COHOMOLOGICAL RIGIDITY OF REAL BOTT MANIFOLDS

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Abstract. A real Bott manifold is the total space of iterated $\mathbb{R}P^1$ bundles starting with a point, where each $\mathbb{R}P^1$ bundle is projectivization of a Whitney sum of two real line bundles. We prove that two real Bott manifolds are diffeomorphic if their cohomology rings with $\mathbb{Z}/2$ coefficients are isomorphic.

A real Bott manifold is a real toric manifold and admits a flat riemannian metric invariant under the natural action of an elementary abelian 2-group. We also prove that the converse is true, namely a real toric manifold which admits a flat riemannian metric invariant under the action of an elementary abelian 2-group is a real Bott manifold.

1. Introduction

A fundamental result in the theory of toric varieties says that the categories of toric varieties (over the complex numbers $\mathbb{C}$) and fans are equivalent (see [16]). This reduces the classification of toric varieties to that of fans. Among toric varieties, compact smooth toric varieties which we call toric manifolds are well studied and the classification as varieties is completed for some classes of toric manifolds (see [10], [16], [18] for example).

However, not much is known for the topological classification of toric manifolds, and the following problem is addressed in [14] (see also [4], [13]).

Cohomological rigidity problem for toric manifolds. Are two toric manifolds diffeomorphic (or homeomorphic) if their cohomology rings with integer coefficients are isomorphic as graded rings?

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As is well-known, there are many closed smooth manifolds which are not homeomorphic but have isomorphic cohomology rings. So the problem above seems unlikely but no counterexample is known and there are some partial affirmative solutions to the problem (see [4], [13], [14]).

The set $X(\mathbb{R})$ of real points in a toric manifold X is called a real toric manifold. It appears as the fixed point set of the complex conjugation on X. For example, when X is a complex projective space $\mathbb{C}P^n$, $X(\mathbb{R})$ is a real projective space $\mathbb{R}P^n$. It is known that

$$H^*(X(\mathbb{R});\mathbb{Z}/2) \cong H^{2*}(X;\mathbb{Z}) \otimes \mathbb{Z}/2$$

for any toric manifold X where $\mathbb{Z}$ denotes the integers and $\mathbb{Z}/2 = \{0, 1\}$, and one may ask the same question as the rigidity problem above for real toric manifolds with $\mathbb{Z}/2$ coefficients, namely

Cohomological rigidity problem for real toric manifolds. Are two real toric manifolds diffeomorphic (or homeomorphic) if their cohomology rings with $\mathbb{Z}/2$ coefficients are isomorphic as graded rings?

In this paper we are concerned with a sequence of $\mathbb{R}P^1$ bundles

$$(1.1) \qquad M_n \xrightarrow{\mathbb{R}P^1} M_{n-1} \xrightarrow{\mathbb{R}P^1} \cdots \xrightarrow{\mathbb{R}P^1} M_1 \xrightarrow{\mathbb{R}P^1} M_0 = \{\text{a point}\}$$

such that $M_i \to M_{i-1}$ for $i = 1, \dots, n$ is the projective bundle of a Whitney sum of two real line bundles over M_{i-1}, where one of the two line bundles may be assumed to be trivial without loss of generality. Grossberg-Karshon [8] considered the sequence above in the complex case and named it a *Bott tower* of height n. Following them, we call the sequence above a *real Bott tower* of height n. The top manifold M_n of a real Bott tower is a real toric manifold. We call it a *real Bott manifold*. The main purpose of this paper is to prove the following which answers the cohomological rigidity problem affirmatively for real Bott manifolds.

Theorem 1.1. *Two real Bott manifolds are diffeomorphic if their cohomology rings with $\mathbb{Z}/2$ coefficients are isomorphic as graded rings.*

Although real toric manifolds have similar properties to toric manifolds, there is one major difference, that is, a real toric manifold is not simply connected while a toric manifold is simply connected. Real toric manifolds provide many examples of aspherical manifolds and real Bott manifolds are examples of flat riemannian manifolds. In fact, any real toric manifold of dimension n supports an action of an elementary abelian 2-group $T^n(\mathbb{R})$ of rank n and real Bott manifolds of dimension n admit a flat riemannian metric invariant under the action of $T^n(\mathbb{R})$.

The following shows that these are the only examples among real toric manifolds.

Theorem 1.2. *A real toric manifold of dimension n which admits a flat riemannian metric invariant under the action of $T^n(\mathbb{R})$ is a real Bott manifold.*

This paper is organized as follows. We describe the cohomology ring and the fundamental group of a real Bott manifold in Sections 2 and 3. In Section 4 we find necessary and sufficient conditions for an isomorphism between cohomology rings of real Bott manifolds to satisfy in terms of matrices. Using the conditions, we construct a monomorphism between the fundamental groups of the real Bott manifolds in Section 5. It may not be an isomorphism but the existence of the monomorphism implies that the fundamental groups are isomorphic, which is done in Section 6 by studying group extensions. Since real Bott manifolds are flat riemannian manifolds, the isomorphism of the fundamental groups implies Theorem 1.1 by a theorem of Bieberbach. In Section 7 we enumerate diffeomorphism classes in real Bott manifolds of dimension up to 4. This result is obtained in [15] independently by a different method. Theorem 1.2 is proved in Section 8. In Section 9 we view real Bott manifolds from a viewpoint of small covers introduced in [5]. In the Appendix, we give a proof on a (probably known) fact used in Section 6.

2. Cohomology rings

We shall describe the cohomology ring of the real Bott manifold M_n in the tower (1.1).

We recall a general well-known fact. Let $E \to X$ be a real vector bundle of rank m over a topological space X and let $P(E)$ be the projectivization of E. As is well-known, $H^*(P(E);\mathbb{Z}/2)$ is an algebra over $H^*(X)$ through the projection map from $P(E)$ to X and the algebra structure is described as

$$H^*(P(E);\mathbb{Z}/2) = H^*(X;\mathbb{Z}/2)[x]\Big/\big(\sum_{i=0}^{m} w_i(E)x^{m-i}\big) \tag{2.1}$$

where $w_i(E)$ denotes the i-th Stiefel-Whitney class of E and x is given by the first Stiefel-Whitney class of the canonical line bundle over $P(E)$. Moreover, the Stiefel-Whitney class of $T_fP(E)$ the tangent bundle along the fibers of $P(E)$ is given by

$$w(T_f(P(E))) = \sum_{i=0}^{m} w_i(E)(1+x)^{m-i},$$

in particular,

$$w_1(T_f(P(E))) = w_1(E) \tag{2.2}$$

when m is even.

