$H(2)$-UNKNOTTING NUMBER OF A KNOT

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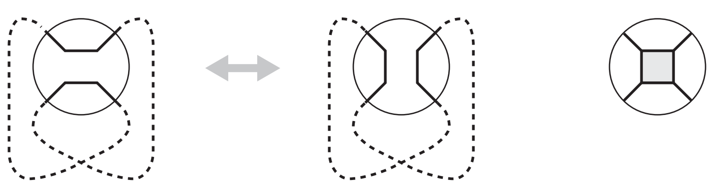
Abstract. An $H(2)$-move is a local move of a knot which is performed by adding a half-twisted band. It is known an $H(2)$-move is an unknotting operation. We define the $H(2)$-*unknottiing number* of a knot K to be the minimum number of $H(2)$-moves needed to transform K into a trivial knot. We give several methods to estimate the $H(2)$-unknottiing number of a knot. Then we give tables of $H(2)$-unknottiing numbers of knots with up to 9 crossings.

1. Introduction

An $H(2)$-*move* is a change in a knot projection as shown in Fig. 1(a); note that both diagrams are taken to represent single component knots, and so the strings are connected as shown in dotted arcs. Since we obtain the diagram by adding a twisted band to each of these knots as shown in Fig. 1(b), it can be said that each of the knots is obtained from the other by *adding a twisted band*. It is easy to see that an $H(2)$-move is an unknotting operation; see [9, Theorem 1]. We call the minimum number of $H(2)$-moves needed to transform a knot K into another knot K' the $H(2)$-*Gordian distance* from K to K', denoted by $\mathrm{d}_2(K, K')$. In particular, the $H(2)$-*unknottiing number* of K is the $H(2)$-Gordian distance from K to a trivial knot, denoted by $\mathrm{u}_2(K)$.



(a) (b)

Figure 1. $H(2)$-move

In this paper, we give several criteria on the $H(2)$-unknottiing number and then we give tables of the $H(2)$-unknottiing numbers of knots with up to 9 crossings.

Date: November 24, 2008.

Key words and phrases. Knot, $H(2)$-move, $H(2)$-unknottiing number, signature, Arf invariant, Jones polynomial, Q polynomial.

The second author was partially supported by Grant-in-Aid for Scientific Research (C) (No. 20540079), Japan Society for the Promotion of Science.

Lickorish [16] was the first to consider an $H(2)$-unknotting number one knot; he has given a criterion for the $H(2)$-unknottiing number one knot using the linking form on the first homology group of the double cover of S^3 branched along the knot. As an application he showed that the $H(2)$-unknottiing number of the figure-eight knot is two, which had been conjectured by Riley. Similarly, he showed that among the knots in the prime knot table with at most seven crossings 4_1, 6_3, 7_2, 7_5 and 7_7 are the only knots with $H(2)$-unknottiing number greater than one.

Then, Hoste, Nakanishi and Taniyama [9] have given an inequality estimating a lower bound of the $H(2)$-unknottiing number, which uses the minimum number of generators of the first homology group of the cyclic branched covering space of the knot (Theorem 2.1); they studied an $H(2)$-move in a more general context. Besides, Yasuhara [36] has given a criterion on an $H(2)$-unknotting number one knot using the signature and the Arf invariant as an application of the theorem on a surface in a 4-dimensional manifold (Theorem 4.5). He mentions that this criterion proves $\mathrm{u}_2(4_1)$, $\mathrm{u}_2(3_1\#3_1) > 1$. We prove Theorem 4.5 from 3-dimensional point of view using polynomial invariants. This leads to a further criterion for the $H(2)$-unknotting number one knot which does not cover Theorem 4.5 (Theorem 5.7); however, it requires that the determinant $\equiv 0 \pmod 3$. The proof uses some relations among the Jones polynomial, the signature, and the Conway polynomial in [21], which is based on the Casson invariant of the double branched covering space of a knot. Furthermore, using the Jones-Rong value [10, 32] of the Brandt-Lickorish-Millett-Ho Q polynomial [2, 8] we introduce another method to calculate the $H(2)$-unknotting number (Theorem 8.1), which is motivated by Stoimenow [33], where he calculated the unknotting number.

On the other hand, Nakajima [26] has listed the $H(2)$-unknotting numbers of prime knots with up to 10 crossings. He uses the above-mentioned criteria due to Hoste et al. and Yasuhara to give a lower bound and a relation with the usual unknotting number (Theorem 3.1) to give an upper bound. In this paper, we list the $H(2)$-unknotting numbers of knots with up to 9 crossings including composite knots (Tables 4–6), which improves Nakajima's table.

This paper is organized as follows: In Sec. 2, we review a criterion for the $H(2)$-unknotting number using the first homology group of the cyclic branched covering space, and give the definitions and some properties of the polynomial invariants and the signature. In Sec. 3, we give a relation between the $H(2)$-unknotting number and usual unknotting number due to Nakajima [26], which gives an upper bound of the $H(2)$-unknotting number. In Secs. 4–7, we give several criteria for the $H(2)$-unknotting number one knot using the signature, Arf invariant, and some values about the Jones polynomial, which is summarized in Table 2. In Sec. 8, we give a criterion for the $H(2)$-Gordian distance using the Q polynomial. In Sec. 9, we give tables of the $H(2)$-unknotting numbers of knots with up to 9 crossings including composite knots.

Notation. For knots with up to 10 crossings we use Rolfsen notation [31] with the correction by Perko [29]; $10_{161} = 10_{162}$.

2. Preliminaries

In this section, first we review a criterion on the $H(2)$-unknotting number using the first homology group of the cyclic branched covering space, and then we give the definitions and some properties of the polynomial invariants and the signature.

Let $\Sigma(K)$ be the double branched cover of S^3 branched over a knot K and $\det(K)$ the determinant of K. Then in the criterion [16, Theorem 1] on the $H(2)$-unknotting number one knot using the linking form on $H_1(\Sigma(K); \boldsymbol{Z})$ Lickorish has shown that $H_1(\Sigma(K); \boldsymbol{Z})$ is cyclic of order $\det(K)$. Moreover, Hoste et al. [9, Theorem 4] have shown:

Theorem 2.1. *Let $\mu(K, r)$ be the minimum number of generators of the first integral homology group of the r-fold cyclic branched cover of S^3 branched over a knot K. Then*

$$\mu(K, r)/(r-1) \leq \mathrm{u}_2(K). \tag{1}$$

Next, we review the polynomial invariants. The *Conway polynomial* $\nabla_L(z) \in \boldsymbol{Z}[z]$ [4], the *Jones polynomial* $V(L; t) \in \boldsymbol{Z}[t^{\pm 1/2}]$ [11], and the *HOMFLYPT polynomial* $P(L; v, z) \in \boldsymbol{Z}[v^{\pm 1}, z^{\pm 1}]$ [6, 11, 30] are invariants of the isotopy type of an oriented link L, which are defined by the following formulas:

$$\nabla(U; z) = 1; \tag{2}$$

$$\nabla(L_+; z) - \nabla(L_-; z) = z\nabla(L_0; z); \tag{3}$$

$$V(U; t) = 1; \tag{4}$$

$$t^{-1}V(L_+; t) - tV(L_-; t) = \left(t^{1/2} - t^{-1/2}\right) V(L_0; t); \tag{5}$$

$$P(U; v, z) = 1; \tag{6}$$

$$v^{-1}P(L_+; v, z) - vP(L_-; v, z) = zP(L_0; v, z), \tag{7}$$

where U is the unknot and L_+, L_-, L_0 are three links that are identical except near one point where they are as in Fig. 2; we call (L_+, L_-, L_0) a *skein triple.*

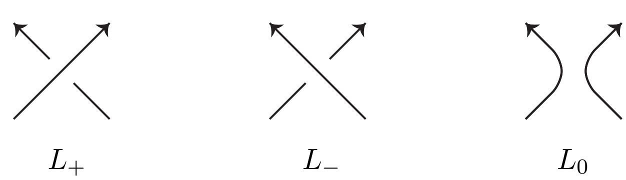


Figure 2. A skein triple.

By [18, Proposition 22], the HOMFLYPT polynomial of an r-component link L is of the form

$$P(L;v,z)=\sum_{q\geq 0}P_{2q-r+1}(L;v)z^{2q-r+1}, \tag{8}$$

where each $P_{2q-r+1}(L;v)\in \boldsymbol{Z}[v^{\pm 1}]$ is called the *coefficient polynomial*. If $0\leq q\leq r-2$, then

$$P_{2q-r+1}(L;1)=0. \tag{9}$$

In fact, the Conway polynomial is obtained from the HOMFLYPT polynomial by

$$\nabla_L(z)=P(L;1,z), \tag{10}$$

which is of the form

$$\nabla_L(z)=\sum_{q\geq 0}a_{2q+r-1}(L)z^{2q+r-1}, \tag{11}$$

where $a_{2q+r-1}(L)=P_{2q+r-1}(L;1)$.

