# COHOMOLOGICAL NON-RIGIDITY OF GENERALIZED REAL BOTT MANIFOLDS OF HEIGHT 2

MIKIYA MASUDA

Abstract. We investigate when two generalized real Bott manifolds of height 2 have isomorphic cohomology rings with $\mathbb{Z}/2$ coefficients and also when they are diffeomorphic. It turns out that cohomology rings with $\mathbb{Z}/2$ coefficients do not distinguish those manifolds up to diffeomorphism in general. This gives a counterexample to the cohomological rigidity problem for real toric manifolds posed in [5]. We also prove that generalized real Bott manifolds of height 2 are diffeomorphic if they are homotopy equivalent.

## 1. Introduction

A toric manifold is a compact smooth toric variety and a real toric manifold is the set of real points of a toric manifold. In [7] we asked whether toric manifolds are diffeomorphic if their cohomology rings with $\mathbb{Z}$ coefficients are isomorphic as graded rings, which is now called *cohomological rigidity problem for toric manifolds*. No counterexample and some partial affirmative solutions are known to the problem (see [3], [7]). If $X$ is a toric manifold and $X(\mathbb{R})$ is the real toric manifold associated to $X$, then $H^*(X(\mathbb{R}); \mathbb{Z}/2)$ is isomorphic to $H^{2*}(X; \mathbb{Z}) \otimes \mathbb{Z}/2$ as graded rings. Motivated by this, we posed in [5] the following analogous problem.

**Cohomological rigidity problem for real toric manifolds.** Are two real toric manifolds are diffeomorphic if their cohomology rings with $\mathbb{Z}/2$ coefficients are isomorphic as graded rings?

We say that *cohomological rigidity over* $\mathbb{Z}/2$ holds for a family of closed smooth manifolds if the manifolds in the family are distinguished up to diffeomorphism by their cohomology rings with $\mathbb{Z}/2$ coefficients. A real Bott manifold is the total space of an iterated $\mathbb{R}P^1$ bundles where each $\mathbb{R}P^1$ bundle is the projectivization of a Whitney sum of two real line bundles. A real Bott manifold is not only a real toric manifold but also a compact flat riemannian manifold. We proved in [5] (and [6]) that cohomological rigidity over $\mathbb{Z}/2$ holds for the family of real Bott manifolds.

---

*Date*: September 23, 2008.

2000 *Mathematics Subject Classification.* Primary 57R91; Secondary 14M25.

*Key words and phrases.* real toric manifold, cohomological rigidity, generalized real Bott manifold of height 2.

The author was partially supported by Grant-in-Aid for Scientific Research 19204007.

In this paper we consider real toric manifolds obtained as the total spaces of projectivization of Whitney sums of real line bundles over a real projective space. We call such a real toric manifold a *generalized real Bott manifold of height 2*. In this paper we will investigate when those two manifolds have isomorphic cohomology rings with $\mathbb{Z}/2$ coefficients and also when they are diffeomorphic. As a result, we will see that cohomological rigidity over $\mathbb{Z}/2$ fails to hold for some family of generalized real Bott manifolds of height 2, which gives a negative answer to the cohomological rigidity problem for real toric manifolds above. We also prove that generalized real Bott manifolds of height 2 are diffeomorphic if they are homotopy equivalent.

The author thanks Y. Nishimura for pointing out a mistake in an earlier version of this paper and T. Panov for his comments.

## 2. Cohomological condition

Let $a, b$ be positive integers and we fix them. Let $\gamma$ be the tautological line bundle over $\mathbb{R}P^{a-1}$ and let $\mathbf{1}$ denote a trivial real line bundle over an appropriate space. For a real vector bundle $E$, we denote by $P(E)$ the total space of the projectivization of $E$. For an integer $q$ such that $0 \leq q \leq b$, we set

$$M(q) := P(q\gamma \oplus (b-q)\mathbf{1}).$$

Note that

$$M(q) \text{ is diffeomorphic to } M(b-q) \tag{2.1}$$

because $P(E \otimes L)$ and $P(E)$ are diffeomorphic for any smooth vector bundle $E$ and line bundle $L$ over a smooth manifold.

A simple computation shows that

$$H^*(M(q); \mathbb{Z}/2) = \mathbb{Z}/2[x, y]/(x^a, (x+y)^q y^{b-q}) \tag{2.2}$$

where $x$ is the pullback of the first Stiefel-Whitney class of $\gamma$ to $M(q)$ and $y$ is the first Stiefel-Whitney class of the tautological line bundle over $M(q)$. One easily sees that a set $\{x^i y^j \mid 0 \leq i < a,\ 0 \leq j < b\}$ is an additive basis of $H^*(M(q); \mathbb{Z}/2)$.