Now we return to the tower (1.1). By definition $M_j = P(L_{j-1} \oplus \underline{\mathbb{R}})$ with some line bundle L_{j-1} over M_{j-1} for $j = 1, \dots, n$, where $\underline{\mathbb{R}}$ denotes the trivial line bundle. Let γ_j be the canonical line bundle over M_j and set $x_j = w_1(\gamma_j)$. We use the same notation γ_j (resp. x_j) for the pullback of γ_j (resp. x_j) by compositions of projections $M_k \to M_{k-1} \to \cdots \to M_j$ where $k > j$. Then the repeated use of (2.1) shows

(2.3)
$$H^*(M_k; \mathbb{Z}/2) = \mathbb{Z}/2[x_1, \dots, x_k] / \big(x_j(x_j + w_1(L_{j-1})) \mid j = 1, \dots, k\big).$$

Since $H^1(M_{j-1}; \mathbb{Z}/2)$ is additively generated by $x_1, \dots, x_{j-1}$ and L_{j-1} is a line bundle over M_{j-1}, one can uniquely write

$$w_1(L_{j-1}) = \sum_{i=1}^{j-1} A^i_j x_i \quad \text{with } A^i_j \in \mathbb{Z}/2 \tag{2.4}$$

where $j = 2, \dots, n$. As is well-known, line bundles are classified by their first Stiefel-Whitney classes and the first Stiefel-Whitney class behaves additively for tensor products of line bundles; so it follows from (2.4) that

$$L_{j-1} = \gamma_1^{A^1_j} \otimes \cdots \otimes \gamma_{j-1}^{A^{j-1}_j}. \tag{2.5}$$

For convenience, we set $A^i_j = 0$ unless $i < j$ and form a square matrix A of size n with A^i_j as an (i, j) entry. A is an upper triangular matrix with zero diagonal entries.

The observation above implies that the tower (1.1) is completely determined by the matrix A. So we may denote M_n by $M(A)$. For later use we record the ring structure of $H^*(M(A); \mathbb{Z}/2)$ as a lemma which follows from (2.3) and (2.4).

Lemma 2.1. *Let A and $M(A)$ be as above. Then $H^*(M(A); \mathbb{Z}/2)$ is generated by degree one elements $x_1, \dots, x_n$ as a graded ring with n relations*

$$x_j^2 = x_j \sum_{i=1}^{n} A^i_j x_i \quad \textit{for } j = 1, \dots, n.$$

We conclude this section with the following lemma.

Lemma 2.2. *The real Bott manifold $M(A)$ is orientable if and only if the sum of entries is zero in $\mathbb{Z}/2$ for each row of A.*

Proof. The repeated use of (2.2) together with (2.4) shows that

$$\begin{aligned} w_1(M(A)) &= \sum_{j=1}^{n} w_1(L_{j-1} \oplus \underline{\mathbb{R}}) = \sum_{j=1}^{n} w_1(L_{j-1}) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{j-1} A^i_j x_i = \sum_{i=1}^{n} (\sum_{j=1}^{n} A^i_j) x_i. \end{aligned}$$

Since $M(A)$ is orientable if and only if $w_1(M(A)) = 0$, the lemma follows from the identity above. □

3. Fundamental groups

A general description of the fundamental group of an arbitrary real toric manifold is given in [19] motivated by the work [5]. In this section, we shall describe the fundamental group of $M(A)$ in a direct way.

Let s_i ($i = 1, \dots, n$) be an Euclidean motion on $\mathbb{R}^n$ defined by

(3.1)

$$\begin{aligned} s_i(u_1, \dots, u_n) &= (u_1, \dots, u_{i-1}, u_i + \frac{1}{2}, (-1)^{A^i_{i+1}} u_{i+1}, \dots, (-1)^{A^i_n} u_n) \\ &= ((-1)^{A^i_1} u_1, \dots, (-1)^{A^i_n} u_n) + \frac{1}{2} e^i \end{aligned}$$

where $e^1, \dots, e^n$ denote the standard basis of $\mathbb{R}^n$. The group $\Gamma(A)$ generated by $s_1, \dots, s_n$ is a crystallographic group. In fact, the subgroup generated by $s_1^2, \dots, s_n^2$ consists of all translations by $\mathbb{Z}^n$. The action of $\Gamma(A)$ on $\mathbb{R}^n$ is free and the orbit space $\mathbb{R}^n/\Gamma(A)$ is compact.

Lemma 3.1. *$\mathbb{R}^n/\Gamma(A)$ is diffeomorphic to $M(A)$. Therefore $M(A)$ is a riemannian flat manifold with $\Gamma(A)$ as the fundamental group.*

Proof. Let Γ_k ($k = 1, \dots, n$) be a subgroup of $\Gamma(A)$ generated by $s_1, \dots, s_k$. It acts on $\mathbb{R}^k$ by restricting the action of $\Gamma(A)$ on $\mathbb{R}^n$. We claim that a sequence of projections

$$\mathbb{R}^n/\Gamma_n \to \mathbb{R}^{n-1}/\Gamma_{n-1} \to \cdots \to \mathbb{R}^1/\Gamma_1 \to \{0\}$$

agrees with the real Bott tower (1.1). The lemma follows from the claim.

We shall prove the claim by induction on height. It is obviously true up to height one. Suppose it is true up to height $j-1$. We note that the line bundle γ_i over M_{j-1} for $i \le j-1$ is obtained as the quotient of $\mathbb{R}^{j-1} \times \mathbb{R}$ by the diagonal action of Γ_{j-1} where the action of Γ_{j-1} on the second factor $\mathbb{R}$ is given through a homomorphism $\Gamma_{j-1} \to \{\pm 1\}$ sending s_i to -1 and the others s_ℓ ($\ell \neq i$) to 1. This together with (2.5) shows that the line bundle L_{j-1} in (2.5) is obtained as the quotient of

$\mathbb{R}^{j-1} \times \mathbb{R}$ by the diagonal action of Γ_{j-1} where the action of Γ_{j-1} on the second factor $\mathbb{R}$ is given through a homomorphism $\Gamma_{j-1} \to \{\pm 1\}$ sending s_i to $(-1)^{A^i_j}$ for $i \le j-1$. Therefore the action of Γ_{j-1} on $\mathbb{R}^{j-1} \times \mathbb{R} = \mathbb{R}^j$ is nothing but the restriction of the action of Γ_j to Γ_{j-1} while the action of s_j on $\mathbb{R}^j$ is trivial on the first $(j-1)$ coordinates and translation by $1/2$ on the last coordinate.

We consider a map

$$\begin{aligned} \mathbb{R}^j = \mathbb{R}^{j-1} \times \mathbb{R} &\to (\mathbb{R}^{j-1} \times \mathbb{R})/\Gamma_{j-1} \oplus \underline{\mathbb{R}} = L_{j-1} \oplus \underline{\mathbb{R}}; \\ (x, \theta) &\mapsto \big([x, \sin 2\pi\theta], \cos 2\pi\theta\big). \end{aligned}$$

Since $s_i(x,\theta) = (s_i x, (-1)^{A^i_j}\theta)$ for $i \le j-1$ and $s_j^2(x,\theta) = (x, \theta+1)$, the map above is invariant under the action of Γ_{j-1} and s_j^2 and factors through a diffeomorphism from the orbit space $\mathbb{R}^j/\langle \Gamma_{j-1}, s_j^2\rangle$ onto the unit circle bundle of $L_{j-1} \oplus \underline{\mathbb{R}}$. Furthermore, since $\Gamma_j = \langle \Gamma_{j-1}, s_j\rangle$ and $s_j(x,\theta) = (x, \theta + \frac{1}{2})$, the map induces a diffeomorphism from $\mathbb{R}^j/\Gamma_j$ onto the projectivization $P(L_{j-1} \oplus \underline{\mathbb{R}}) = M_j$. This shows that the projection $\mathbb{R}^j/\Gamma_j \to \mathbb{R}^{j-1}/\Gamma_{j-1}$ agrees with the projection $M_j \to M_{j-1}$, completing the induction step. □

We shall investigate the structure of $\Gamma(A)$.