In particular, if L_+, L_- are knots and L_0 is a 2-component link, then Eq. (3) implies

$$a_2(L_+)-a_2(L_-)=\mathrm{lk}(L_0), \tag{12}$$

since $a_1(L_0)=\mathrm{lk}(L_0)$, the linking number of L_0; cf. [14, Proposition 3.12]. Also, the *Arf invariant* of a knot K, $\mathrm{Arf}(K)\in \boldsymbol{Z}_2$, is given by

$$\mathrm{Arf}(K)\equiv a_2(K) \pmod 2. \tag{13}$$

In addition, the Jones polynomial is obtained from the HOMFLYPT polynomial by

$$V(L;t)=P(L;t,t^{1/2}-t^{-1/2}). \tag{14}$$

Lastly, we give some properties of the signature. We denote the signature and the nullity of an oriented link L by $\sigma(L)$ and $\mathrm{n}(L)$, respectively. The following is due to Giller [7]; cf. [25, Theorem 6.4.7]. Note that the signature of a knot is an even integer.

Proposition 2.2. *The signature of a knot can be determined by the following three axioms.*

(i) *For the trivial knot U, $\sigma(U)=0$.*

(ii) *If (L_+,L_-,L_0) is a skein triple with $L_\pm$ knots, then*

$$\sigma(L_-)-2\leq\sigma(L_+)\leq\sigma(L_-). \tag{15}$$

(iii) *Let* $\mathrm{sign}V(K;-1)=V(K;-1)/|V(K;-1)|$. *Then*

$$(-1)^{\sigma(K)/2}=\mathrm{sign}V(K;-1). \tag{16}$$

For the signature and nullity, Murasugi has shown the following [23, Lemma 7.1].

Proposition 2.3. *For a skein triple* (L_+, L_-, L_0)*, we have*

$$|\sigma(L_+) - \sigma(L_0)| + |\mathrm{n}(L_+) - \mathrm{n}(L_0)| = 1. \tag{17}$$

Murasugi has shown that for a link L, $\sigma(L) + \mathrm{lk}(L)$ is an invariant of an unoriented link type [24, Theorem 1], which implies the following proposition.

Proposition 2.4. *Let L be an oriented 2-component link and L' a link obtained from L by reversing the orientation of one component. Then*

$$\sigma(L') = \sigma(L) + 2\mathrm{lk}(L). \tag{18}$$

3. Relation to the usual unknotting number

Let K be a knot. We denote by $\mathrm{u}(K)$ the (usual) unknotting number of K. Then K has a diagram such that changing $\mathrm{u}(K)$ crossings in this diagram turns K into the trivial knot. Let u_+ and u_- be the numbers of positive and negative crossings in these crossings; so $\mathrm{u}(K) = u_+ + u_-$. Then Nakajima has proved the following [26, Theorem 3.2.3], which is useful to decide the $H(2)$-unknotting number.

Theorem 3.1.

$$\mathrm{u}_2(K) \leq \mathrm{u}(K) + 1. \tag{19}$$

Moreover, if both u_+ and u_- are even, then

$$\mathrm{u}_2(K) \leq \mathrm{u}(K). \tag{20}$$

Proof. Let K be a knot with diagram as above, and $u = \mathrm{u}(K)$, and $p = u_+$, $0 \leq p \leq u$. Then we may deform it into a diagram having a $(u+1)$-string tangle as shown in Fig. 3 so that K can be unknotted by changing the positive crossings $c_1, \ldots, c_p$, and negative crossings $c_{p+1}, \ldots, c_u$ simultaneously; cf. [34, Lemma 1]. Then the results follow from Lemma 3.2 below. □

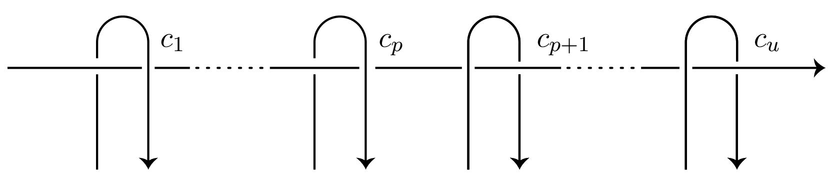


Figure 3. A $(u+1)$-string tangle.

In order to prove this theorem, we consider the five moves as shown in Fig. 4; the moves M_+ and M_- are equivalent to the crossing changes.

Then we have:

Lemma 3.2. (i) *Each of the moves M_+ and M_- is realized by two $H(2)$-moves.*

(ii) *Each of the moves M_{++} and M_{--} is realized by two $H(2)$-moves.*

(iii) *The move M_{+-} is realized by three $H(2)$-moves.*

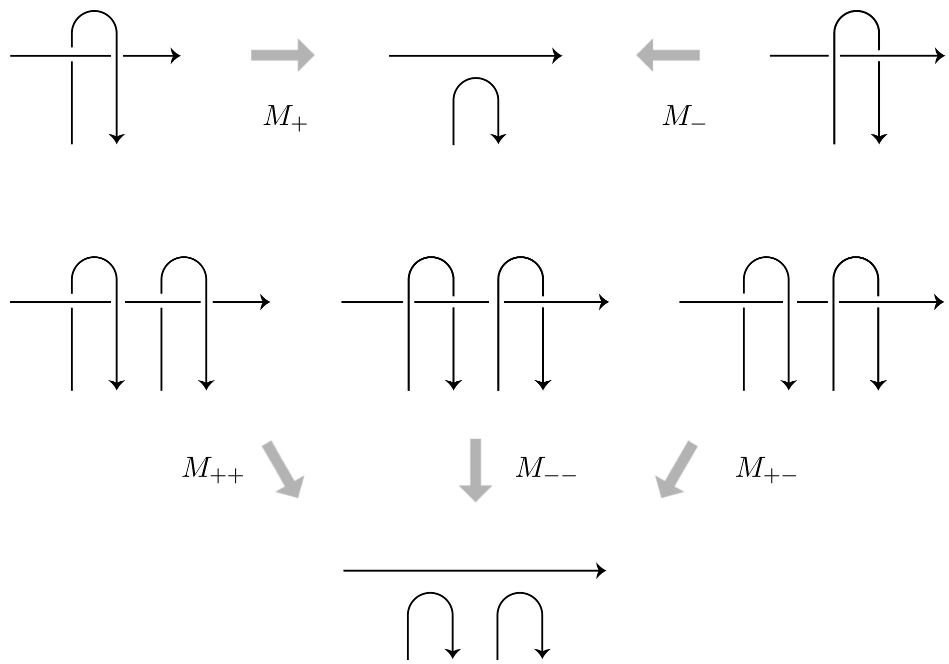


FIGURE 4. The moves M_+, M_-, M_{++}, M_{--}, M_{+-}.

Proof. Fig. 5 shows that a single M_+-move is realized by two $H(2)$-moves; cf. the proof of Lemma 1 in [9]. Fig. 6 shows that a single M_{++}-move is realized by two $H(2)$-moves. Fig. 7 shows that a single M_{+-}-move is realized by three $H(2)$-moves. For other moves, we can show similarly. □

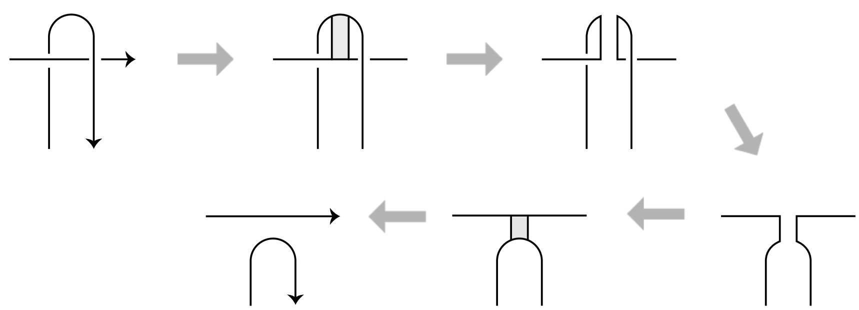


FIGURE 5. The move M_+.

Remark 3.1. In [9, Theorem 2], the formulas $\mathrm{u}_2(K) \leq 2\mathrm{u}(K)$ and $\mathrm{u}_2(K) \leq \mathrm{cr}(K) - 2$ are given, where $\mathrm{cr}(K)$ is the minimum crossing number. However, Eq. (20) is sharper than these formulas. In fact, $\mathrm{u}(K) \leq (\mathrm{cr}(K) - 1)/2$; cf. [27, Eq. (0.1)].

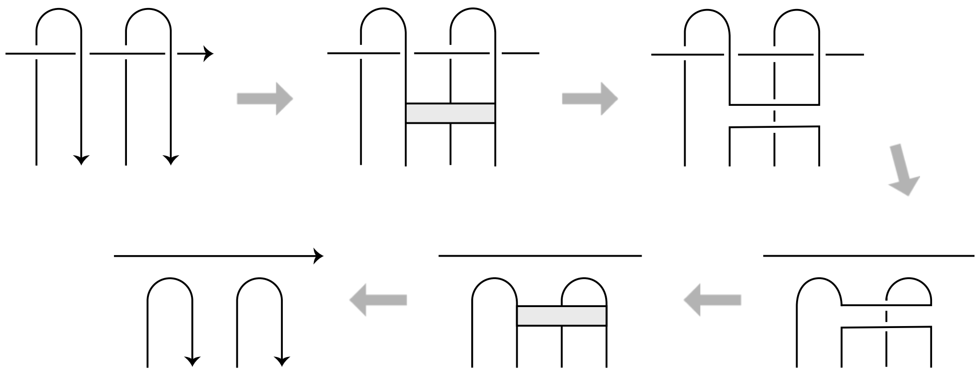


FIGURE 6. The move M_{++}.