**Lemma 2.1.** *If $0 < q < b$, then both $y^a$ and $(x+y)^a$ are non-zero.*

*Proof.* Suppose $y^a = 0$. Then it follows from (2.2) that there are constants $c, d \in \mathbb{Z}/2$ and a homogeneous polynomial $f(x, y)$ in $x, y$ over $\mathbb{Z}/2$ such that

$$y^a = \begin{cases} cx^a & \text{if } a < b \\ dx^a + f(x,y)(x+y)^q y^{b-q} & \text{if } a \geq b \end{cases}$$

as polynomials in $x, y$. Clearly the former does not occur and the latter also does not occur because $q > 0$ by assumption. This is a contradiction, so $y^a \neq 0$.

If we set $X = x$ and $Y = x + y$, then $x + y = Y$ and $y = X + Y$, so that the role of $x$ and $x + y$ will be interchanged. Since $b - q > 0$ by assumption, the above argument applied to $Y$ instead of $y$ proves that $(x + y)^a \neq 0$. □

**Definition.** $h(a) := \min\{n \in \mathbb{N} \cup \{0\} \mid 2^n \geq a\}$.

For example,

$$h(1) = 0,\ h(2) = 1,\ h(3) = h(4) = 2,\ h(5) = h(6) = h(7) = h(8) = 3,$$
$$h(9) = \cdots = h(16) = 4,\ \ldots.$$

**Lemma 2.2.** *Let $q$ and $q'$ be non-negative integers. Then $\binom{q'}{i} \equiv \binom{q}{i} \pmod 2$ for $0 \leq \forall i < a$ if and only if $q' \equiv q \pmod{2^{h(a)}}$, where $\binom{n}{m}$ is understood to be $0$ when $n < m$ as usual.*

*Proof.* When $q' = q$, the lemma is trivial. We may assume that $q' > q$ without loss of generality. We note that the former congruence relations in the lemma are equivalent to the following congruence relation of polynomials in $t$ with $\mathbb{Z}/2$ coefficients:

$$(1+t)^{q'-q} \equiv 1 \pmod{t^a}. \tag{2.3}$$

We shall prove the "if" part first. Suppose $q' \equiv q \pmod{2^{h(a)}}$. Then $q' - q = 2^{h(a)}R$ with some positive integer $R$ and the left hand side of (2.3) turns into

$$(1+t)^{q'-q} = (1+t^{2^{h(a)}})^R \equiv 1 \pmod{t^a}$$

where the last congruence relation holds because $2^{h(a)} \geq a$. This verifies (2.3).

We shall prove the "only if" part by induction on $a$. When $a = 1$, $2^{h(a)} = 1$ and hence the congruence relation $q' \equiv q \pmod{2^{h(a)}}$ trivially holds. Suppose that the induction assumption is satisfied for $a-1$ with $a \geq 2$ and that (2.3) holds for $a$. Then (2.3) holds for $a-1$, so the induction assumption implies $q' \equiv q \pmod{2^{h(a-1)}}$. When $a-1$ is not a power of 2, $h(a-1) = h(a)$; so the congruence relation $q' \equiv q \pmod{2^{h(a)}}$ holds for $a$. When $a-1$ is a power of 2, say $2^s$,

$$h(a-1) = s, \quad h(a) = s+1$$

and $q' - q = 2^s Q$ with some positive integer $Q$ because $q' \equiv q \pmod{2^{h(a-1)}}$. Therefore

$$(1+t)^{q'-q} = (1+t^{2^s})^Q = 1 + Qt^{2^s} + \text{higher degree terms}.$$

Since this is congruent to 1 modulo $t^a$ and $a > 2^s = a-1$, $Q$ must be even. This shows that $q' \equiv q \pmod{2^{s+1}}$, proving the induction assumption for $a$ because $s+1 = h(a)$. This completes the induction step and proves the "only if" part. □

**Theorem 2.3.** *Let $0 \leq q, q' \leq b$. Then $H^*(M(q);\mathbb{Z}/2)$ and $H^*(M(q');\mathbb{Z}/2)$ are isomorphic as graded rings if and only if $q' \equiv q$ or $b - q \pmod{2^{h(a)}}$.*

*Proof.* If both $q$ and $q'$ are in $\{0, b\}$, then the theorem is trivial. So we may assume $0 < q < b$ without loss of generality. We denote by $x'$ and $y'$ the generators in $H^*(M(q');\mathbb{Z}/2)$ corresponding to $x$ and $y$.

