak{m}}$ is orthogonally decomposed as
$$\tilde{\mathfrak{m}} = \mathfrak{m} + \mathfrak{m}^\perp,$$
where subspaces $\mathfrak{m}$ and $\mathfrak{m}^\perp$ correspond to the tangent space $T_o M$ of M and the normal space $N_o M$, respectively. Hence, we have an orthogonal decomposition
$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} + \mathfrak{m}^\perp.$$
Any $Z \in \mathfrak{g}$ can be decomposed as
$$Z = Z_{\mathfrak{h}} + Z_{\mathfrak{m}} + Z^\perp$$
according to the above orthogonal decomposition. Define two linear mappings $\Psi_1 : \mathfrak{g} \to \mathfrak{m}^\perp$ and $\Psi_2 : \mathfrak{g} \to \mathrm{Hom}(\mathfrak{m}, \mathfrak{m}^\perp)$ by
$$\Psi_1(Z) := Z^\perp \quad \text{and} \quad \Psi_2(Z)(X) := (\mathrm{ad}_{\mathfrak{g}}(Z_{\mathfrak{h}})X)^\perp - B(X, Z_{\mathfrak{m}}) \qquad (X \in \mathfrak{m}),$$
where $B : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}^\perp$ is the bilinear mapping corresponding to the second fundamental form of M at o. Then the following theorem has been proven by T. Gotoh.

Theorem 4.1 (Gotoh [3]). *Let G/K be a compact Riemannian symmetric space and M a compact connected submanifold of G/K. Then, the Killing nullity of M satisfies the inequality*
$$\mathrm{nul}_K(M) \geq \mathrm{codim}(M) + \dim \mathrm{Im}(\Psi_2|_{\mathfrak{h}}). \qquad (4.1)$$
Moreover, if M satisfies the equality in (4.1), then M is an orbit of a closed subgroup of G, i.e., M is a homogeneous submanifold of G/K.

Remark 4.2. We note that the Killing nullity $\mathrm{nul}_K(M)$ is a global invariant of M. On the other hand, the right hand side of (4.1) is determined at the origin $o \in M$, because $\mathrm{Im}(\Psi_2|_{\mathfrak{h}})$ is only depend on the choice of a subspace $\mathfrak{m}$ in $\tilde{\mathfrak{m}}$.

5   An estimate for the case of $S^2 \times S^2$

In this section we shall give an estimate of (4.1) in the case of $S^2 \times S^2$ explicitly.

We set

$$S^2 \times S^2 := \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|x\| = \|y\| = 1\} \subset \mathbb{R}^3 \times \mathbb{R}^3,$$

and assume that $S^2 \times S^2$ is equipped with a complex structure $J := J_0 \oplus J_0$, where J_0 is the canonical complex structure of S^2. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and put $o := (e_1, e_1) \in S^2 \times S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$. Then J acts on the basis $\{(e_2, 0), (e_3, 0), (0, e_2), (0, e_3)\}$ of the tangent space $T_o(S^2 \times S^2)$ of $S^2 \times S^2$ at the origin o as:

$$J(e_2, 0) = (e_3, 0), \qquad J(e_3, 0) = (-e_2, 0),$$
$$J(0, e_2) = (0, e_3), \qquad J(0, e_3) = (0, -e_2).$$

A compact Lie group $G := SO(3) \times SO(3)$ acts on $S^2 \times S^2$ transitively and isometrically. Then the isotropy subgroup at o is

$$H = \left\{ \left(\begin{bmatrix} 1 & O \\ O & A \end{bmatrix}, \begin{bmatrix} 1 & O \\ O & B \end{bmatrix} \right) \,\middle|\, A, B \in SO(2) \right\} \cong SO(2) \times SO(2).$$

Therefore $S^2 \times S^2$ can be identified with a homogeneous space G/H in the following manner:

$$\begin{aligned} S^2 \times S^2 \quad &\cong \quad G/H = (SO(3) \times SO(3))/(SO(2) \times SO(2)), \\ g \cdot o \quad &\longleftrightarrow \quad gH. \end{aligned}$$

We denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Here $\mathfrak{g}$ and $\mathfrak{h}$ can be expressed as the following:

$$\begin{aligned} \mathfrak{g} \;&=\; \mathfrak{so}(3) \oplus \mathfrak{so}(3) \\ &= \left\{ \left(\begin{bmatrix} 0 & -x_1 & -x_2 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -y_1 & -y_2 \\ y_1 & 0 & -y_3 \\ y_2 & y_3 & 0 \end{bmatrix} \right) \,\middle|\, x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R} \right\}, \end{aligned}$$

$$\begin{aligned} \mathfrak{h} \;&=\; \mathfrak{so}(2) \oplus \mathfrak{so}(2) \\ &= \left\{ \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -z_1 \\ 0 & z_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -z_2 \\ 0 & z_2 & 0 \end{bmatrix} \right) \,\middle|\, z_1, z_2 \in \mathbb{R} \right\}. \end{aligned}$$

We set a subspace $\tilde{\mathfrak{m}}$ of $\mathfrak{g}$ as

$$\tilde{\mathfrak{m}} = \left\{ \left(\begin{bmatrix} 0 & -x_1 & -x_2 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -y_1 & -y_2 \\ y_1 & 0 & 0 \\ y_2 & 0 & 0 \end{bmatrix} \right) \,\middle|\, x_1, x_2, y_1, y_2 \in \mathbb{R} \right\}.$$

Then we have a canonical decomposition $\mathfrak{g} = \mathfrak{h} \oplus \tilde{\mathfrak{m}}$. The tangent space $T_o(S^2 \times S^2)$ can be identified with $\tilde{\mathfrak{m}}$ in a natural manner. More precisely, the bases of these spaces correspond with each other as in the following:

$$
\begin{array}{rcl}
T_o(S^2 \times S^2) & \cong & \tilde{\mathfrak{m}} \\[6pt]
(e_2, 0) & \longleftrightarrow & \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right), \\[18pt]
(e_3, 0) & \longleftrightarrow & \left(\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right), \\[18pt]
(0, e_2) & \longleftrightarrow & \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right), \\[18pt]
(0, e_3) & \longleftrightarrow & \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right).
\end{array}
$$

We denote the Grassmannian manifold of all oriented 2-planes in $T_o(G/H)$ by $\tilde{G}_2(T_o(G/H))$. The action of the rotation group $SO(T_o(G/H)) =: G'$ on $T_o(G/H)$ induces a transitive action of G' on $\tilde{G}_2(T_o(G/H))$. We can express G' as a matrix group $SO(4)$ with respect to the basis $\{(e_2, 0), (e_3, 0), (0, e_2), (0, e_3)\}$ of $T_o(G/H)$. Then the isotropy subgroup of the action of G' at $V_o := \mathrm{span}_{\mathbb{R}}\{(e_2, 0), (e_3, 0)\} \in \tilde{G}_2(T_o(G/H))$ is

$$
H' = \left\{ \begin{bmatrix} A & O \\ O & B \end{bmatrix} \,\middle|\, A, B \in SO(2) \right\} \cong SO(2) \times SO(2).
$$

Therefore $\tilde{G}_2(T_o(G/H))$ can be identified with a homogeneous space

$$
\tilde{G}_2(T_o(G/H)) \cong G'/H' = SO(4)/(SO(2) \times SO(2))
$$

with the origin V_o. We also denote Lie algebras of G' and H' by $\mathfrak{g}'$ and $\mathfrak{h}'$, respectively, i.e.,

$$
\begin{aligned}
\mathfrak{g}' &= \mathfrak{so}(4) = \{X \in M_4(\mathbb{R}) \mid {}^t X = -X\}, \\
\mathfrak{h}' &= \left\{ \begin{bmatrix} X & O \\ O & Y \end{bmatrix} \,\middle|\, X, Y \in \mathfrak{so}(2) \right\}.
\end{aligned}
$$

Now we set a subspace $\mathfrak{m}'$ of $\mathfrak{g}'$ as

$$
\mathfrak{m}' := \left\{ \begin{bmatrix} O & -{}^t X \\ X & O \end{bmatrix} \,\middle|\, X \in M_2(\mathbb{R}) \right\}.
$$

Then we have a canonical decomposition $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$. The tangent space of $\tilde{G}_2(T_o(G/H))$ at V_0 can be identified with $\mathfrak{m}'$. Take a maximal abelian subspace $\mathfrak{a}'$ of $\mathfrak{m}'$ as

$$
\mathfrak{a}' := \left\{ \begin{bmatrix} O & -{}^t X \\ X & O \end{bmatrix} \,\middle|\, X = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} \right\}.
$$