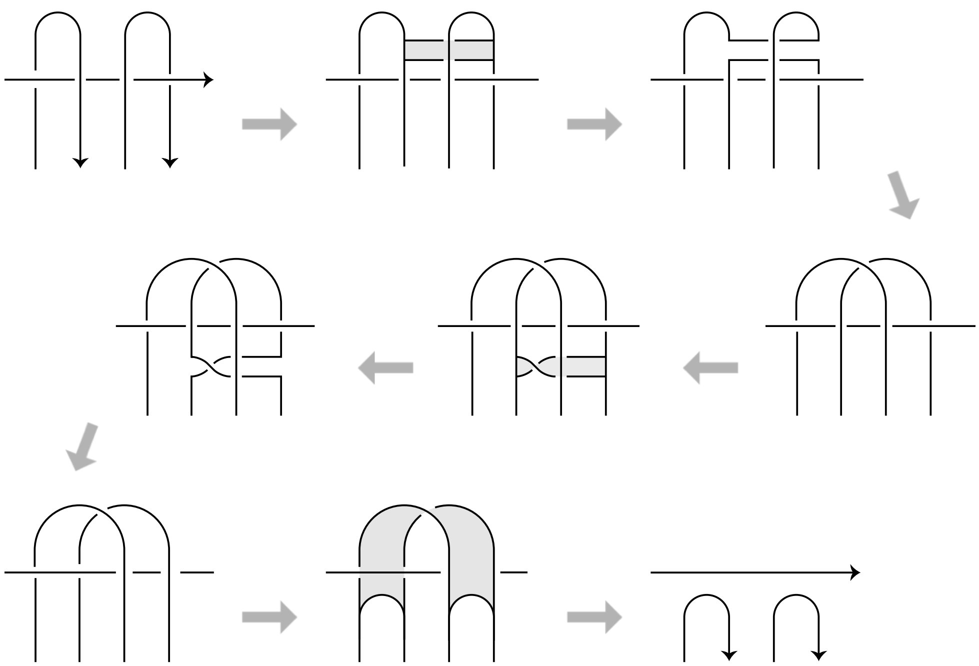


FIGURE 7. The move M_{+-}.

4. $H(2)$-UNKNOTTING NUMBER ONE KNOT WITH SIGNATURE $\equiv 0 \pmod 4$

The main purpose of this section is to prove Theorem 4.5, a criterion for the $H(2)$-unknotting number one knot using the signature and the Arf invariant. This theorem follows from Proposition 5.1 in Yasuhara [36], which is given as an application of the main theorem on a surface in a 4-dimensional manifold. However, we prove from 3-dimensional point of view using polynomial invariants, which will continue to further criteria in Secs. 5–7.

Let L_+, L_-, L_0, L_∞ be oriented four links that are identical except within a ball B where they are as in Fig. 8. We suppose that L_+, L_- are knots and L_0 is a 2-component link; then L_∞ is a knot. Note that outside B the orientation of one of the strands of L_∞ is the reverse of that of the three links L_+, L_-, L_0. In this situation, we call $(L_+, L_-, L_0, L_\infty)$ an *oriented skein quadruple.*

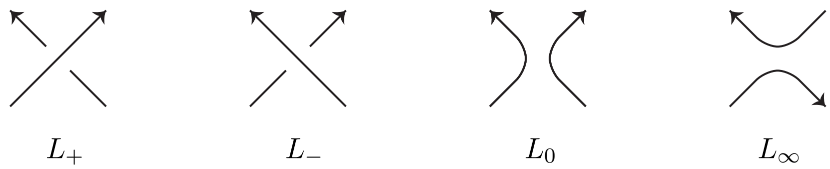


FIGURE 8. An oriented skein quadruple.

Lemma 4.1. *If K is an oriented knot with* $\mathrm{u}_2(K) = 1$*, then there exists an oriented skein quadruple* (L_+, L_-, L_0, K) *with L_- a trivial knot. Hence* $\mathrm{u}(L_+) = 1$.

Proof. Since a trivial knot U and K are related by an $H(2)$-move, we can assume that K and U are identical except in a ball B where they are as in Fig. 9; the orientation of one of the strands of U outside B is the reverse of that of K. Consider an oriented skein quadruple $(L_+, L_-, L_0, L_\infty)$ which are obtained by placing each of the tangles shown in Fig. 9 inside B and using the same configuration as before in $S^3 \setminus B$. Since L_- and L_∞ are isotopic to U and K, respectively, this oriented skein quadruple is the desired one. □

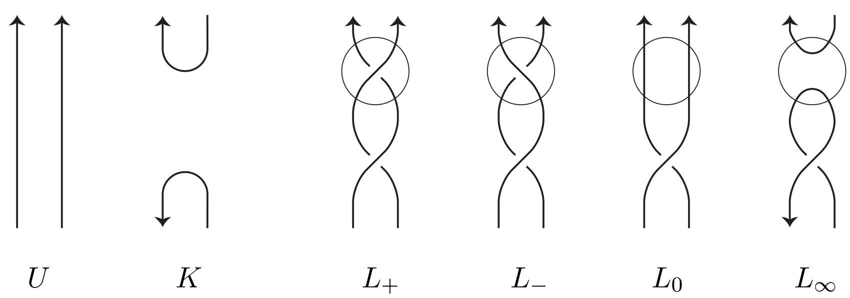


FIGURE 9

The following is due to Birman and Kanenobu; see Corollary in [1].

Proposition 4.2. *Let* $(L_+, L_-, L_0, L_\infty)$ *be an oriented skein quadruple. Then*

$$V(L_+; t) - tV(L_-; t) + t^{3\mathrm{lk}(L_0)}(t-1)V(L_\infty; t) = 0. \tag{21}$$

Lemma 1 in [12] implies the following.

Proposition 4.3. *Let $(L_+, L_-, L_0, L_\infty)$ be an oriented skein quadruple and J_1, J_2 be the components of L_0. Then*

$$a_2(L_\infty) = -\frac{1}{2}\left(a_2(L_+) + a_2(L_-)\right) + 2\left(a_2(J_1) + a_2(J_2)\right) + \frac{1}{2}\mathrm{lk}(L_0)^2 \tag{22}$$

Lemma 4.4. *Let $(L_+, L_-, L_0, L_\infty)$ be an oriented skein quadruple with L_- a trivial knot. Then*

$$\sigma(L_+) = 0 \text{ or } -2; \tag{23}$$

$$\mathrm{lk}(L_0) = \frac{1}{2}\left(\sigma(L_\infty) - \sigma(L_+)\right); \tag{24}$$

$$\mathrm{Arf}(L_\infty) \equiv -\frac{1}{2}\left(\mathrm{lk}(L_0) - \mathrm{lk}(L_0)^2\right) \pmod 2. \tag{25}$$

Proof. Eq. (23) follows from Proposition 2.2(ii).

Next we prove Eq. (24). Put $\lambda = \mathrm{lk}(L_0)$. Let L_0' be an oriented link obtained from L_0 by reversing orientation of one component of L_0. Then by Proposition 2.4 we have

$$\sigma(L_0') = \sigma(L_0) + 2\lambda. \tag{26}$$

By Proposition 2.3, for two pairs of links (L_+, L_0) and (L_∞, L_0') we have

$$|\sigma(L_+) - \sigma(L_0)| + |\mathrm{n}(L_+) - \mathrm{n}(L_0)| = 1; \tag{27}$$

$$|\sigma(L_\infty) - \sigma(L_0')| + |\mathrm{n}(L_\infty) - \mathrm{n}(L_0')| = 1. \tag{28}$$

Putting $x = \sigma(L_\infty) - \sigma(L_0')$ and $y = \sigma(L_+) - \sigma(L_0)$, we have $|x| \leq 1$, $|y| \leq 1$ and $x - y = \sigma(L_\infty) - \sigma(L_+) - 2\lambda$ from (26). We will show $x - y = 0$. There are four cases:

(i) $\sigma(L_\infty) \equiv 0 \pmod 4$ and $\lambda \equiv 0 \pmod 2$.
(ii) $\sigma(L_\infty) \equiv 0 \pmod 4$ and $\lambda \equiv 1 \pmod 2$.
(iii) $\sigma(L_\infty) \equiv 2 \pmod 4$ and $\lambda \equiv 0 \pmod 2$.
(iv) $\sigma(L_\infty) \equiv 2 \pmod 4$ and $\lambda \equiv 1 \pmod 2$.

We only prove for case (i). From Proposition 2.2(iii), we have

$$\mathrm{sign} V(L_\infty; -1) = (-1)^{\sigma(L_\infty)/2} = 1, \tag{29}$$

and so $V(L_\infty; -1) \geq 1$. From Proposition 4.2, we obtain

$$V(L_+; -1) = 2(-1)^{3\lambda} V(L_\infty; -1) - V(L_-; -1) = 2V(L_\infty; -1) - 1, \tag{30}$$

and so $V(L_+; -1) \geq 1$. Again by Proposition 2.2(iii), we have

$$\mathrm{sign} V(L_+; -1) = (-1)^{\sigma(L_+)/2} > 0, \tag{31}$$

and so $\sigma(L_+) \equiv 0 \pmod 4$. Then by Eq. (23), we obtain $\sigma(L_+) = 0$. Therefore $x - y = \sigma(L_\infty) - 2\lambda \equiv 0 \pmod 4$. Since $|x - y| \leq |x| + |y| \leq 2$, we have $x - y = 0$, completing the proof of Eq. (24).