Lemma 3.2. *For $i < \ell$, $s_\ell s_i = s_i s_\ell^{(-1)^{A^i_\ell}}$, i.e.*

$$s_\ell s_i = \begin{cases} s_i s_\ell^{-1} & \text{if } A^i_\ell = 1, \\ s_i s_\ell & \text{if } A^i_\ell = 0. \end{cases}$$

Proof. Easy to check. □

Lemma 3.3. *Let $\mathcal{G}(A)$ be the group generated by $\sigma_1, \dots, \sigma_n$ with the relations in Lemma 3.2 for σ_j's instead of s_j's. Then the homomorphism $\psi\colon \mathcal{G}(A) \to \Gamma(A)$ defined by $\psi(\sigma_j) = s_j$ for $j = 1, \dots, n$ is an isomorphism.*

Proof. Using the relations, one can express an element σ of $\mathcal{G}(A)$ as $\sigma_1^{a_1}\sigma_2^{a_2}\dots\sigma_n^{a_n}$ with $a_1, \dots, a_n \in \mathbb{Z}$. Suppose $\psi(\sigma) = s_1^{a_1}s_2^{a_2}\dots s_n^{a_n}$ is the identity element. Then $\psi(\sigma)$ fixes any element of $\mathbb{R}^n$. But it maps the origin of $\mathbb{R}^n$ to $\frac{1}{2}\sum_{j=1}^n \epsilon_j a_j e_j$, where $\epsilon_j = \pm 1$, and the image must again be the origin, so we have $a_j = 0$ for any j. This shows that σ is the identity and ψ is injective. The surjectivity of ψ is trivial. □

s_j^2 is a translation of $\mathbb{R}^n$ by e_j so that $s_1^2, \dots, s_n^2$ commute with each other and generate a free abelian subgroup N. The images of s_j's in the quotient $\Gamma(A)/N$ commute with each other, which easily follows from Lemma 3.2, so that $\Gamma(A)/N$ is an elementary abelian 2-group.

We identify N with $\mathbb{Z}^n$ and $\Gamma(A)/N$ with $(\mathbb{Z}_2)^n$ in a natural way where $\mathbb{Z}_2 = \{\pm 1\}$ and obtain a short exact sequence:

$$0 \to \mathbb{Z}^n \to \Gamma(A) \to (\mathbb{Z}_2)^n \to 1. \tag{3.2}$$

One may think of $M(A) = \mathbb{R}^n/\Gamma(A)$ as the orbit space of the torus $\mathbb{R}^n/\mathbb{Z}^n$ by the induced action of $\Gamma(A)/\mathbb{Z}^n = (\mathbb{Z}_2)^n$. We shall explicitly describe the action using complex numbers $\mathbb{C}$. Let S^1 denote the unit circle of $\mathbb{C}$. We identify $\mathbb{R}/\mathbb{Z}$ with S^1 (and hence $\mathbb{R}^n/\mathbb{Z}^n$ with $(S^1)^n$) through the exponential map sending $u \in \mathbb{R}$ to $\exp(2\pi\sqrt{-1}u) \in \mathbb{C}$. For $z \in S^1$ and $a \in \mathbb{Z}/2 = \{0, 1\}$ we define

$$z(a) := \begin{cases} z & \text{if } a = 0, \\ \bar{z} & \text{if } a = 1. \end{cases}$$

Then the induced action of s_i defined in (3.1) on $(S^1)^n$ is given by

$$(z_1, \dots, z_n) \to (z_1, \dots, z_{i-1}, -z_i, z_{i+1}(A^i_{i+1}), \dots, z_n(A^i_n)).$$

4. An isomorphism between cohomology rings

As is described in Lemma 2.1, $H^*(M(A); \mathbb{Z}/2) = R_1$ is a graded algebra over $\mathbb{Z}/2$ generated by degree one elements $x_1, \dots, x_n$ with n relations

$$x_j^2 = x_j \sum_{i=1}^{n} A^i_j x_i \quad (j = 1, \dots, n). \tag{4.1}$$

The set V_1 of degree one elements in R_1 with vanishing squares forms a vector space over $\mathbb{Z}/2$ of positive dimension. Set $n_1 = \dim V_1$. Permuting the suffixes of $x_1, \dots, x_n$, we may assume that the first n_1 elements $x_1, \dots, x_{n_1}$ form a basis of V_1. We consider the quotient graded ring $R_2 = R_1/(V_1)$ where (V_1) denotes the ideal in R_1 generated by V_1. Similarly, the set V_2 of degree one elements in R_2 with vanishing squares forms a vector space over $\mathbb{Z}/2$ of positive dimension. Set $n_2 = \dim V_2$. Permuting the suffixes of $x_{n_1+1}, \dots, x_n$, we may assume that the image of $x_{n_1+1}, \dots, x_{n_1+n_2}$ in the quotient ring R_2 forms a basis of V_2. Then consider the quotient graded ring $R_3 = R_2/(V_2)$ and repeat the same argument, and so on. This procedure will terminate at a finite steps, say q steps, so that we obtain a sequence of natural numbers $(n_1, \dots, n_q)$, which is an invariant of the cohomology ring. We call this sequence the *type* of A or $H^*(M(A); \mathbb{Z}/2)$. The argument above shows that through a suitable permutation of suffixes of $x_1, \dots, x_n$ we may assume that the upper triangular matrix A decomposed into $q \times q$ blocks according to the type $(n_1, \dots, n_q)$ has zero matrices of sizes $n_1, \dots, n_q$ as the diagonal q blocks, i.e.

$$A = \begin{pmatrix} O_{n_1} & & & * \\ & O_{n_2} & & \\ & & \ddots & \\ 0 & & & O_{n_q} \end{pmatrix} \tag{4.2}$$

where O_m denotes the zero matrix of size m and every column vector in $(i, i+1)$-block is non-zero for $i = 1, \dots, q-1$. We note that permuting suffixes of $x_1, \dots, x_n$ corresponds to conjugating the matrix A by a permutation matrix.

Let B be an upper triangular matrix of the same type and same form as (4.2) and let

$$\varphi\colon H^*(M(A); \mathbb{Z}/2) \to H^*(M(B); \mathbb{Z}/2)$$

be an isomorphism as graded rings. We denote by $y_1, \dots, y_n$ the generators of $H^*(M(B); \mathbb{Z}/2)$. Since $\varphi(x_i)^2 = \varphi(x_i{}^2) = 0$ for $1 \le i \le n_1$, $\varphi(x_i)$ is a linear combination of $y_1, \dots, y_{n_1}$. In general, one easily sees that $\varphi(x_i)$ for $n_{j-1} + 1 \le i \le n_j$ is a linear combination of $y_1, \dots, y_{n_j}$. This means that if we view $P \in \mathrm{GL}(n; \mathbb{Z}/2)$ defined by

$$(\varphi(x_1), \dots, \varphi(x_n)) = (y_1, \dots, y_n)P \tag{4.3}$$

as a $q \times q$ block matrix of type $(n_1, \dots, n_q)$, then P is an upper triangular block matrix. Since P is non-singular, every diagonal block of P is also non-singular. Therefore, we may assume that the diagonal entries of P are all one if necessary by permuting the suffixes of the generators y_i's in each block. With this understood, we have

Lemma 4.1. *$B = PA$ and*

$$P^\ell_j B^i_\ell = P^i_j B^\ell_j + P^\ell_j B^i_j + P^\ell_j B^\ell_j B^i_\ell \quad \textit{for } i < \ell.$$