The "if" part easily follows from (2.2) and Lemma 2.2. We shall prove the "only if" part. Suppose that there is an isomorphism

$$\varphi\colon H^*(M(q');\mathbb{Z}/2)\to H^*(M(q);\mathbb{Z}/2)$$

as graded rings. Since $\varphi(x')^a=\varphi(x'^a)=0$, $\varphi(x')$ is neither $y$ nor $x+y$ by Lemma 2.1. Therefore $\varphi(x')=x$ and hence $\varphi(y')=y$ or $x+y$. Suppose $\varphi(y')=y$. (When $\varphi(y')=x+y$, the role of $q$ and $b-q$ will be interchanged.) Then $(x'+y')^{q'}y'^{b-q'}$ maps to $(x+y)^{q'}y^{b-q'}$ by $\varphi$ and it is zero in $H^*(M(q);\mathbb{Z}/2)$, so there are constants $c,d\in\mathbb{Z}/2$ and a homogeneous polynomial $f(x,y)$ in $x,y$ over $\mathbb{Z}/2$ such that

$$(x+y)^{q'}y^{b-q'}=\begin{cases} c(x+y)^q y^{b-q} & \text{in case } a>b\\ d(x+y)^q y^{b-q}+f(x,y)x^a & \text{in case } a\le b\end{cases}$$

as polynomials in $x,y$. Clearly $c$ is non-zero, so $c=1$. Therefore $q'=q$ in case $a>b$. If $d=0$, then the right-hand side of the identity above in case $a\le b$ is divisible by $x$ as $a\ge 1$ while the left-hand side is not. Therefore $d=1$ and the identity above in case $a\le b$ implies the former congruence relations in Lemma 2.2 by comparing the coefficients of $x^iy^{b-i}$ for $i<a$; so $q'\equiv q \pmod{2^{h(a)}}$ by Lemma 2.2. □

**Corollary 2.4.** *Cohomological rigidity over $\mathbb{Z}/2$ holds for $M(q)$'s if $b\le 2^{h(a)}$.*

*Proof.* Suppose that $M(q)$ and $M(q')$ have isomorphic cohomology rings with $\mathbb{Z}/2$ coefficients. Then $q$ and $q'$ must satisfy the congruence relation in Theorem 2.3. But since $b\le 2^{h(a)}$, the congruence implies that $q'=q$ or $b-q$. This together with (2.1) shows that $M(q')$ is diffeomorphic to $M(q)$. □

## 3. KO theoretical condition

In this section, we use KO theory to deduce a necessary and sufficient condition on $q$ and $q'$ for $M(q)$ and $M(q')$ to be diffeomorphic. We begin with a general lemma.

**Lemma 3.1.** *Let $E\to X$ be a real smooth vector bundle over a smooth manifold $X$. Let $\pi\colon P(E)\to X$ be the associated real projective bundle and let $\eta$ be the tautological real line bundle over $P(E)$. Then the tangent bundle $\tau P(E)$ of $P(E)$ with $\mathbf{1}$ added is isomorphic to* $\mathrm{Hom}(\eta,\pi^*(E))\oplus\pi^*(\tau X)$.

*Proof.* A point $\ell$ of $P(E)$ is a line in $E$ and the fibers of $\eta$ over $\ell$ are vectors in the line $\ell$, so $\eta$ is a subbundle of $\pi^*(E)$. We give a fiber metric on $E$. It induces a fiber metric on $\pi^*(E)$ and we denote by $\eta^\perp$ the orthogonal complement of $\eta$ in $\pi^*(E)$. Then $\tau_f P(E)$ the tangent bundle along the fiber of $\pi\colon P(E)\to X$ is isomorphic to $\mathrm{Hom}(\eta,\eta^\perp)$. This is proved in [8, Lemma 4.4] when $X$ is a point and the same argument works for any $X$. In fact, the argument is as follows. We note that the unit $S^0$ bundle $S(\eta)$ of $\eta$ can naturally be identified with the unit sphere bundle $S(E)$ of $E$. Let

$v \in S(\eta)$ be in the fiber over $\ell \in P(E)$, that is, $v$ is a vector in the line $\ell$ with unit length. To an element $\psi \in \mathrm{Hom}(\eta, \eta^{\perp})$ over $\ell \in P(E)$, we assign $\psi(v)$. It is tangent to the fiber of $S(E)$ over $\pi(\ell) \in X$ at $v \in S(E) = S(\eta)$ and $\psi(-v) = -\psi(v)$, so $\psi(v)$ defines an element of $\tau_f P(E)$ over $\ell$. This correspondence gives an isomorphism from $\mathrm{Hom}(\eta, \eta^{\perp})$ to $\tau_f P(E)$.