Then the restricted root system Δ of a compact symmetric pair (G', H') is given by

$$\Delta = \{\theta_1 + \theta_2, \theta_1 - \theta_2\},$$

and

$$\mathcal{C} := \left\{ \begin{bmatrix} O & -{}^t X \\ X & O \end{bmatrix} \ \middle| \ X = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} \quad \begin{array}{l} 0 \le \theta_1 + \theta_2 \le \pi \\ 0 \le \theta_1 - \theta_2 \le \pi \end{array} \right\}$$

is a fundamental domain of $\mathfrak{a}'$. The action of H on $\tilde{G}_2(T_o(G/H)) \cong G'/H'$ is equivalent with the isotropy action of H' on G'/H'. Each orbit of H-action on $\tilde{G}_2(T_o(G/H)) \cong G'/H'$ intersects $\mathrm{Exp}(\mathcal{C})$ with only one point. Hence the orbit space of H-action can be identified with $\mathcal{C}$. This implies that $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$ are invariants of H-action on $\tilde{G}_2(T_o(G/H)) \cong G'/H'$. Geometrically, $\theta_1 - \theta_2$ is the Kähler angle with respect to a complex structure $J_0 \oplus J_0$. On the other hand, $\theta_1 + \theta_2$ is the Kähler angle with respect to $J_0 \oplus (-J_0)$.
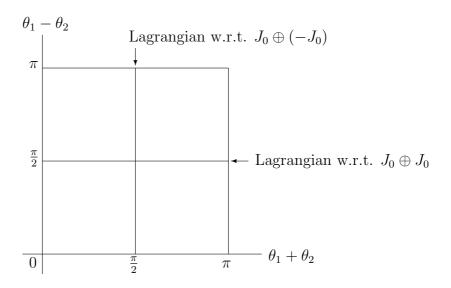

Figure 1: figure of $\mathcal{C}$

Put $X \in \mathcal{C}$ as

$$X = \begin{bmatrix} 0 & 0 & -\theta_1 & 0 \\ 0 & 0 & 0 & -\theta_2 \\ \theta_1 & 0 & 0 & 0 \\ 0 & \theta_2 & 0 & 0 \end{bmatrix}.$$

Then $\mathrm{Exp} X \in G'/H' \cong \tilde{G}_2(T_o(G/H))$ can be expressed as

$$
\begin{aligned}
\mathrm{Exp} X &= \exp X \cdot V_0 \\
&= \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \cdot \mathrm{span}\{(e_2, 0), (e_3, 0)\} \\
&= \mathrm{span}\{\cos\theta_1 (e_2, 0) + \sin\theta_1 (0, e_2), \ \cos\theta_2 (e_3, 0) + \sin\theta_2 (0, e_3)\}.
\end{aligned}
$$

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Hereafter we consider Lagrangian surfaces with respect to a complex structure $J_0 \oplus J_0$. Hence we assume that $\theta_1 - \theta_2 = \frac{\pi}{2}$ and put

$$\theta := \theta_1 = \theta_2 + \frac{\pi}{2} \qquad \left(\frac{\pi}{4} \leq \theta \leq \frac{3}{4}\pi \right).$$

Then

$$\mathrm{Exp}X = \mathrm{span}\{\cos\theta(e_2, 0) + \sin\theta(0, e_2), \ \sin\theta(e_3, 0) - \cos\theta(0, e_3)\}.$$

We denote by $\mathfrak{m}_\theta$ a 2-dimensional subspace of $\tilde{\mathfrak{m}}$ which corresponds to $\mathrm{Exp}X$ by the identification of $T_o(S^2 \times S^2)$ and $\tilde{\mathfrak{m}}$. Then

$$
\mathfrak{m}_\theta \;=\; \mathrm{span}\left\{ \left(\begin{bmatrix} 0 & -\cos\theta & 0 \\ \cos\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right), \right.
$$
$$
\left. \left(\begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & 0 \\ \sin\theta & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \cos\theta \\ 0 & 0 & 0 \\ -\cos\theta & 0 & 0 \end{bmatrix} \right) \right\}
$$
$$
=\; \left\{ \left(\begin{bmatrix} 0 & -x_1\cos\theta & -x_2\sin\theta \\ x_1\cos\theta & 0 & 0 \\ x_2\sin\theta & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -x_1\sin\theta & x_2\cos\theta \\ x_1\sin\theta & 0 & 0 \\ -x_2\cos\theta & 0 & 0 \end{bmatrix} \right) \right.
$$
$$
\left. \Big| \; x_1, x_2 \in \mathbb{R} \right\}.
$$

We have an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}_\theta + \mathfrak{m}_\theta^\perp.$$