Finally, we prove Eq. (25). Since L_- is a trivial knot, $a_2(L_-) = 0$, and by Eq. (12) we have $a_2(L_+) = \lambda$. Then using Proposition 4.3, we obtain Eq. (25). □

The following theorem is the main result in this section.

Theorem 4.5. *Let K be a knot with $\mathrm{u}_2(K) = 1$. Then*
(i) *If $\sigma(K) \equiv 0 \pmod 8$, then $\mathrm{Arf}(K) = 0$.*
(ii) *If $\sigma(K) \equiv 4 \pmod 8$, then $\mathrm{Arf}(K) = 1$.*

Proof. Since $\mathrm{u}_2(K) = 1$, from Lemma 4.1 there exists an oriented skein quadruple (L_+, L_-, L_0, K) such that L_- is a trivial knot.

First we prove (i). Since $\sigma(K) \equiv 0 \pmod 8$, by Eqs. (23) and (24) in Lemma 4.4 we obtain

$$\mathrm{lk}(L_0) \equiv 0 \ \text{ or } \ 1 \pmod 4, \tag{32}$$

which implies $\mathrm{Arf}(K) = 0$ by Eq. (25) in Lemma 4.4.

Next we prove (ii). Since $\sigma(K) \equiv 4 \pmod 8$, by Eqs. (23) and (24) in Lemma 4.4 we obtain

$$\mathrm{lk}(L_0) \equiv 2 \ \text{ or } \ 3 \pmod 4, \tag{33}$$

which implies $\mathrm{Arf}(K) = 1$ by Eq. (25) in Lemma 4.4. This completes the proof. □

5. $H(2)$-unknotting number one knot with signature $\equiv 2 \pmod 4$

The main purpose of this section is to prove Theorem 5.7, a criterion for the $H(2)$-unknotting number one knot which does not cover Theorem 4.5. However, it requires that the determinant $\equiv 0 \pmod 3$. First we consider the value $V'(K;-1)$, the first derivative of the Jones polynomial of a knot K at $t = -1$.

Lemma 5.1. *For a knot K,*

$$V'(K;-1) \equiv 0 \pmod 8. \tag{34}$$

Proof. Jones [11, Proposition 12.5] has shown that

$$V(K;t) = 1 - (1-t)(1-t^3)W(K;t) \tag{35}$$

for some Laurent polynomial $W(K;t)$, from which we have

$$V'(K;-1) = 8W(K;-1) - 4W'(K;-1). \tag{36}$$

Since it is easy to see that $W'(K;-1) \equiv W'(K;1) \pmod 2$, we have only to show $W'(K;1) \equiv 0 \pmod 2$.

Taking the second and third derivatives at $t = 1$ of each side of Eq. (35), we obtain

$$V^{(2)}(K;1) = -6W(K;1); \tag{37}$$

$$V^{(3)}(K;1) = -18\left(W(K;1) + W'(K;1)\right). \tag{38}$$

On the other hand, by [22], and Eq. (2), Lemma 2 in [21] we have

$$V^{(2)}(K;1) = -6a_2(K); \tag{39}$$

$$V^{(3)}(K;1) = \frac{3}{4}P_0^{(3)}(K;1) = 18\left(a_2(K) - P_2'(K;1)\right), \tag{40}$$

where $P_0^{(3)}(K;1)$ and $P_2'(K;1)$ are the third derivative of the coefficient polynomial $P_0(K;v)$ at $v=1$ and the first derivative of the coefficient polynomial $P_2(K;v)$ at $v=1$, respectively; see (8). Using these formulas, we obtain

$$W'(K;1) = P_2'(K;1) - 2a_2(K). \tag{41}$$

Furthermore, since $P_2'(K;1) \equiv a_2(K)^2 + a_2(K) - 2a_4(K) \pmod 4$ from [21, Proposition 2], we have

$$P_2'(K;1) \equiv 0 \pmod 2, \tag{42}$$

and thus $W'(K;1) \equiv 0 \pmod 2$, completing the proof. □

Lemma 5.2. *Let $(L_+, L_-, L_0, L_\infty)$ be an oriented skein quadruple with L_- a trivial knot. Suppose that $\sigma(L_\infty) \equiv 2\varepsilon \pmod 8$, $\varepsilon = \pm 1$. Then*

(i) *If $\mathrm{Arf}(L_\infty) = 0$, then $\mathrm{lk}(L_0) \equiv (\varepsilon+1)/2 \pmod 4$ and $\sigma(L_+) = \varepsilon - 1$.*
(ii) *If $\mathrm{Arf}(L_\infty) = 1$, then $\mathrm{lk}(L_0) \equiv (-\varepsilon+5)/2 \pmod 4$ and $\sigma(L_+) = -\varepsilon - 1$.*

Proof. If $\mathrm{Arf}(L_\infty) = 0$, then by Eq. (25) in Lemma 4.4 we have $\mathrm{lk}(L_0) \equiv 0$ or $1 \pmod 4$. Then since $\sigma(L_\infty) \equiv 2\varepsilon \pmod 8$, by Eqs. (23) and (24) in Lemma 4.4 we have:

- If $\mathrm{lk}(L_0) \equiv 0 \pmod 4$, then $\varepsilon = -1$, $\sigma(L_+) = -2$.
- If $\mathrm{lk}(L_0) \equiv 1 \pmod 4$, then $\varepsilon = 1$, $\sigma(L_+) = 0$.

This proves (i).

Next, if $\mathrm{Arf}(L_\infty) = 1$, then by Eq. (25) in Lemma 4.4 we have $\mathrm{lk}(L_0) \equiv 2$ or $3 \pmod 4$. Then since $\sigma(L_\infty) \equiv 2\varepsilon \pmod 8$, by Eqs. (23) and (24) in Lemma 4.4 we have:

- If $\mathrm{lk}(L_0) \equiv 2 \pmod 4$, then $\varepsilon = 1$, $\sigma(L_+) = -2$.
- If $\mathrm{lk}(L_0) \equiv 3 \pmod 4$, then $\varepsilon = -1$, $\sigma(L_+) = 0$.

This proves (ii). □

Lemma 5.3. *Let $(L_+, L_-, L_0, L_\infty)$ be an oriented skein quadruple with L_- a trivial knot. Put $\lambda = \mathrm{lk}(L_0)$. Then*

$$V'(L_+;-1) = 1 - (6\lambda+1)(-1)^\lambda V(L_\infty;-1) + 2(-1)^\lambda V'(L_\infty;-1). \tag{43}$$

Proof. Since L_- is a trivial knot, by Proposition 4.2 we have

$$V(L_+;t) = t - t^{3\lambda}(t-1)V(L_\infty;t). \tag{44}$$

Taking the first derivative at $t=-1$, we obtain Eq. (43). □

Lemma 5.4. *Let K be an unknotting number one knot which can be unknotted by changing a positive crossing to a negative crossing, and C be a surgical knot for $\Sigma(K)$, the double cover of S^3 branched over K. Then we have*

$$a_2(C) = -\frac{1}{24}V'(K;-1) + \frac{1}{96}(V(K;-1)-1)(V(K;-1)-5). \tag{45}$$

Proof. By Proposition 2.2(ii), $\sigma(K) = 0$ or -2. By [21, Theorem 6] we have

$$a_2(C) = \begin{cases} -\frac{1}{24}V'(K;-1) + \frac{1}{96}(\delta-1)(\delta-5) & \text{if } \sigma(K) = 0; \\ \\ -\frac{1}{24}V'(K;-1) + \frac{1}{96}(\delta+1)(\delta+5) & \text{if } \sigma(K) = -2, \end{cases} \tag{46}$$

where $\delta = |V(K;-1)| = \det(K)$. If $\sigma(K) = 0$, then by Proposition 2.2(iii) $V(K;-1) > 0$, and so $\delta = V(K;-1)$, which implies Eq. (45). If $\sigma(K) = -2$, then by Proposition 2.2(iii) $V(K;-1) < 0$, and so $\delta = -V(K;-1)$, which implies Eq. (45). This completes the proof. □

Lemma 5.5. *Let $(L_+, L_-, L_0, L_\infty)$ be an oriented skein quadruple with L_- a trivial knot. Let $\lambda = \mathrm{lk}(L_0)$. Then*

$$V'(L_\infty;-1) \equiv \frac{(-1)^\lambda}{2}\Big((-1)^\lambda V(L_\infty;-1) - 1\Big)\Big((-1)^\lambda V(L_\infty;-1) + 6\lambda - 5\Big) \pmod{24}. \tag{47}$$

Proof. Put $\alpha = V(L_+;-1)$. By Lemma 5.4 we have

$$a_2(C) = -\frac{1}{24}V'(L_+;-1) + \frac{1}{96}(\alpha-1)(\alpha-5), \tag{48}$$

where C is a surgical knot for $\Sigma(L_+)$. On the other hand, by [19, Lemmas 2 and 6] we have

$$a_2(C) \equiv \frac{1}{4}\left(\frac{\alpha-1}{4} + a_2(L_+)\right) \pmod{2}. \tag{49}$$

Combining Eqs. (48) and (49), we have

$$V'(L_+;-1) \equiv \frac{1}{4}(\alpha-1)(\alpha-11) - 6a_2(L_+) \pmod{48}. \tag{50}$$

Putting $\beta = V(L_\infty;-1)$, Eq. (44) implies

$$\alpha = 2(-1)^\lambda\beta - 1 \tag{51}$$

Using this and $a_2(L_+) = \lambda$, by Lemma 5.3 we have

$$2\varepsilon V'(L_\infty;-1) \equiv \Big((-1)^\lambda\beta - 1\Big)\Big((-1)^\lambda\beta + 6\lambda - 5\Big) \pmod{48}, \tag{52}$$

completing the proof. □

Since $V(K;-1) = \nabla(K; 2\sqrt{-1})$ for a knot K, it is easy to see the following.