Proof. It follows from (4.3) that

$$\varphi(x_k) = \sum_{i=1}^n P^i_k y_i \quad \text{for} \quad k = 1, \dots, n. \tag{4.4}$$

We plug this in (4.1) mapped by φ to obtain

$$\begin{aligned} \Big(\sum_{i=1}^n P^i_j y_i\Big)^2 &= \Big(\sum_{i=1}^n P^i_j y_i\Big)\Big(\sum_{k=1}^n \sum_{i=1}^n A^k_j P^i_k y_i\Big) \\ &= \Big(\sum_{i=1}^n P^i_j y_i\Big)\Big(\sum_{i=1}^n (PA)^i_j y_i\Big) \end{aligned} \tag{4.5}$$

Comparing the coefficients of y_iy_j for $i<j$ at both sides above and noting that $P^j_j=1$ and $(PA)^j_j=0$, we obtain

$$B^i_j=(PA)^i_j \quad \text{for } i<j.$$

(Note that the term y_iy_j may appear in ${y_j}^2$ but not in ${y_i}^2$ because B is assumed to be upper triangular.) The identity above holds even for $i\ge j$ because the both sides then vanish. Therefore $B=PA$.

More generally, comparing the coefficients of y_iy_ℓ for $i<\ell$ at the both sides of (4.5) and replacing PA by B, we obtain the latter identity in the lemma. □

5. A monomorphism between fundamental groups

Let A,B and P be as in Section 4. In this section we construct a monomorphism between the fundamental groups $\Gamma(B)$ and $\Gamma(A)$ using P.

Any element $s\in\Gamma(A)$ can be expressed uniquely as $s=s_1^{a_1}s_2^{a_2}\dots s_n^{a_n}$ with integers a_i's by Lemma 3.2. We denote the exponent a_j of s_j by $\mathcal{E}_j(s)$.

Lemma 5.1. *If $p_i,q_i\in\mathbb{Z}$ for $i=1,\dots,n$, then*

$$\mathcal{E}_j((s_1^{p_1}s_2^{p_2}\dots s_n^{p_n})(s_1^{q_1}s_2^{q_2}\dots s_n^{q_n}))=(-1)^{\sum_{k=1}^{j-1}q_kB^k_j}p_j+q_j$$
$$\mathcal{E}_j((s_1^{p_1}s_2^{p_2}\dots s_n^{p_n})(s_n^{-q_n}\dots s_2^{-q_2}s_1^{-q_1}))=(-1)^{\sum_{k=1}^{j-1}q_kB^k_j}(p_j-q_j)$$

Proof. Using Lemma 3.2, we see

$$s_\ell^p s_k^q=s_k^q s_\ell^{p(-1)^{qB^k_\ell}} \quad \text{for } \ell>k, \text{ and } p,q\in\mathbb{Z} \tag{5.1}$$

and the repeated use of this identity implies the lemma. □

We use notation t_i's for $\Gamma(B)$ in place of s_i's for $\Gamma(A)$. We regard P as an *integer* matrix and define

$$\rho(t_r)=s_1^{P^r_1}s_2^{P^r_2}\dots s_n^{P^r_n} \quad (r=1,\dots,n). \tag{5.2}$$

We shall check that ρ preserves the relations in Lemma 3.2 for $\Gamma(B)$ so that ρ induces a homomorphism from $\Gamma(B)$ to $\Gamma(A)$ by Lemma 3.3. It follows from Lemma 5.1 that

$$\mathcal{E}_j(\rho(t_\ell t_i))=(-1)^{\sum_{k=1}^{j-1}P^i_kA^k_j}P^\ell_j+P^i_j=(-1)^{B^i_j}P^\ell_j+P^i_j\in\mathbb{Z} \tag{5.3}$$

where we used the fact $PA=B$ and $A^k_j=0$ for $k\ge j$ in the latter identity. Similarly we have

$$\mathcal{E}_j(\rho(t_it_\ell))=(-1)^{B^\ell_j}P^i_j+P^\ell_j\in\mathbb{Z} \tag{5.4}$$

and

$$\mathcal{E}_j(\rho(t_i t_\ell^{-1})) = (-1)^{B_j^\ell}(P_j^i - P_j^\ell) \in \mathbb{Z}. \tag{5.5}$$

Now suppose $i < \ell$. When $B_\ell^i = 0$, we have $t_\ell t_i = t_i t_\ell$ by Lemma 3.2 for $\Gamma(B)$ and

$$P_j^\ell B_j^i = P_j^i B_j^\ell \in \mathbb{Z}/2$$

by Lemma 4.1. An elementary case-by-case check (according to the values of B_j^i and B_j^ℓ) shows that the identity above ensures that the right hand sides at (5.3) and (5.4) coincide. When $B_\ell^i = 1$, we have $t_\ell t_i = t_i t_\ell^{-1}$ by Lemma 3.2 for $\Gamma(B)$ and

$$P_j^\ell = P_j^i B_j^\ell + P_j^\ell B_j^i + P_j^\ell B_j^\ell \in \mathbb{Z}/2 \quad \text{for } i < \ell$$

by Lemma 4.1. A similar elementary case-by-case check shows that the identity above ensures that the right hand sides at (5.3) and (5.5) coincide. In any case the map ρ preserves the relations for $\Gamma(B)$ and $\Gamma(A)$ and hence induces a homomorphism from $\Gamma(B)$ to $\Gamma(A)$.

Lemma 5.2. *The homomorphism $\rho\colon \Gamma(B) \to \Gamma(A)$ is injective and*

1. *$\rho(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and $\mathbb{Z}^n/\rho(\mathbb{Z}^n)$ is of order $\det P$ (which is odd),*
2. *ρ induces an isomorphism from $\Gamma(B)/\mathbb{Z}^n$ onto $\Gamma(A)/\mathbb{Z}^n$.*

Therefore ρ is an isomorphism if and only if $\det P = \pm 1$.

Proof. It follows from (5.3) with $\ell = i$ that

$$\mathcal{E}_j(\rho(t_i^2)) = \begin{cases} 2P_j^i & \text{if } B_j^i = 0, \\ 0 & \text{if } B_j^i = 1. \end{cases}$$

Therefore ρ maps the normal subgroup $\mathbb{Z}^n$ of $\Gamma(B)$ to that of $\Gamma(A)$, so that ρ maps the short exact sequence (3.2) for $\Gamma(B)$ to that for $\Gamma(A)$. The above fact also shows that the map ρ restricted to $\mathbb{Z}^n$ agrees with P for i, j with $B_j^i = 0$, in particular, if we view the restricted map as a block matrix as before, then it is an upper triangular block matrix and the diagonal blocks agree with those of P. Therefore the determinant of the restricted map is equal to $\det P$. This proves (1).

On the other hand, it follows from the definition (5.2) of ρ that the map induced from ρ on $\Gamma(B)/\mathbb{Z}^n = (\mathbb{Z}/2)^n$ is nothing but P, so it is an isomorphism, proving (2). These imply that ρ is always injective and an isomorphism if and only if $\det P = \pm 1$. □

6. Group extension

A square $(0,1)$-matrix of size m is in $\mathrm{GL}(m;\mathbb{Z})$ if and only if it is in $\mathrm{GL}(m;\mathbb{Z}/2)$ when $m \le 3$. Therefore, if $n_i \le 3$ for all i, where $(n_1, \dots, n_q)$ is the type of A and B, then $\det P = \pm 1$ and ρ in Lemma 5.2

is an isomorphism. In general ρ may not be an isomorphism, but we prove the following using the existence of ρ.