Now we shall determine $\mathrm{Ker}(\Psi_2|_\mathfrak{h})$ under these notations. For

$$
Z \;=\; \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -z_1 \\ 0 & z_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -z_2 \\ 0 & z_2 & 0 \end{bmatrix} \right) \in \mathfrak{h},
$$
$$
X \;=\; \left(\begin{bmatrix} 0 & -x_1\cos\theta & -x_2\sin\theta \\ x_1\cos\theta & 0 & 0 \\ x_2\sin\theta & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -x_1\sin\theta & x_2\cos\theta \\ x_1\sin\theta & 0 & 0 \\ -x_2\cos\theta & 0 & 0 \end{bmatrix} \right) \in \mathfrak{m}_\theta,
$$

we have

$$\Psi_2(Z)X = (\mathrm{ad}(Z)X)^\perp$$
$$
=\; \left(\begin{bmatrix} 0 & x_2 z_1 \sin\theta & -x_1 z_1 \cos\theta \\ -x_2 z_1 \sin\theta & 0 & 0 \\ x_1 z_1 \cos\theta & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -x_2 z_2 \cos\theta & -x_1 z_2 \sin\theta \\ x_2 z_2 \cos\theta & 0 & 0 \\ x_1 z_2 \sin\theta & 0 & 0 \end{bmatrix} \right)^\perp.
$$

Thus for

$$
Y = \left(\begin{bmatrix} 0 & -y_1\sin\theta & -y_2\cos\theta \\ y_1\sin\theta & 0 & 0 \\ y_2\cos\theta & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y_1\cos\theta & -y_2\sin\theta \\ -y_1\cos\theta & 0 & 0 \\ y_2\sin\theta & 0 & 0 \end{bmatrix} \right) \in \mathfrak{m}_\theta^\perp,
$$

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we have

$$\begin{aligned}\langle\Psi_2(Z)X,Y\rangle &= \langle[Z,X]^\perp,Y\rangle = \langle[Z,X],Y\rangle = -\frac{1}{2}\mathrm{tr}(ZXY - XZY)\\ &= x_1y_2(z_1\cos^2\theta + z_2\sin^2\theta) - x_2y_1(z_1\sin^2\theta + z_2\cos^2\theta).\end{aligned}$$

Note that $Z \in \mathfrak{h}$ is in $\mathrm{Ker}(\Psi_2|_{\mathfrak{h}})$ if and only if $\langle\Psi_2(Z)X,Y\rangle = 0$ for any $X \in \mathfrak{m}_\theta, Y \in \mathfrak{m}_\theta^\perp$. Thus

$$\begin{aligned}Z \in \mathrm{Ker}(\Psi_2|_{\mathfrak{h}}) &\iff \langle\Psi_2(Z)X,Y\rangle = 0 \qquad (^\forall X \in \mathfrak{m}_\theta,\ ^\forall Y \in \mathfrak{m}_\theta^\perp)\\ &\iff \begin{cases} z_1\cos^2\theta + z_2\sin^2\theta = 0\\ z_1\sin^2\theta + z_2\cos^2\theta = 0 \end{cases}\\ &\iff \begin{cases} z_1 = -z_2 & (\text{if }\cos^2\theta = \sin^2\theta)\\ z_1 = z_2 = 0 & (\text{if }\cos^2\theta \neq \sin^2\theta). \end{cases}\end{aligned}$$

This yields that when $\theta = \frac{\pi}{4}$ or $\theta = \frac{3}{4}\pi$

$$\mathrm{Ker}(\Psi_2|_{\mathfrak{h}}) = \left\{ \left(\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & -z\\ 0 & z & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & z\\ 0 & -z & 0 \end{bmatrix} \right) \ \middle|\ z \in \mathbb{R} \right\},$$

hence $\dim(\mathrm{Im}(\Psi_2|_{\mathfrak{h}})) = 1$. Otherwise $\mathrm{Ker}(\Psi_2|_{\mathfrak{h}}) = \{0\}$, hence $\dim(\mathrm{Im}(\Psi_2|_{\mathfrak{h}})) = 2$.