Lemma 5.6. *Let K be a knot. Then*
(i) $\mathrm{Arf}(K) = 0$ *if and only if* $V(K;-1) \equiv 1 \pmod 8$.
(ii) $\mathrm{Arf}(K) = 1$ *if and only if* $V(K;-1) \equiv 5 \pmod 8$.

The following theorem is the main result in this section.

Theorem 5.7. *Let K be a knot with $\mathrm{u}_2(K) = 1$. Suppose that $\det(K) \equiv 0 \pmod 3$ and $\sigma(K) \equiv 2\varepsilon \pmod 8$, $\varepsilon = \pm 1$. Then*
(i) *If* $\mathrm{Arf}(K) = 0$, *then* $V'(K;-1) \equiv 8\varepsilon \pmod{24}$.
(ii) *If* $\mathrm{Arf}(K) = 1$, *then* $V'(K;-1) \equiv -8\varepsilon \pmod{24}$.

Proof. Put $\beta = V(K;-1)$. Using the assumption $\det(K) = |\beta| \equiv 0 \pmod 3$ and Lemma 5.6, we have

$$\beta \equiv \begin{cases} 9 \pmod{24} & \text{if } \mathrm{Arf}(K) = 0; \\ 21 \pmod{24} & \text{if } \mathrm{Arf}(K) = 1. \end{cases} \tag{53}$$

Since $\mathrm{u}_2(K) = 1$, from Lemma 4.1 there exists an oriented skein quadruple (L_+, L_-, L_0, K) such that L_- is a trivial knot. Put $\lambda = \mathrm{lk}(L_0)$.

We prove (i). By Lemma 5.2(i) $\lambda \equiv (\varepsilon+1)/2 \pmod 4$. Then $(-1)^\lambda = -\varepsilon$, and so using Eq. (47) in Lemma 5.5, we obtain

$$V'(K;-1) \equiv \frac{-\varepsilon}{2}(-\varepsilon\beta - 1)(-\varepsilon\beta + 6\lambda - 5) \pmod{24}. \tag{54}$$

Then using Eq. (53), we obtain $V'(K;-1) \equiv 8\varepsilon \pmod{24}$.

Similarly, we may prove (ii), and so the proof is complete. □

Example 5.1. The $H(2)$-unknotting number of the knot 8_{21} is 2; $\mathrm{u}_2(8_{21}) = 2$. In fact, $\sigma(8_{21}) = 2$, $\mathrm{Arf}(8_{21}) = 0$, $\det(8_{21}) = 15 \equiv 0 \pmod 3$, $V(8_{21};t) = 2t^{-1} - 2t^{-2} + 3t^{-3} - 3t^{-4} + 2t^{-5} - 2t^{-6} + t^{-7}$, $V(8_{21};-1) = -15 \equiv 9 \pmod{24}$, $V'(8_{21};-1) = -56 \equiv -8 \pmod{24}$, and thus by the criterion in Theorem 5.7 we have $\mathrm{u}_2(8_{21}) > 1$. On the other hand, since it is known that $\mathrm{u}(8_{21}) = 1$, by Theorem 3.1 we have $\mathrm{u}_2(8_{21}) \leq 2$, obtaining the result.

6. The value of the Jones polynomial at $t = e^{i\pi/3}$

In this section, we add a condition on the value $V(K; e^{i\pi/3})$ to the criterion of Theorem 5.7, giving Corollary 6.3. Also, we give a criterion for the product of the $H(2)$-unknotting and usual unknotting numbers (Theorem 6.5).

Let $d = \dim H_1(\Sigma(K); \boldsymbol{Z}_3)$, where $\Sigma(K)$ is the double cover of S^3 branched over a knot K. Then Theorem 2.1 implies

$$d \leq \mathrm{u}_2(K). \tag{55}$$

Lickorish and Millett [17, Theorem 3] have shown:

Proposition 6.1.

$$V(K; e^{i\pi/3}) = \pm(i\sqrt{3})^d. \tag{56}$$

This value is closely related to $V'(K;-1)$, the value of the first derivative of the Jones polynomial at $t=-1$.

Theorem 6.2.

(i) *If* $d=0$*, then* $V'(K;-1) \equiv 0 \pmod{24}$*,* $V(K;-1) \equiv \varepsilon \pmod 3$*, and* $V(K; e^{i\pi/3}) = \varepsilon$*, where* $\varepsilon = \pm 1$*.*

(ii) *If* $d=1$*, then* $V'(K;-1) \equiv 8\varepsilon \pmod{24}$ *and* $V(K; e^{i\pi/3}) = i\varepsilon\sqrt{3}$*, where* $\varepsilon = \pm 1$*.*

(iii) *If* $d \geq 2$*, then* $V'(K;-1) \equiv 0 \pmod{24}$*.*

Proof. There are polynomials $P(t)$ in $\boldsymbol{Z}[t^{\pm 1}]$ and integers a, b such that

$$V(K;t) = (t^2 - t + 1)P(t) + at + b. \tag{57}$$

Then we have

$$V(K; e^{i\pi/3}) = \left(\frac{a}{2} + b\right) + \frac{\sqrt{3}a}{2} i; \tag{58}$$

$$V'(K;-1) \equiv a \pmod 3. \tag{59}$$

If d is even, then from Eqs. (56) and (58) we obtain $a=0$, and so by Eq. (59) and Lemma 5.1 we obtain $V'(K;-1) \equiv 0 \pmod{24}$. Furthermore, if $d=0$ then from Eqs. (57), (58) and (56) we have $V(K;-1) \equiv b \pmod 3$ and $V(K; e^{i\pi/3}) = b = \pm 1$.

If d is odd, then from Eqs. (56) and (58) we obtain $a = -2b$ and $b = \pm 3^{(d-1)/2}$. Furthermore, if $d > 1$ then by Eq. (59) and Lemma 5.1 we obtain $V'(K;-1) \equiv 0 \pmod{24}$. If $d=1$, then $b = \pm 1$, and so from Eq. (58) we obtain $V(K; e^{i\pi/3}) = -bi\sqrt{3}$ and from Eq. (59) we obtain $V'(K;-1) \equiv -2b \equiv b \pmod 3$. Using Lemma 5.1 we obtain $V'(K;-1) \equiv -8b \pmod{24}$. This completes the proof. □

Theorem 5.7 together with Theorem 6.2 imply the following:

Corollary 6.3. *Let* K *be a knot with* $\mathrm{u}_2(K) = 1$*. Suppose that* $\det(K) \equiv 0 \pmod 3$ *and* $\sigma(K) \equiv 2\varepsilon \pmod 8$, $\varepsilon = \pm 1$*. Then*

(i) *If* $\mathrm{Arf}(K) = 0$*, then* $V'(K;-1) \equiv 8\varepsilon \pmod{24}$ *and* $V(K; e^{i\pi/3}) = i\varepsilon\sqrt{3}$*.*

(ii) *If* $\mathrm{Arf}(K) = 1$*, then* $V'(K;-1) \equiv -8\varepsilon \pmod{24}$ *and* $V(K; e^{i\pi/3}) = -i\varepsilon\sqrt{3}$*.*

Proof. For a knot K with $\mathrm{u}_2(K) = 1$ and $\det(K) \equiv 0 \pmod 3$, from Eq. (55) we have $\dim H_1(\Sigma(K); \boldsymbol{Z}_3) = 1$, and so combining Theorems 5.7 and 6.2(ii), we obtain the result. □

Example 6.1. In Example 5.1 we have shown $\mathrm{u}_2(8_{21}) > 1$ using Theorem 5.7, which is also proved by using the value of the Jones polynomial at $t = e^{i\pi/3}$

in Corollary 6.3. In fact, $\det(8_{21}) = 15$, $\sigma(8_{21}) = 2$, $\mathrm{Arf}(8_{21}) = 0$, and $V(8_{21}; e^{i\pi/3}) = -i\sqrt{3}$.