Lemma 6.1. *If $H^*(M(A);\mathbb{Z}/2)$ is isomorphic to $H^*(M(B);\mathbb{Z}/2)$ as graded rings, then $\Gamma(A)$ is isomorphic to $\Gamma(B)$.*

We admit the lemma above for the moment and complete the proof of Theorem 1.1 in the Introduction.

Proof of Theorem 1.1. Real Bott manifolds are compact riemannian flat manifolds by Lemma 3.1, hence by a theorem of Bieberbach they are diffeomorphic if and only if their fundamental groups are isomorphic (see [20], Theorem 3.3.1 in p.105). Therefore, Theorem 1.1 follows from Lemma 6.1. □

The rest of this section is devoted to the proof of Lemma 6.1. Remember the group extension (3.2)

$$0\to\mathbb{Z}^n\to\Gamma(A)\to(\mathbb{Z}_2)^n\to 1.$$

Conjugation action of $\Gamma(A)$ on $\mathbb{Z}^n$ induces a homomorphism

$$\phi_A : (\mathbb{Z}_2)^n\to \mathrm{Aut}(\mathbb{Z}^n).$$

We remark that the $(\mathbb{Z}_2)^n$-module $\mathbb{Z}^n$ via ϕ_A decomposes into sum of rank one $(\mathbb{Z}_2)^n$-modules, which follows from Lemma 3.2. There is a 2-cocycle

$$f_A : (\mathbb{Z}_2)^n \times (\mathbb{Z}_2)^n\to\mathbb{Z}^n$$

whose cohomology class $[f_A] \in H^2_{\phi_A}((\mathbb{Z}_2)^n;\mathbb{Z}^n)$ represents the above group extension, that is, $\Gamma(A)$ is the product $\mathbb{Z}^n \times (\mathbb{Z}_2)^n$ with group law:

$$(\ell,\alpha)(m,\beta) = (\ell + \phi_A(\alpha)(m) + f_A(\alpha,\beta), \alpha\beta). \tag{6.1}$$

Similarly we have ϕ_B and f_B for the group $\Gamma(B)$.

Lemma 5.2 shows that there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \Gamma(B) & \longrightarrow & (\mathbb{Z}_2)^n & \longrightarrow & 1 \\ & & {\scriptstyle\rho}\downarrow & & {\scriptstyle\rho}\downarrow & & {\scriptstyle\bar\rho}\downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \Gamma(A) & \longrightarrow & (\mathbb{Z}_2)^n & \longrightarrow & 1. \end{array}$$

where $\bar\rho$ is an isomorphism. We write

$$\rho(0,\alpha) = (\lambda(\alpha), \bar\rho(\alpha)).$$

Then, for $(\ell,\alpha)\in\Gamma(B)$ we have

$$\begin{aligned}\rho(\ell,\alpha) &= \rho((\ell,1)(0,\alpha)) = \rho(\ell,1)\rho(0,\alpha)\\ &= (\rho(\ell),1)(\lambda(\alpha),\bar\rho(\alpha))\\ &= (\rho(\ell)+\lambda(\alpha),\bar\rho(\alpha)).\end{aligned}\tag{6.2}$$

Therefore, applying ρ to the both sides of the identity $(0,\alpha)(0,\beta) = (f_B(\alpha,\beta),\alpha\beta)$, we have

$$\begin{aligned}\rho((0,\alpha)(0,\beta)) &= (\lambda(\alpha),\bar\rho(\alpha))(\lambda(\beta),\bar\rho(\beta))\\ &= (\lambda(\alpha)+\phi_A(\bar\rho(\alpha))(\lambda(\beta))+f_A(\bar\rho(\alpha),\bar\rho(\beta)),\bar\rho(\alpha\beta)),\end{aligned}$$

while we have

$$\rho(f_B(\alpha,\beta),\alpha\beta) = (\rho(f_B(\alpha,\beta))+\lambda(\alpha\beta),\bar\rho(\alpha\beta))$$

by (6.2). It follows that

$$\rho(f_B(\alpha,\beta)) = \lambda(\alpha)+\phi_A(\bar\rho(\alpha))(\lambda(\beta))-\lambda(\alpha\beta)+\bar\rho^* f_A(\alpha,\beta).\tag{6.3}$$

Similarly, applying ρ to the both sides of the identity $(0,\alpha)(\ell,1) = (\phi_B(\alpha)(\ell),\alpha)$, we have

$$\begin{aligned}\rho((0,\alpha)(\ell,1)) &= (\lambda(\alpha),\bar\rho(\alpha))(\rho(\ell),1)\\ &= (\lambda(\alpha)+\phi_A(\bar\rho(\alpha))(\rho(\ell)),\bar\rho(\alpha)),\end{aligned}$$

while

$$\rho(\phi_B(\alpha)(\ell),\alpha) = (\rho(\phi_B(\alpha)(\ell))+\lambda(\alpha),\bar\rho(\alpha)).$$

It follows that

$$\rho(\phi_B(\alpha)(\ell)) = \phi_A(\bar\rho(\alpha))(\rho(\ell)).$$

We regard elements in $\mathbb{Z}^n$ as column vectors and represent the homomorphism $\rho:\mathbb{Z}^n\to\mathbb{Z}^n$ by an integral matrix Q. Then the identity above is equivalent to

$$Q\cdot\phi_B(\alpha) = \phi_A(\bar\rho(\alpha))\cdot Q.$$

We note that $\tilde Q = (\det Q)Q^{-1}$ is an integral matrix, where $\det Q$ that is the order of $\mathbb{Z}^n/\rho(\mathbb{Z}^n)$ is odd by Lemma 5.2. It follows from the identity above that

$$\phi_B(\alpha)\cdot\tilde Q = \tilde Q\cdot\phi_A(\bar\rho(\alpha)).\tag{6.4}$$

Applying $\tilde Q$ to the both sides of (6.3), we have

$$\begin{aligned}&\tilde Q Q f_B(\alpha,\beta)\\ =&\tilde Q\lambda(\alpha)+\tilde Q\phi_A(\bar\rho(\alpha))(\lambda(\beta))-\tilde Q\lambda(\alpha\beta)+\tilde Q\bar\rho^* f_A(\alpha,\beta)\\ =&\tilde Q\lambda(\alpha)+\phi_B(\alpha)(\tilde Q\lambda(\beta))-\tilde Q\lambda(\alpha\beta)+\tilde Q\bar\rho^* f_A(\alpha,\beta)\\ =&\delta_B(\tilde Q\lambda)(\alpha,\beta)+\tilde Q\bar\rho^* f_A(\alpha,\beta)\end{aligned}$$

where we used (6.4) at the second identity and the definition of coboundary δ_B at the last identity. Since $\tilde{Q}Q$ is $\det Q$ times the identity matrix, the identity above implies that

$$[\det Q \cdot f_B] = [\tilde{Q}\bar{\rho}^* f_A] \in H^2_{\phi_B}((\mathbb{Z}_2)^n, \tilde{Q}\mathbb{Z}^n). \tag{6.5}$$

Here $\tilde{Q}\mathbb{Z}^n$ is viewed as a $(\mathbb{Z}_2)^n$-module via ϕ_B. It decomposes into the direct sum of rank one $(\mathbb{Z}_2)$-modules because we have (6.4) and the $(\mathbb{Z}_2)^n$-module $\mathbb{Z}^n$ via ϕ_A decomposes into the direct sum of rank one $(\mathbb{Z}_2)$-modules. Therefore

$$H^2_{\phi_B}((\mathbb{Z}_2)^n, \tilde{Q}\mathbb{Z}^n) \cong \bigoplus_{i=1}^n H^2_{\phi_i}((\mathbb{Z}_2)^n; \mathbb{Z}) \tag{6.6}$$

where $\phi_i\colon (\mathbb{Z}_2)^n \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$ is a homomorphism.