Let L be a Lagrangian surface of $S^2 \times S^2$ with respect to a complex structure $J_0 \oplus J_0$. Assume that L contains $o = (e_1, e_1) \in S^2 \times S^2$. Then from Theorem 4.1 we have

$$\mathrm{nul}_K(L) \geq \mathrm{codim}(L) + \dim(\mathrm{Im}(\Psi_2|_{\mathfrak{h}})) \geq 3. \tag{5.1}$$

Since $\dim(\mathrm{Im}(\Psi_2|_{\mathfrak{h}}))$ is invariant under the action of H, the equality of the second inequality of (5.1) holds if and only if $T_o(L)$ is contained in subset $H \cdot \mathfrak{m}_{\pi/4}$ or $H \cdot \mathfrak{m}_{3\pi/4}$ of $\tilde{G}_2(T_o(G/H))$. Let $\tilde{G}_2(T(G/H))$ denote the Grassmannian bundle over G/H whose fiber at each point $p \in G/H$ is $\tilde{G}_2(T_p(G/H))$. Since any point of L can be moved to the origin o by the action of G, we have the following proposition.

Proposition 5.1. *Let L be a Lagrangian surface with respect to a complex structure $J_0 \oplus J_0$ on $S^2 \times S^2$. For any $p \in L$, take $g \in G$ such that $gp = o$. Then the Killing nullity of L satisfies the inequality*

$$\mathrm{nul}_K(L) = \mathrm{nul}_K(gL) \geq \mathrm{codim}(gL) + \dim\mathrm{Im}(\Psi_2|_{\mathfrak{h}}) \geq 3. \tag{5.2}$$

Moreover, the equality condition of the last inequality in (5.2) holds for all $p \in L$ if and only if the tangent bundle TL of L is contained in the subbundle $G \cdot \mathfrak{m}_{\pi/4}$ or $G \cdot \mathfrak{m}_{3\pi/4}$ of $\tilde{G}_2(T(G/H))$.

Now we study the condition that the equality of the last inequality of (5.2) will be satisfied. When $\theta = \frac{\pi}{4}$, we have $\theta_1 + \theta_2 = 0$, $\theta_1 - \theta_2 = \frac{\pi}{2}$. When $\theta = \frac{3}{4}\pi$, we have $\theta_1 + \theta_2 = \pi$, $\theta_1 - \theta_2 = \frac{\pi}{2}$. Hence $\mathfrak{m}_{\frac{\pi}{4}}$ and $\mathfrak{m}_{\frac{3}{4}\pi}$ are Lagrangian subspaces of $\tilde{\mathfrak{m}} \cong T_o(G/H)$ with respect to $J_0 \oplus J_0$, and are complex subspaces with respect to $J_0 \oplus (-J_0)$. Therefore the equality of the last inequality of (5.2) holds for all $p \in L$ if

and only if L is a complex submanifold of $S^2 \times S^2$ with respect to a complex structure $J_0 \oplus (-J_0)$. A complex submanifold of a Kähler manifold is a calibrated submanifold, so it is volume minimizing in its homology class, in particular it is a stable minimal submanifold. Castro and Urbano [1] obtained the following result for stable minimal Lagrangian surfaces in $S^2 \times S^2$.

Theorem 5.2 (Castro-Urbano [1])**.** *The only stable compact minimal Lagrangian surface of $S^2 \times S^2$ is the totally geodesic Lagrangian sphere*

$$\mathbf{M}_0 := \{(x, -x) \in S^2 \times S^2 \mid x \in S^2\}.$$

Hence, we have

Corollary 5.3. *Let L be a compact connected Lagrangian surface in $S^2 \times S^2$ with respect to a complex structure $J_0 \oplus J_0$. When we move any point of L to the origin o, the inequality*

$$\mathrm{nul}_K(L) \geq \mathrm{codim}(L) + \dim \mathrm{Im}(\Psi_2|_{\mathfrak{h}}) \geq 3$$

is satisfied. Moreover, the equality of the second inequality in the above formula holds for all points of L if and only if L is congruent to $\mathbf{M}_0$.

6 Classification of Lagrangian surfaces with low Killing nullities

In this section, using the inequality in Proposition 5.1, let us classify Lagrangian surfaces of $S^2 \times S^2$ with low Killing nullities.

6.1 The case where $\mathrm{nul}_K(L) = 3$

Let L be a compact connected Lagrangian surface in $S^2 \times S^2$. Assume that $\mathrm{nul}_K(L) = 3$. Then, by Corollary 5.3, we have

$$3 = \mathrm{nul}_K(L) \geq \mathrm{codim}(L) + \dim \mathrm{Im}(\Psi_2|_{\mathfrak{h}}) \geq 3$$

for all points of L. Hence, the equality condition of the second inequality holds. Using Corollary 5.3 again, L must be congruent to the totally geodesic Lagrangian sphere $\mathbf{M}_0$. It is known that $\mathbf{M}_0$ is globally tight.