In Table 1 we list all prime knots K with up to 10 crossings and their invariants such that we can prove $\mathrm{u}_2(K) > 1$ in a similar way; all of them are eventually known to be of $H(2)$unknotting number two. For the knots 9_{35}, 9_{40}, 9_{47}, 9_{48} and 10_{74} we can evaluate using the minimum number of generators of the first integral homology group of the double cover of S^3, $H_1(\Sigma(K); \boldsymbol{Z})$ (Theorem 2.1).

Table 1. Invariants of knots with $H(2)$-unknotting number two

K	$\mathrm{u}(K)$	$\sigma(K)$	$\mathrm{Arf}(K)$	$V(K;-1)$ (mod 24)	$V'(K;-1)$ (mod 24)	$V(K;e^{i\pi/3})$	$H_1(\Sigma(K);\boldsymbol{Z})$
8_{21}	1	2	0	9	-8	$-i\sqrt{3}$	$\boldsymbol{Z}_{15}$
9_2	1	2	0	9	-8	$-i\sqrt{3}$	$\boldsymbol{Z}_{15}$
9_{16}	3	-6	0	9	-8	$-i\sqrt{3}$	$\boldsymbol{Z}_{39}$
9_{35}	3	2	1	21	0	3	$\boldsymbol{Z}_3 \oplus \boldsymbol{Z}_9$
9_{40}	2	2	1	21	8	$i\sqrt{3}$	$\boldsymbol{Z}_5 \oplus \boldsymbol{Z}_{15}$
9_{47}	2	-2	1	21	0	3	$\boldsymbol{Z}_3 \oplus \boldsymbol{Z}_9$
9_{48}	2	-2	1	21	0	3	$\boldsymbol{Z}_3 \oplus \boldsymbol{Z}_9$
10_9	1	-2	0	9	8	$i\sqrt{3}$	$\boldsymbol{Z}_{39}$
10_{19}	2	2	1	21	8	$i\sqrt{3}$	$\boldsymbol{Z}_{51}$
10_{36}	2	2	1	21	8	$i\sqrt{3}$	$\boldsymbol{Z}_{51}$
10_{74}	2	2	0	9	0	-3	$\boldsymbol{Z}_3 \oplus \boldsymbol{Z}_{21}$
10_{77}	2 or 3	-2	0	9	8	$i\sqrt{3}$	$\boldsymbol{Z}_{63}$
10_{82}	1	2	0	9	-8	$-i\sqrt{3}$	$\boldsymbol{Z}_{63}$
10_{84}	1	-2	0	9	8	$i\sqrt{3}$	$\boldsymbol{Z}_{87}$
10_{89}	2	2	1	21	8	$i\sqrt{3}$	$\boldsymbol{Z}_{99}$
10_{112}	2	2	0	9	-8	$-i\sqrt{3}$	$\boldsymbol{Z}_{87}$
10_{113}	1	-2	0	9	8	$i\sqrt{3}$	$\boldsymbol{Z}_{111}$
10_{136}	1	-2	0	9	8	$i\sqrt{3}$	$\boldsymbol{Z}_{15}$
10_{159}	1	2	0	9	-8	$-i\sqrt{3}$	$\boldsymbol{Z}_{39}$
10_{164}	2	-2	1	21	-8	$-i\sqrt{3}$	$\boldsymbol{Z}_{51}$

We define the sign of a positive crossing as $+1$ and that of a negative crossing as -1.

Lemma 6.4. *Let K be a knot with* $\mathrm{u}(K) = 1$ *and* $\det(K) \equiv 0 \pmod 3$*. If K can be unknotted by changing a crossing of sign ε, then $V(K; e^{i\pi/3}) = i\varepsilon\sqrt{3}$.*

Proof. Since $\mathrm{u}(K) = 1$, $H_1(\Sigma(K); \boldsymbol{Z})$ is cyclic; cf. [15, Theorem 11.5.2]. Thus since $\det(K) \equiv 0 \pmod 3$, we have $\dim H_1(\Sigma(K); \boldsymbol{Z}_3) = 1$, and so by Proposition 6.1 we have $V(K; e^{i\pi/3}) = \pm i\sqrt{3}$.

Suppose that K can be unknotted by changing a positive crossing. Then there is an oriented skein quadruple $(L_+, L_-, L_0, L_\infty)$, where L_+ is isotopic to K and L_- is trivial. Then by Proposition 4.2 we have

$$V(K; t) - t + t^{3\mathrm{lk}(L_0)}(t-1)V(L_\infty; t) = 0, \tag{60}$$

from which, we have

$$V(K; -1) + 1 - 2(-1)^{\mathrm{lk}(L_0)}V(L_\infty; -1) = 0. \tag{61}$$

Then since $V(K; -1) \equiv \det(K) \equiv 0 \pmod 3$, $V(L_\infty; -1) \not\equiv 0 \pmod 3$, and so $\dim H_1(\Sigma(L_\infty); \boldsymbol{Z}_3) = 0$. Then by Proposition 6.1 we have $V(L_\infty; e^{i\pi/3}) = \pm 1$. Putting $V(L_\infty; e^{i\pi/3}) = \theta$, from Eq. (60) we have

$$\begin{aligned} V(K; e^{i\pi/3}) &= e^{i\pi/3} - (-1)^{\mathrm{lk}(L_0)}(e^{i\pi/3} - 1)\theta \\ &= \frac{1}{2}\left(1 + (-1)^{\mathrm{lk}(L_0)}\theta\right) + i\frac{\sqrt{3}}{2}\left(1 - (-1)^{\mathrm{lk}(L_0)}\theta\right), \end{aligned} \tag{62}$$

and therefore $V(K; e^{i\pi/3}) = i\sqrt{3}$.

The other case can be proved similarly, and the proof is complete. □

The following theorem enables us to show $\mathrm{u}_2(K) = 2$ knowing $\mathrm{u}(K) = 1$ without calculating the values $V'(K; -1)$ or $V(K; e^{i\pi/3})$.

Theorem 6.5. *Suppose that K is a knot satisfying* $|\sigma(K)| = 2$, $\mathrm{Arf}(K) = 0$, *and* $\det(K) \equiv 0 \pmod 3$*. Then* $\mathrm{u}(K)\mathrm{u}_2(K) > 1$*. Furthermore, if* $\mathrm{u}(K) = 1$, *then* $\mathrm{u}_2(K) = 2$.

Proof. Suppose that $\mathrm{u}(K) = \mathrm{u}_2(K) = 1$. Let $\sigma(K) = 2\varepsilon$, where $\varepsilon = \pm 1$. Then by Proposition 2.2(ii) K can be unknotted by switching a crossing of sign $-\varepsilon$. Then by Lemma 6.4 we have $V(K; e^{i\pi/3}) = -i\varepsilon\sqrt{3}$. On the other hand, by Corollary 6.3 we have $V(K; e^{i\pi/3}) = i\varepsilon\sqrt{3}$; a contradiction.

Furthermore, if $\mathrm{u}(K) = 1$, then $\mathrm{u}_2(K) \geq 2$ and also by Theorem 3.1 $\mathrm{u}_2(K) \leq 2$, and so we obtain $\mathrm{u}_2(K) = 2$. □

Remark 6.1. Lemma 6.4 follows from Theorem 3.1 in Traczyk [35], which is, however, incorrectly stated. Let $K = 10_{67}$. Using this theorem, Traczyk shows that $\mathrm{u}(K) = 2$ [35, Example 4.8]. We can prove $\mathrm{u}(K) = 2$ using Theorem 6.5 without calculating the value $V(K; e^{i\pi/3})$. In fact, $|\sigma(K)| = 2$, $\mathrm{Arf}(K) = 0$ and $\det(K) = 63$, and so Theorem 6.5 implies $\mathrm{u}(K)\mathrm{u}_2(K) > 1$. Also we know $\mathrm{u}_2(K) = 1$ [26] and $\mathrm{u}(K) \leq 2$, we conclude that $\mathrm{u}(K) = 2$.

7. Discussion for an $H(2)$-unknotting number one knot

In Secs. 4–6, we have given several criteria for an $H(2)$-unknotting number one knot. The situation is quite complicated, and so we summarize in Table 2, where all the possible cases of the following values of an $H(2)$-unknotting number one knot are given; the signature $\pmod 8$, Arf invariant, Jones polynomial at $t = -1 \pmod{24}$, first derivative of the Jones polynomial at $t = -1 \pmod{24}$, Jones polynomial at $t = e^{\pi i/3}$. In addition, we give a criterion for an amphicheiral knot with $H(2)$-unknotting number one using only the determinant (Corollary 7.2).

In order to complete Table 2 we need the following Theorem, which gives a further necessary condition for an $H(2)$-unknotting number one knot with signature $\equiv 0 \pmod 4$ and determinant $\equiv 0 \pmod 3$.

Theorem 7.1. *Let K be a knot with $\mathrm{u}_2(K) = 1$ and $\det(K) \equiv 0 \pmod 3$. If either $\sigma(K) \equiv 0$ or $4 \pmod 8$, then $V'(K; -1) \equiv \pm 8 \pmod{24}$.*

Proof. Since $\mathrm{u}_2(K) = 1$, from Lemma 4.1 there exists an oriented skein quadruple (L_+, L_-, L_0, K) such that L_- is a trivial knot. Suppose that $\sigma(K) \equiv 0 \pmod 8$. Then by Theorem 4.5 we have $\mathrm{Arf}(K) = 0$. Then by Lemma 5.6 $V(K; -1) \equiv 1 \pmod 8$, and since $\det(K) = |V(K; -1)| \equiv 0 \pmod 3$, we obtain $V(K; -1) \equiv 9 \pmod{24}$. On the other hand, by Eqs. (23) and (24) in Lemma 4.4 we have $\mathrm{lk}(L_0) \equiv 0$ or $1 \pmod 4$. Then by Lemma 5.5 we obtain the result.