Fact. $H^2_{\phi}((\mathbb{Z}_2)^n; \mathbb{Z})$ is an elementary abelian 2-group for any homomorphism $\phi\colon (\mathbb{Z}_2)^n \to \mathrm{Aut}(\mathbb{Z})$.

(This fact is probably known but since we do not know the literature, we will give a proof in the Appendix.) It follows from (6.6) and the fact above that $H^2_{\phi_B}((\mathbb{Z}_2)^n, \tilde{Q}\mathbb{Z}^n)$ is an elementary abelian 2-group. Since $\det Q$ is odd as remarked before, the identity (6.5) implies that

$$[f_B] = [\tilde{Q}\bar{\rho}^* f_A] \in H^2_{\phi_B}((\mathbb{Z}_2)^n, \tilde{Q}\mathbb{Z}^n). \tag{6.7}$$

The group Γ corresponding to the cocycle $\tilde{Q}\bar{\rho}^* f_A$ is the product $\tilde{Q}\mathbb{Z}^n \times (\mathbb{Z}_2)^n$ with group law:

$$\begin{aligned} &(\tilde{Q}\ell, \alpha)(\tilde{Q}m, \beta) \\ =&(\tilde{Q}\ell + \phi_B(\alpha)(\tilde{Q}m) + \tilde{Q}f_A(\bar{\rho}(\alpha), \bar{\rho}(\beta)), \alpha\beta) \end{aligned} \tag{6.8}$$

in which we note that $\tilde{Q}\ell + \phi_B(\alpha)(\tilde{Q}m) + \tilde{Q}f_A(\bar{\rho}(\alpha), \bar{\rho}(\beta)) \in \tilde{Q}\mathbb{Z}^n$. In fact, using (6.4)

$$\begin{aligned} &\tilde{Q}\ell + \phi_B(\alpha)(\tilde{Q}m) + \tilde{Q}f_A(\bar{\rho}(\alpha), \bar{\rho}(\beta)) \\ =&\tilde{Q}\ell + \tilde{Q}\phi_A(\bar{\rho}(\alpha))(m) + \tilde{Q}f_A(\bar{\rho}(\alpha), \bar{\rho}(\beta)) \\ =&\tilde{Q}(\ell + \phi_A(\bar{\rho}(\alpha))(m) + f_A(\bar{\rho}(\alpha), \bar{\rho}(\beta))). \end{aligned} \tag{6.9}$$

Since $\Gamma(B)$ is isomorphic to Γ by (6.7), it suffices to prove that Γ is isomorphic to $\Gamma(A)$.

Define a map $\mathcal{T} : \Gamma(A) \to \Gamma$ by

$$\mathcal{T}(\ell, \alpha) = (\tilde{Q}\ell, \bar{\rho}^{-1}(\alpha)).$$

This is clearly a bijection. Using (6.8), (6.9) and (6.1), we have

$$\begin{aligned}
&\mathcal{T}(\ell,\alpha)\mathcal{T}(m,\beta)\\
=&(\tilde{Q}\ell,\bar{\rho}^{-1}(\alpha))(\tilde{Q}m,\bar{\rho}^{-1}(\beta))\\
=&(\tilde{Q}(\ell+\phi_A(\alpha)(m)+f_A(\alpha,\beta)),\bar{\rho}^{-1}(\alpha\beta))\\
=&\mathcal{T}((\ell+\phi_A(\alpha)(m)+f_A(\alpha,\beta),\alpha\beta))\\
=&\mathcal{T}((\ell,\alpha)(m,\beta)).
\end{aligned}$$

Hence $\mathcal{T}$ is an isomorphism of $\Gamma(A)$ onto Γ. This completes the proof of Lemma 6.1.

Remark. The above proof of Lemma 6.1 actually proves that any subgroup of $\Gamma(A)$ with odd index is isomorphic to $\Gamma(A)$.

7. Classification of real Bott manifolds of low dimension

Real Bott manifolds are determined by upper triangular square $(0,1)$-matrices with zero diagonal entries and the diffeomorphism classification of real Bott manifolds reduces to the isomorphism classification of associated cohomology rings with $\mathbb{Z}/2$ coefficients by our main Theorem 1.1. As observed in Section 4, we may assume that our matrices are of the form (4.2) which we call a *normal* form. Therefore, it suffices to check which matrices of normal form produce isomorphic cohomology rings and this can be done by an elementary computation when the size n of matrices, that is the dimension of real Bott manifolds, is up to 4. We remember that permuting the suffixes of the cohomology generators $x_1,\dots,x_n$ in Section 4 corresponds to conjugating our matrices by a permutation matrix. So the cohomology rings associated with conjugate matrices by permutation matrices are isomorphic. This decreases necessary computations. Below are the results. The same results are obtained in [15] independently but the method is different from ours.

The case $n=2$. Real Bott manifolds of dimension 2 are the torus $(S^1)^2$ or the Klein bottle and the corresponding matrices of normal form are respectively the zero matrix of size 2 and $\begin{pmatrix}0&1\\0&0\end{pmatrix}$.

The case $n=3$. There are four diffeomorphism classes in real Bott manifolds of dimension 3 and corresponding matrices of normal form are distinguished by their types as seen below. The number of an item below with $\star$ shows that the corresponding real Bott manifold is orientable (see Lemma 2.2).

$1^\star$. Type (3)

The zero matrix of size 3 and the real Bott manifold is $(S^1)^3$.

2. Type $(2,1)$

$$\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&1\\0&0&1\\0&0&0\end{pmatrix}$$

The real Bott manifold is $S^1 \times$ (Klein bottle).

3⋆. Type $(1,2)$

$$\begin{pmatrix}0&1&1\\0&0&0\\0&0&0\end{pmatrix}$$

4. Type $(1,1,1)$

$$\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\quad\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}$$

Remark. Compact riemannian flat manifolds of dimension 3 are classified. There are ten diffeomorphism classes and six of them are orientable ([20, p.117 and p.120]). One can easily check that the real Bott manifolds in 1⋆ and 3⋆ above are respectively of types $\mathcal{G}_1$ and $\mathcal{G}_2$ in [20, Theorem 3.5.5] and those in 2 and 4 above are respectively of types $\mathcal{B}_1$ and $\mathcal{B}_3$ in [20, Theorem 3.5.9].

The case $n = 4$. There are twelve diffeomorphism classes in real Bott manifolds of dimension 4 and corresponding matrices of normal form are as described below. We list representatives of conjugacy classes in matrices of normal form by permutation matrices. The suffix of a matrix below denotes the number of elements in the conjugacy class represented by the matrix. The number of an item below with ⋆ shows that the corresponding real Bott manifold is orientable as before.