6.2 The case where $\mathrm{nul}_K(L) = 4$

Next, assume that $\mathrm{nul}_K(L) = 4$. Then, by the conclusion of the subsection above, L cannot be congruent to $\mathbf{M}_0$. Therefore, by Corollary 5.3, there exist $p \in L$ and $g \in G$ such that $gp = o$ and

$$4 = \mathrm{nul}_K(L) = \mathrm{nul}_K(gL) \geq \mathrm{codim}(gL) + \dim \mathrm{Im}(\Psi_2|_{\mathfrak{h}}) \geq 4.$$

Hence, the equality condition of the first inequality holds. By Theorem 4.1, L is a homogeneous Lagrangian surface in $S^2 \times S^2$. Here, let us use the following recent result concerning with homogeneous Lagrangian surfaces in $S^2 \times S^2$.

Theorem 6.1 (Ma-Ohnita [8]). *Let L be a compact homogeneous Lagrangian surface in $S^2 \times S^2$. Then L must be congruent to either the totally geodesic Lagrangian sphere*

$$\mathbf{M}_0 = \{(x, -x) \in S^2 \times S^2 \mid x \in S^2\}$$

or Lagrangian tori obtained by a product of latitude circles in S^2

$$T_{a,b} := \{(x, y) \in S^2 \times S^2 \mid x_1 = a, \ y_1 = b\} \qquad (0 \le a, b < 1).^2$$

Note that $\mathrm{nul}_K(\mathbf{M}_0) = 3$, $\mathrm{nul}_K(T_{a,b}) = 4$. Therefore, Theorem 6.1 implies that the Lagrangian surface L must be congruent to $T_{a,b}$. It is clear that $T_{a,b}$ is tight and, especially, the totally geodesic Lagrangian torus $\mathbf{T} := T_{0,0}$ is globally tight.

Thus we finish the proof of Theorem 1.2 and Corollary 1.3.

Acknowledgements

References

[1] I. Castro and F. Urbano, *Minimal Lagrangian surfaces in $S^2 \times S^2$*, Comm. Anal. Geom. **15** (2007), no. 2, 217–248.

[2] A. B. Givental', *Lagrangian imbeddings of surfaces and unfolded Whitney umbrella*, Funkt. Anal. Prilozhen **20** (1986), 35–41; English transl., Funct. Anal. Appl. **20** (1986), 197–203.

[3] T. Gotoh, *The nullity of a compact minimal real hypersurface in a quaternion projective space*, Geom. Dedicata **76** (1999), 53–64.

[4] H. Iriyeh, H. Ono and T. Sakai, *Integral geometry and Hamiltonian volume minimizing property of a totally geodesic Lagrangian torus in $S^2 \times S^2$*, Proc. Japan Acad., **79**, Ser. A (2003), 167–170. arXiv:mathDG/0310432.

[5] N. H. Kuiper, *Minimal total absolute curvature for immersions*, Invent. Math. **10** (1970), 209–238.

[6] N. H. Kuiper, *On convex maps*, Nieuw Archief voor Wisk **10** (1962), 147–164.

[7] J. A. Little, and W. F. Pohl, *On tight immersions of maximal codimension*, Invent. Math. **13** (1971), 179–204.

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^{2}The symbols $\mathbf{M}_0, T_{a,b}$ and $\mathbf{T}$ were introduced in Castro and Urbano's paper [1].

[8] H. Ma and Y. Ohnita, *On Lagrangian submanifolds in complex hyperquadrics and isoparametric hypersurfaces in spheres*, to appear in Math. Z.

[9] Y.-G. Oh, *Second variation and stabilities of minimal lagrangian submanifolds in Kähler manifolds*, Invent. Math. **101** (1990), 501–519.

[10] Y.-G. Oh, *Tight Lagrangian submanifolds in* $\mathbb{C}P^n$, Math. Z. **207** (1991), 409–416.

Hiroshi Iriyeh
School of Science and Technology for Future Life
Tokyo Denki University
2-2 Kanda-Nishiki-Cho, Chiyoda-Ku
Tokyo, 101-8457, Japan
  E-mail address : hirie@im.dendai.ac.jp

Takashi Sakai
Graduate School of Science
Osaka City University
3-3-138, Sugimoto, Sumiyoshi-ku
Osaka, 558-8585, Japan
  E-mail address : tsakai@sci.osaka-cu.ac.jp