Similarly, if $\sigma(K) \equiv 4 \pmod 8$, then we have $\mathrm{Arf}(K) = 1$, $V(K; -1) \equiv 21 \pmod{24}$ and $\mathrm{lk}(L_0) \equiv 2$ or $3 \pmod 4$, and then we obtain the result, completing the proof. □

Remark 7.1. Let K be a knot satisfying either

(i) $\sigma(K) \equiv 0 \pmod 8$ and $\mathrm{Arf}(K) = 0$; or
(ii) $\sigma(K) \equiv 4 \pmod 8$ and $\mathrm{Arf}(K) = 1$.

Then we can not deduce $\mathrm{u}_2(K) > 1$ only by Theorem 4.5. If further $\det(K) \equiv 0 \pmod 3$ and $V'(K; -1) \not\equiv \pm 8 \pmod{24}$, then by Theorem 7.1, we obtain $\mathrm{u}_2(K) > 1$. However, this also follows from Theorem 2.1. In fact, using Theorem 6.2, we have $\dim H_1(\Sigma(K); \boldsymbol{Z}_3) \geq 2$.

For example, let $K = 9_{46}$. Then $\sigma(K) = 0$ and $\mathrm{Arf}(K) = 0$. From $V(K; t) = 2 - t^{-1} + t^{-2} - 2t^{-3} + t^{-4} - t^{-5} + t^{-6}$, we have $V(K; -1) = 9$ and $V'(K; -1) = 24 \equiv 0 \pmod{24}$. By Theorem 7.1, we obtain $\mathrm{u}_2(K) > 1$. On the other hand, since $H_1(\Sigma(K); \boldsymbol{Z}_3) = \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3$, by Theorem 2.1 or Eq. (55) we have $\mathrm{u}_2(K) > 1$. Since K can be unknotted by applying two $H(2)$-moves, K has $H(2)$-unknotting number 2; see Sec. 9.2.

Now we can complete Table 2 summarizing the previous results: Theorem 4.5, Lemma 5.6, Theorem 5.7, Lemma 5.1, Theorem 6.2, Corollary 6.3, Theorem 7.1. Note that if a knot K satisfies the conditions in Case (Xn) (X=A, B, C; n=1, 2, 3), then its mirror image $K!$ satisfies the conditions in Case (Xn!) except for (A1), (A2), (B1), (B2).

Table 2. A knot K with $u_2(K) = 1$.

Case	$\sigma(K)$ (mod 8)	$\mathrm{Arf}(K)$	$V(K;-1)$ (mod 24)	$V'(K;-1)$ (mod 24)	$V(K;e^{\pi i/3})$
A1	0	0	1	0	1
A2	0	0	17	0	-1
A3	0	0	9	8	$i\sqrt{3}$
A3!	0	0	9	-8	$-i\sqrt{3}$
B1	4	1	5	0	-1
B2	4	1	13	0	1
B3	4	1	21	8	$i\sqrt{3}$
B3!	4	1	21	-8	$-i\sqrt{3}$
C1	2	0	1	0	1
C2	2	0	17	0	-1
C3	2	0	9	8	$i\sqrt{3}$
D1	2	1	5	0	-1
D2	2	1	13	0	1
D3	2	1	21	-8	$-i\sqrt{3}$
C1!	-2	0	1	0	1
C2!	-2	0	17	0	-1
C3!	-2	0	9	-8	$-i\sqrt{3}$
D1!	-2	1	5	0	-1
D2!	-2	1	13	0	1
D3!	-2	1	21	8	$i\sqrt{3}$

Example 7.1. Let T_m be the torus knot of type $(2, 2m+1)$ having $(2m+1)$ negative crossings if $m > 0$ and $(-2m-1)$ positive crossings if $m < 0$. Then it is easy to see $u_2(T_m) = 1$ if $m \neq 0, -1$. We have:

$$\mathrm{Arf}(T_m) = m(m+1)/2 \pmod 2; \tag{63}$$

$$\sigma(T_m) = \begin{cases} 2m & \text{if } m > 0; \\ 2m+2 & \text{if } m < 0; \end{cases} \tag{64}$$

cf. [13, Eq. (10)], [25, Theorem 7.5.1]. Also using Proposition 11.9 in [11], we have

$$V(T_m;t) = t^{-3m-1}\left(t^{2m} + \frac{t^{2m+2}-1}{t+1}\right), \tag{65}$$

which yields:

$$V(T_m;-1) = (-1)^m(2m+1); \tag{66}$$

$$V'(T_m;-1) \equiv \begin{cases} -8(-1)^k \pmod{24} & \text{if } m = 3k+1; \\ 0 \pmod{24} & \text{otherwise;} \end{cases} \tag{67}$$

$$V(T_m;e^{i\pi/3}) = \begin{cases} (-1)^m & \text{if } m \equiv 0 \pmod 3; \\ i(-1)^m\sqrt{3} & \text{if } m \equiv 1 \pmod 3; \\ (-1)^{m+1} & \text{if } m \equiv 2 \pmod 3. \end{cases} \tag{68}$$

Thus each T_m satisfies the conditions in Table 2 as shown in Table 3, from which we see that each case in Table 2 can occur.

TABLE 3. 2-braid torus knot T_m.

$2m+1(>0) \pmod{24}$	Case	$2m+1(<0) \pmod{24}$	Case
1	A1	-1	A1
3	D3	-3	D3!
5	B1	-5	B1
7	C2!	-7	C2
9	A3	-9	A3!
11	D2	-11	D2!
13	B2	-13	D2
15	C3!	-15	C3
17	A2	-17	A2
19	D1	-19	D1!
21	B3	-21	B3!
23	C1!	-23	C1

As a corollary of Theorem 7.1, we have the following, which gives a condition for the determinant of an amphicheiral knot with $H(2)$-unknotting number one.

Corollary 7.2. *Let K be an amphicheiral knot. If* $\det(K) \equiv 5, 9, 13,$ *or* $21 \pmod{24}$*, then* $\mathrm{u}_2(K) > 1$.

Proof. Let K be an amphicheiral knot with $\mathrm{u}_2(K) = 1$. Then $\sigma(K) = 0$, and so by Theorem 4.5 and Lemma 5.6, $V(K;-1) \equiv 1 \pmod 8$. Thus if $\det(K) \equiv 5$, 13, or 21; or $\det(K) \equiv 9$ and $V(K;-1) \equiv 15 \pmod{24}$, then $\mathrm{u}_2(K) > 1$.

Suppose that $V(K;-1) \equiv 9 \pmod{24}$, then since K is amphicheiral, we have $V(K;t) = V(K;t^{-1})$ and thus $V'(K;-1) = 0$. Therefore, by Theorem 7.1 we obtain $\mathrm{u}_2(K) > 1$. □

Example 7.2. Among all prime amphicheiral knots with up to 10 crossings, using Corollary 7.2, we may conclude each of the following knots has $H(2)$-unknotting number > 1:

$$4_1,\ 6_3,\ 8_{17},\ 8_{18},\ 10_{37},\ 10_{79},\ 10_{81},\ 10_{88},\ 10_{99},\ 10_{109},\ 10_{115}.$$

8. The Q polynomial

The *Q polynomial* $Q(L;z) \in \boldsymbol{Z}[z^{\pm 1}]$ [2, 8] is an invariant of the isotopy type of an unoriented link L, which is defined by the following formulas:

$$Q(U;z) = 1; \tag{69}$$

$$Q(L_+;z) + Q(L_-;z) = z\left(Q(L_0;z) + Q(L_\infty;z)\right). \tag{70}$$

where U is the unknot and L_+, L_-, L_0, L_∞ are four unoriented links that are identical except near one point where they are as in Fig. 10. We call $(L_+, L_-, L_0, L_\infty)$ an *unoriented skein quadruple.*

Figure 10. An unoriented skein quadruple.

Let $\rho(K) = Q\left(K; (\sqrt{5}-1)/2\right)$. Then Jones [10] has shown:

$$\rho(K) = \pm\sqrt{5}^{r}, \tag{71}$$

where $r = \dim H_1(\Sigma(K); \boldsymbol{Z}_5)$ with $\Sigma(K)$ the double cover of S^3 branched over K. His proof uses the Birman-Wenzel algebra. Furthermore, Rong [32] has given a topological proof, from which he deduced some information on the values $\rho(L_-)/\rho(L_\infty)$, $\rho(L_0)/\rho(L_\infty)$, $\rho(L_+)/\rho(L_\infty)$, where $(L_+, L_-, L_0, L_\infty)$ is an unoriented skein quadruple. Using these values, Stoimenow [33, Theorem 4.1] has given some criterion on the unknotting number of a knot. Similarly, we can obtain a criterion on the $H(2)$-Gordian distance and $H(2)$-unknotting number.