1⋆. Type (4)

The zero matrix of size 4 and the real Bott manifold is $(S^1)^4$.

2. Type $(3,1)$

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}_3\quad\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&1\\0&0&0&0\end{pmatrix}_3\quad\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&0\end{pmatrix}_1$$

The real Bott manifold is $(S^1)^2 \times$ (Klein bottle).

3⋆. Type $(2,2)$

$$\begin{pmatrix}0&0&0&0\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}_2\quad\begin{pmatrix}0&0&1&1\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}_1$$

The real Bott manifold is the product of S^1 and the 3-dimensional real Bott manifold of Type $(1,2)$.

4. Type $(2,2)$

$$\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}_2 \quad \begin{pmatrix}0&0&1&0\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}_4$$

The real Bott manifold is (Klein bottle)×(Klein bottle).

5. Type $(2,1,1)$

$$\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}_2 \quad \begin{pmatrix}0&0&0&0\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}_2 \quad \begin{pmatrix}0&0&1&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}_1 \quad \begin{pmatrix}0&0&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}_1$$

The real Bott manifold is the product of S^1 and the 3-dimensional real Bott manifold of Type $(1,1,1)$.

6. Type $(2,1,1)$

$$\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}_2 \quad \begin{pmatrix}0&0&1&0\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}_2 \quad \begin{pmatrix}0&0&0&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}_2$$

7. Type $(1,3)$

$$\begin{pmatrix}0&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}_1$$

8. Type $(1,2,1)$

$$\begin{pmatrix}0&1&1&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}_2 \quad \begin{pmatrix}0&1&1&0\\0&0&0&1\\0&0&0&1\\0&0&0&0\end{pmatrix}_1 \quad \begin{pmatrix}0&1&1&1\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}_2 \quad \begin{pmatrix}0&1&1&1\\0&0&0&1\\0&0&0&1\\0&0&0&0\end{pmatrix}_1$$

$9^\star$. Type $(1,1,2)$

$$\begin{pmatrix}0&1&1&0\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}_2$$

10. Type $(1,1,2)$

$$\begin{pmatrix}0&1&0&0\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}_1 \quad \begin{pmatrix}0&1&1&1\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}_1$$

11. Type $(1,1,1,1)$

$$\begin{pmatrix}0&1&0&1\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}_1 \quad \begin{pmatrix}0&1&1&0\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}_1 \quad \begin{pmatrix}0&1&1&1\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}_1$$

12. Type $(1,1,1,1)$

$$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}_1 \quad \begin{pmatrix}0&1&1&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}_1 \quad \begin{pmatrix}0&1&0&0\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}_1 \quad \begin{pmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}_1$$

8. Riemannian flat real toric manifolds

A toric manifold X of complex dimension n supports an action of $(\mathbb{C}^*)^n$ and its real part $X(\mathbb{R})$ supports an action of $(\mathbb{R}^*)^n$, where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. Let T be the maximal compact toral subgroup of $(\mathbb{C}^*)^n$ and $T^n(\mathbb{R})$ be the maximal elementary abelian 2-subgroup of $(\mathbb{R}^*)^n$. The orbit space $X(\mathbb{R})/T^n(\mathbb{R})$ can naturally be identified with X/T. When X is projective, the orbit space can be identified with a simple n-polytope via a moment map.

The action of $T^n(\mathbb{R})$ on the real Bott manifold $M(A) = \mathbb{R}^n/\Gamma(A)$ is given as follows. Let r_j $(j = 1, \dots, n)$ be an involution on $\mathbb{R}^n$ defined by

$$r_j(x^1, \dots, x^n) = (x^1, \dots, x^{j-1}, -x^j, x^{j+1}, \dots, x^n).$$

As easily checked

$$r_j s_i = \begin{cases} s_i r_j & \text{if } i \neq j, \\ s_i^{-1} r_j & \text{if } i = j, \end{cases}$$

where s_i is the Euclidean motion on $\mathbb{R}^n$ defined in (3.1), so r_j induces an involution $\bar{r}_j$ on $M(A) = \mathbb{R}^n/\Gamma(A)$. Obviously $\bar{r}_j$'s commute with each other so that they generate an elementary abelian 2-group of rank n and this gives the action of $T^n(\mathbb{R})$.

We remark that the action of $T^n(\mathbb{R})$ on $M(A) = \mathbb{R}^n/\Gamma(A)$ preserves the flat riemannian metric on it. The group generated by s_i's and r_j's agrees with the group generated by r_j's and translations by $\frac{1}{2}e^1, \dots, \frac{1}{2}e^n$ where $e^1, \dots, e^n$ are the standard basis of $\mathbb{R}^n$ as before. It follows that the orbit space $M(A)/T^n(\mathbb{R})$ can be identified with an n-cube

$$\{(x^1, \dots, x^n) \in \mathbb{R}^n \mid 0 \le x^1 \le 1/2, \dots, 0 \le x^n \le 1/2\}.$$

The purpose of this section is to prove Theorem 1.2 in the Introduction, that is

Theorem 8.1. *A real toric manifold of dimension n which admits a flat riemannian metric invariant under the action of $T^n(\mathbb{R})$ is a real Bott manifold.*

We recall some results for the proof of the theorem above. Let X be a toric manifold and let X_i $(1 \le i \le m)$ be a connected complex codimension-one closed submanifold of X fixed pointwise under some circle subgroup T_i of the torus T. We call X_i a *characteristic submanifold* of X. Then

$$K_X := \{I \subset \{1, \dots, m\} \mid \cap_{i \in I} X_i \neq \emptyset\}$$

is the underlying abstract simplicial complex of the fan of X.

Let $X(\mathbb{R})$ be the real part of X. The intersection $X_i \cap X(\mathbb{R})$ is a connected real codimension-one closed submanifold of $X(\mathbb{R})$ fixed pointwise under the order two subgroup $T_i \cap T^n(\mathbb{R})$ of $T^n(\mathbb{R})$. Conversely any connected real codimension-one closed submanifold of $X(\mathbb{R})$ fixed pointwise under an order two subgroup of $T^n(\mathbb{R})$ is the intersection of $X(\mathbb{R})$ with some X_i. We call those closed submanifolds *characteristic submanifolds* of $X(\mathbb{R})$ as well. This observation says that there is a bijective correspondence between characteristic submanifolds of X and those of $X(\mathbb{R})$. Hence one can also define K_X using the characteristic submanifolds of $X(\mathbb{R})$.

We say that a simplicial complex is a *crosscomplex* of dimension $n-1$ if it is the boundary complex of a crosspolytope of dimension n, where a crosspolytope of dimension n is the dual (or polar) of an n-cube. We recall two facts from [13]. The first lemma below is stated in [13, Corollary 3.5] in the complex case but it also holds in the real case as stated because of the observation above.

Lemma 8.2 (Corollary 3.5 in [13])**.** *A real toric manifold $X(\mathbb{R})$ is a real Bott manifold if and only if the simplicial complex K_X associated with $X(\mathbb{R})$ is a crosscomplex.*

Lemma 8.3 (Lemma 4.7 in [13])**.** *Let K be a connected simplicial complex of dimension $k \geq 2$. If the link of each vertex of K is a crosscomplex of dimension $k-1$, then K is a crosscomplex.*

Proof of Theorem 8.1. We shall prove the theorem by induction on the dimension n. The theorem is obvious when $n = 1$. A closed surface which admits a flat riemannian metric is a torus or a Klein bottle and they are real Bott manifolds, so the theorem also holds when $n = 2$.