Thus we obtain

$$\tau_f P(E) \oplus \mathbf{1} \cong \mathrm{Hom}(\eta, \eta^{\perp}) \oplus \mathrm{Hom}(\eta, \eta) \cong \mathrm{Hom}(\eta, \pi^*(E)).$$

This implies the lemma because $\tau P(E) \cong \tau_f P(E) \oplus \pi^*(\tau X)$. □

**Definition.** $k(a) := \#\{n \in \mathbb{N} \mid 0 < n < a \text{ and } n \equiv 0, 1, 2, 4 \pmod 8\}$.

For example,

$$k(1) = 0,\ k(2) = 1,\ k(3) = k(4) = 2,\ k(5) = k(6) = k(7) = k(8) = 3,$$
$$k(9) = 4,\ k(10) = 5,\ k(11) = k(12) = 6, \dots$$

It is known that $\widetilde{KO}(\mathbb{R}P^{a-1})$ is a cyclic group of order $2^{k(a)}$ generated by $\gamma - \mathbf{1}$ ([1, Theorem 7.4]). This implies that $2^{k(a)}\gamma$ is trivial because the fiber dimension (that is $2^{k(a)}$) is strictly larger than the dimension of the base space (that is $a - 1$).

**Theorem 3.2.** *Let $0 \le q, q' \le b$. Then $M(q)$ and $M(q')$ are diffeomorphic if and only if $q' \equiv q$ or $b - q \pmod{2^{k(a)}}$.*

*Proof.* We shall prove the "if" part first. If $2^{k(a)} \ge b$ (this is the case when $a \ge b$), then $q' = q$ or $b - q$ and hence $M(q) \cong M(q')$ by (2.1). Suppose $2^{k(a)} < b$. Then $a < b$ so that the bundles $q'\gamma \oplus (b-q')\mathbf{1}$ and $q\gamma \oplus (b-q)\mathbf{1}$ are in the stable range and these bundles are isomorphic because $\widetilde{KO}(\mathbb{R}P^{a-1})$ is a cyclic group of order $2^{k(a)}$ generated by $\gamma - \mathbf{1}$ and $q' \equiv q \pmod{2^{k(a)}}$. Hence $M(q) \cong M(q')$.

We shall prove the "only if" part. Suppose $M(q) \cong M(q')$ and let $f \colon M(q) \to M(q')$ be a diffeomorphism. Then

$$f^*(\tau M(q')) = \tau M(q) \quad \text{in } \widetilde{KO}(M(q)).$$

Since $\tau(\mathbb{R}P^{a-1}) \oplus \mathbf{1} \cong a\gamma$, it follows from Lemma 3.1 that the identity above implies

$$\begin{aligned} f^*\big(\mathrm{Hom}(\eta', q'\gamma \oplus (b-q')\mathbf{1}) \oplus a\gamma\big) \\ = \mathrm{Hom}(\eta, q\gamma \oplus (b-q)\mathbf{1}) \oplus a\gamma \quad \text{in } \widetilde{KO}(M(q)) \end{aligned} \tag{3.1}$$

where $\eta$ and $\eta'$ denote the tautological line bundles over $M(q)$ and $M(q')$ respectively and $\gamma$ is regarded as a line bundle over $M(q)$ and $M(q')$ through the projections onto $\mathbb{R}P^{a-1}$.

If both $q$ and $q'$ are in $\{0, b\}$, then the "only if" part is obviously satisfied. Therefore we may assume that $0 < q < b$. Then $f^*(x') = x$ and $f^*(y') = y$ or $x + y$ by Lemma 2.1. Therefore $f^*(\gamma) = \gamma$ and $f^*(\eta') = \eta$ or $\gamma\eta$. Suppose

$f^*(\eta') = \eta$ occurs. (When $f^*(\eta') = \gamma\eta$ occurs, the role of $q$ and $b-q$ will be interchanged.) Then (3.1) reduces to

$$\mathrm{Hom}(\eta, q'\gamma \oplus (b-q')\mathbf{1}) = \mathrm{Hom}(\eta, q\gamma \oplus (b-q)\mathbf{1}) \quad \text{in } \widetilde{KO}(M(q)).$$

The fibration $M(q) \to \mathbb{R}P^{a-1}$ has a cross-section and we send the identity above to $\widetilde{KO}(\mathbb{R}P^{a-1})$ through the cross-section. Then $\eta$ becomes trivial or $\gamma$ because a line bundle over $\mathbb{R}P^{a-1}$ is either trivial or $\gamma$. In any case, the identity above reduces to

$$(q'-q)(\gamma - \mathbf{1}) = 0 \quad \text{in } \widetilde{KO}(\mathbb{R}P^{a-1}) \tag{3.2}$$

and this implies $q' \equiv q \pmod{2^{k(a)}}$. □

One easily sees that $h(a) \le k(a)$ for any $a$ and the equality holds if and only if $a \le 9$. Corollary 2.4 can be improved as follows.

**Theorem 3.3.** *Cohomological rigidity over $\mathbb{Z}/2$ holds for $M(q)$'s if and only if $a \le 9$ or $b \le 2^{h(a)}$.*

*Proof.* If $a \le 9$, then $h(a) = k(a)$. So the "if" part follows from Theorems 2.3 and 3.2 when $a \le 9$ and from Corollary 2.4 when $b \le 2^{h(a)}$.

Suppose $a \ge 10$ (so $k(a) > h(a) \ge 4$) and $b > 2^{h(a)}$. Then we take

$$(q, q') = \begin{cases} (1, 2^{h(a)}+1) & \text{when } b \text{ is a multiple of } 2^{h(a)}, \\ (0, 2^{h(a)}) & \text{when } b \text{ is not a multiple of } 2^{h(a)}. \end{cases}$$

In both cases above, $q' \equiv q \pmod{2^{h(a)}}$ but $q'$ is not congruent to neither $q$ nor $b-q$ modulo $2^{k(a)}$ since $k(a) > h(a) \ge 4$. Therefore $M(q)$ and $M(q')$ are not diffeomorphic by Theorem 3.2 while they have isomorphic cohomology rings with $\mathbb{Z}/2$ coefficients by Theorem 2.3. □

## 4. Homotopical rigidity

Cohomological rigidity over $\mathbb{Z}/2$ does not hold for $M(q)$'s in general, but the following holds.

**Theorem 4.1.** *If $M(q)$ and $M(q')$ are homotopy equivalent, then they are diffeomorphic.*

*Proof.* For a finite CW complex $X$, $J(X)$ denotes the $J$ group of $X$ and $J\colon \widetilde{KO}(X) \to J(X)$ denotes the $J$ homomorphism. Let $f\colon M(q) \to M(q')$ be a homotopy equivalence. Then

$$J(f^*(\tau M(q'))) = J(\tau M(q)) \quad \text{in } J(M(q))$$

by a theorem of Atiyah ([2, Theorem 3.6]). The same argument as in the latter part of the proof of Theorem 3.2 shows that we may assume that $0 < q < b$ and $f^*(\eta') = \eta$, and then

$$J\big((q'-q)(\gamma - \mathbf{1})\big) = 0 \quad \text{in } J(\mathbb{R}P^{a-1}).$$

This implies (3.2) because $J\colon \widetilde{KO}(\mathbb{R}P^{a-1}) \to J(\mathbb{R}P^{a-1})$ is an isomorphism (see [4, Theorem 13.9]). Hence $M(q)$ and $M(q')$ are diffeomorphic. □

Theorem 4.1 motivates us to ask whether two real toric manifolds are diffeomorphic (or homeomorphic) if they are homotopy equivalent, which we may call *homotopical rigidity problem for real toric manifolds*.

## References

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math. 75 (1962), 603–632.
[2] M. F. Atiyah, *Thom complexes*, Proc. London Math. Soc. (3) 11 (1961), 291–310.
[3] S. Choi, M. Masuda and D. Y. Suh, *Topological classification of generalized Bott towers*, preprint, arXiv:0807.4334.
[4] D. Husemoller, *Fiber Bundles*, Third Edition, Graduate Texts in Math. 20, Springer-Verlag 1993.
[5] Y. Kamishima and M. Masuda, *Cohomological rigidity of real Bott manifolds*, preprint, arXiv:0807.4263.
[6] M. Masuda, *Classification of real Bott manifolds*, preprint, arXiv:0809.2178.
[7] M. Masuda and D. Y. Suh, *Classification problems of toric manifolds via topology*, Proc. of Toric Topology, Contemp. Math. 460 (2008), 273–286, arXiv:0709.4579.
[8] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Ann. of Math. Studies 76, Princeton Univ. Press 1974.

Department of Mathematics, Osaka City University, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
*E-mail address*: masuda@sci.osaka-cu.ac.jp