Now suppose the theorem holds for $n-1 \geq 2$ and let $X(\mathbb{R})$ be a real toric manifold of dimension n which satisfies the assumption in the theorem. Let $X(\mathbb{R})_1, \dots, X(\mathbb{R})_m$ be the characteristic submanifolds of $X(\mathbb{R})$. A vertex of the simplicial complex K_X associated with $X(\mathbb{R})$ corresponds to some $X(\mathbb{R})_i$ and the link of the vertex is the simplicial complex associated with $X(\mathbb{R})_i$. Since $X(\mathbb{R})$ admits a riemannian flat metric invariant under the action of $T^n(\mathbb{R})$, each $X(\mathbb{R})_i$ is again a riemannian flat manifold because it is fixed pointwise under a subgroup of $T^n(\mathbb{R})$. Therefore $X(\mathbb{R})_i$ is a real Bott manifold by the inductive assumption and hence the link of the vertex of K_X is crosscomplex by

Lemma 8.2. Since $\dim K_X = n - 1 \geq 2$, K_X is a crosscomplex by Lemma 8.3 and hence X is a real Bott manifold by Lemma 8.2. This completes the induction step and the proof of the theorem. □

9. Small cover

Let $T^n(\mathbb{R})$ be an elementary abelian 2-group of rank n as before. A closed smooth manifold M of dimension n with a smooth action of $T^n(\mathbb{R})$ is called locally standard if each point of M has an invariant open neighborhood equivariantly diffeomorphic to an invariant open subset of a faithful real $T^n(\mathbb{R})$-module of dimension n. The orbit space of a locally standard $T^n(\mathbb{R})$-manifold M is a manifold with corners because the orbit space of a faithful $T^n(\mathbb{R})$-module of dimension n is homeomorphic to the product of n half lines. A convex polytope of dimension n is called simple if there are exactly n edges meeting at each vertex, and a simplex convex polytope is a typical example of a manifold with corners. If M is locally standard and the orbit space is identified with a simple convex polytope P, then M is called a *small cover* over P ([5]).

A real toric manifold $X(\mathbb{R})$ with the natural $T^n(\mathbb{R})$-action is locally standard and its orbit space is often a simple convex polytope. In fact, this is the case when X is projective, so a real toric manifold $X(\mathbb{R})$ is a small cover when X is projective. However there are many small covers which do not arise this way. For example, every closed surface becomes a small cover but only the torus $S^1 \times S^1$ is a real toric manifold among orientable closed surfaces ([17]). We may think of small covers as a topological counterpart to real toric manifolds and may ask the same question as in the Introduction for small covers. We remark that equivariant homeomorphism types of small covers can be distinguished by their equivariant cohomology algebras with $\mathbb{Z}/2$ coefficients ([12]).

When $X(\mathbb{R})$ is a real Bott manifold, the orbit space is an n-cube as observed in Section 8; so a real Bott manifold of dimension n becomes a small cover over an n-cube and the converse is known to be true up to homeomorphism.

Theorem 9.1 ([13], [3])**.** *A small cover over an n-cube is homeomorphic to a real Bott manifold of dimension n.*

The number Q_n of equivariant homeomorphism classes in small covers over an n-cube is computed in [2] for any n, e.g.

$$Q_1 = 1,\ Q_2 = 6,\ Q_3 = 259,\ Q_4 = 87360,\ Q_5 = 236240088, \dots .$$

However, the number H_n of (non-equivariant) homeomorphism classes in small covers over an n-cube is unknown although

$$H_1 = 1,\ H_2 = 2,\ H_3 = 4,\ H_4 = 12$$

as described in Section 7.

As is well-known, regular simple polytopes of dimension $n \geq 3$ are an n-cube and an n-simplex in each dimension n, the dodecahedron in dimension 3 and the 120-cell in dimension 4. The homeomorphism type of a small cover over an n-simplex is unique, that is the real projective space of dimension n. Small covers over the dodecahedron and the 120-cell admit hyperbolic metrics and are studied in [7] from this point of view. In particular, it is proved in the paper that there are exactly 25 small covers over the dodecahedron up to isometry (equivalently up to homeomorphism by Mostow rigidity).

Appendix

In this appendix we give a proof of the Fact used in Section 6. In fact we will prove a more precise statement. It is well-known that $H^2_\phi((\mathbb{Z}_2)^n;\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/2)^n$ when ϕ is trivial. We prove

Theorem. *If ϕ is non-trivial, $H^2_\phi((\mathbb{Z}_2)^n;\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/2)^{n-1}$.*

We recall the following Hochschild-Serre spectral sequence, see [11, p.355] or [9].

Proposition. *Let $1 \to \Gamma \to \Pi \to \Pi/\Gamma \to 1$ be a group extension and let A be a Π-module through a homomorphism $\phi : \Pi \to \mathrm{Aut}(A)$. Suppose $m \geq 1$ and $H^q_\phi(\Gamma, A) = 0$ for $1 < q < m$. For $0 < q < m$, there is the exact sequence*

$$\begin{aligned} &0 \to H^1_\phi(\Pi/\Gamma, A^\Gamma) \to H^1_\phi(\Pi, A) \to H^0_\phi(\Pi/\Gamma, H^1_\phi(\Gamma, A)) \to \\ &\cdots \to H^q_\phi(\Pi/\Gamma, A^\Gamma) \to H^q_\phi(\Pi, A) \to H^{q-1}_\phi(\Pi/\Gamma, H^1_\phi(\Gamma, A)) \\ &\to H^{q+1}_\phi(\Pi/\Gamma, A^\Gamma) \to H^{q+1}_\phi(\Pi, A) \to \cdots \end{aligned} \tag{A.1}$$

We take $\Pi = (\mathbb{Z}_2)^n$ $(n \geq 2)$ and $A = \mathbb{Z}$ as a Π-module through $\phi : \Pi \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$. Choose an order two subgroup $\Gamma \subset (\mathbb{Z}_2)^n$ such that $\phi(\Gamma) = \{\pm 1\}$. Clearly

$$\Pi = \Gamma \times \mathrm{Ker}\,\phi.$$

It is known and easy to check that $H^2_\phi(\Gamma, A) = 0$, so the assumption in the proposition above is satisfied for $m = 3$. As $A^\Gamma = 0$ by our condition, $H^r_\phi(\Pi/\Gamma, A^\Gamma) = 0$ for any $r \geq 0$. Then the exact sequence (A.1) becomes

$$0 \to H^2_\phi(\Pi, A) \to H^1_\phi(\Pi/\Gamma, H^1_\phi(\Gamma, A)) \to 0. \tag{A.2}$$

On the other hand, it is also known and easy to check that $H^1_\phi(\Gamma, A) \cong \mathbb{Z}/2$, so the action of Π/Γ on $H^1_\phi(\Gamma, A)$ must be trivial. It follows that

$$\begin{aligned} H^1_\phi(\Pi/\Gamma, H^1_\phi(\Gamma, A)) &\cong H^1(\Pi/\Gamma, \mathbb{Z}/2) \\ &\cong H^1((\mathbb{Z}_2)^{n-1}, \mathbb{Z}/2) \\ &\cong (\mathbb{Z}/2)^{n-1}. \end{aligned}$$

This together with (A.2) implies the theorem.

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