Theorem 8.1. *For knots K and K', if $\mathrm{d}_2(K,K') = n$, then*

$$\rho(K)/\rho(K') \in \left\{ \pm\sqrt{5}^{k},\ \sqrt{5}^{\pm n} \;\middle|\; k = 0, \pm 1, \ldots, \pm(n-1) \right\}. \tag{72}$$

Proof. We prove by induction on n. Suppose that K and K' are related by a single $H(2)$-move; $\mathrm{d}_2(K,K')=1$. Then there exist two links L_+, L_- such that $(L_+,L_-;K,K')$ is an unoriented skein quadruple. So we have $\rho(L_+)+\rho(L_-)=((\sqrt{5}-1)/2)\,(\rho(K)+\rho(K'))$. Then from the proof of Theorem 2 in [32], we have $\rho(K)/\rho(K')\in\left\{\ \pm 1,\ \sqrt{5}^{\pm 1}\right\}$.

Next, assume that Eq. (72) holds for knots K, K' with $\mathrm{d}_2(K,K')\leq n$. Let K_0, K_1 be knots with $\mathrm{d}_2(K_0,K_1)=n+1$. Then there is a knot K_2 with $\mathrm{d}_2(K_0,K_2)=n$ and $\mathrm{d}_2(K_1,K_2)=1$, and so

$$\rho(K_0)/\rho(K_2)\in\left\{\ \pm\sqrt{5}^{k},\ \sqrt{5}^{\pm n}\ \middle|\ k=0,\pm 1,\ldots,\pm(n-1)\ \right\}; \tag{73}$$

$$\rho(K_2)/\rho(K_1)\in\left\{\ \pm 1,\ \sqrt{5}^{\pm 1}\ \right\}, \tag{74}$$

which imply

$$\rho(K_0)/\rho(K_1)\in\left\{\ \pm\sqrt{5}^{k},\ \sqrt{5}^{\pm(n+1)}\ \middle|\ k=0,\pm 1,\ldots,\pm(n-1),\ \pm n\ \right\}, \tag{75}$$

completing the proof. □

Theorem 2.1 implies that if $\rho(K)=\pm\sqrt{5}^{n}$, then $\mathrm{u}_2(K)\geq n$. Furthermore, we obtain the following immediately from Theorem 8.1.

Corollary 8.2. *For a knot K, if $\rho(K)=-\sqrt{5}^{n}$, then $\mathrm{u}_2(K)\geq n+1$.*

Example 8.1. The $H(2)$-unknotting number of each of the knots 8_9 and 8_{21} is 2; $\mathrm{u}_2(8_9)=\mathrm{u}_2(8_{21})=2$. In fact, since $Q(8_9;z)=-7+4z+16z^2-10z^3-16z^4+4z^5+8z^6+2z^7$, $Q(8_{21};z)=-7+8z+6z^2-12z^3-2z^4+6z^5+2z^6$, we have $\rho(8_9)=\rho(8_{21})=-\sqrt{5}$, and so by Corollary 8.2 we have $\mathrm{u}_2(8_9)$, $\mathrm{u}_2(8_{21})>1$. Since it is known that $\mathrm{u}(8_9)=\mathrm{u}(8_{21})=1$, by Theorem 3.1 we have $\mathrm{u}_2(8_9)$, $\mathrm{u}_2(8_{21})\leq 2$, and so we obtain the results. Note that since $\sigma(8_9)=0$, $\mathrm{Arf}(8_9)=0$, for the knot 8_9 we can not use the criterion in Theorem 4.5, however for the knot 8_{21} we can use Corollary 6.3 (Examples 5.1 and 6.1).

For each of the following 15 knots, Nakajima [26] has shown that the $H(2)$-unknotting number is ≤ 2, and the value of the ρ-invariant is $-\sqrt{5}$, and so from Corollary 8.2 we can conclude that each of them has $H(2)$-unknotting number 2:

$8_9,\ 8_{21},\ 9_2,\ 9_{12},\ 9_{39},\ 10_{18},\ 10_{33},\ 10_{58},\ 10_{59},\ 10_{129},\ 10_{136},$
$10_{137},\ 10_{138},\ 10_{156},\ 10_{163}$ (with determinant 35).

Similarly, for the following two knots, Nakajima [26] has shown that the $H(2)$-unknotting number is ≤ 3 , and the value of the ρ-invariant is -5, and so from Corollary 8.2 we can conclude that each of them has $H(2)$-unknotting number 3:

$$9_{49},\quad 10_{103}.$$

9. Tables of $H(2)$-unknotting numbers of knots with up to 9 crossings

In this section, we give tables of $H(2)$-unknotting numbers of knots with up to 9 crossings. This improves Nakajima's table [26], which lists the $H(2)$-unknotting numbers of prime knots with up to 10 crossings containing the result of Lickorish. He uses Theorem 2.1 with $r = 2$ and Theorem 4.5 to give lower bounds, and Theorem 3.1 to give upper bounds. He also gives several diagrams of knots with twisted bands to show upper bounds.

9.1. **Prime knots with up to 8 crossings.** Table 4 lists the $H(2)$-unknotting numbers of prime knots with up to 8 crossings together with the absolute values of the signatures, Arf invariants, unknotting numbers, and the method to decide the $H(2)$-unknotting number. The signatures, Arf invariants ($\equiv a_2(K) \pmod 2$), and unknotting numbers are taken from the tables in [5, Appendices B and C]. We use the following methods to decide the lower bound of the $H(2)$-unknotting number of the knots in Table 4.

- (I_n) If the minimum number of generators of $H_1(\Sigma(K); \boldsymbol{Z})$ is n, then $\mathrm{u}_2(K) \geq n$.
- (II) If $\sigma(K) = 0$, $\mathrm{Arf}(K) = 1$, then $\mathrm{u}_2(K) > 1$.
- (II$'$) In addition, if $\mathrm{u}(K) = 1$, then we can conclude $\mathrm{u}_2(K) = 2$.
- (III) If $\sigma(K) = \pm 4$, $\mathrm{Arf}(K) = 0$, then $\mathrm{u}_2(K) > 1$.
- (III$'$) In addition, if $\mathrm{u}(K) = 2$, then we can conclude $\mathrm{u}_2(K) = 2$.
- (IV) Suppose that $\sigma(K) \equiv 2\epsilon \pmod 8$, $\mathrm{Arf}(K) = 0$, and $\det(K) \equiv 0 \pmod 3$. If either $V'(K; -1) \not\equiv 8\varepsilon \pmod{24}$ or $V(K; e^{i\pi/3}) \neq \varepsilon i\sqrt{3}$, then $\mathrm{u}_2(K) > 1$.
- (IV$'$) In addition, if $\mathrm{u}(K) = 1$, then we can conclude $\mathrm{u}_2(K) = 2$.
- (V_n) If $\rho(K) = -\sqrt{5}^{n-1}$, then $\mathrm{u}_2(K) \geq n$.
- (V'_n) In addition, if $\mathrm{u}(K) = n - 1$, then we can conclude $\mathrm{u}_2(K) = n$.

Method I_n follows from Theorem 2.1. For $H_1(\Sigma(K); \boldsymbol{Z})$, we use Table 1 in Appendix C of [3]. Methods II and III follow from Theorem 4.5. Method IV follows from Corollary 6.3. Method V_n follows from Corollary 8.2; see Example 8.1 for 8_9, 8_{21}. Methods II$'$, III$'$ (Propositions 3.2.11 and 3.2.12 in [26]), and Methods IV'_n, V'_n follow from Theorem 3.1. Notice that if $\mathrm{u}(K) = 2$ and $\sigma(K) = \pm 4$, then either $u_+ = 2$ or $u_- = 2$.

For the $H(2)$-unknotting number one knots except the 2-braid torus knots (3_1, 5_1, 7_1), we show diagrams with twisted bands that change to the unknot in Fig. 11; 5_2, 6_1, 6_2, 7_3, 7_4, 7_6 are due to Lickorish, 8_3–8_6, 8_8, 8_{19}, 8_{20} are due to Nakajima [26]. For $\mathrm{u}_2(8_2)$, $\mathrm{u}_2(8_{12}) \leq 2$, we show in Fig. 12; adding twisted bands shown there, 8_2, 8_{12} becomes 3_1, 7_6, respectively, both of which are of $H(2)$-unknotting number one.

Remark 9.1. For the knots with up to 7 crossings we can decide the $H(2)$-unkotting number only by using Lickorish's method [16]. In fact, he mentions that among prime knots with up to 7 crossings 4_1, 6_3, 7_2, 7_4, 7_5 and

Table 4. $H(2)$-unknotting numbers of prime knots with up to 8 crossings

K	$u_2(K)$	$\lvert\sigma\rvert$	Arf	$u(K)$	Method	K	$u_2(K)$	$\lvert\sigma\rvert$	Arf	$u(K)$	Method
3_1	1	2	1	1		8_5	1	4	1	2	
4_1	2	0	1	1	II′ or V'_2	8_6	1	2	0	2	
5_1	1	4	1	2		8_7	1	2	0	1	
5_2	1	2	0	1		8_8	1	0	0	2	
6_1	1	0	0	1		8_9	2	0	0	1	V'_2
6_2	1	2	1	1		8_{10}	1	2	1	2	
6_3	2	0	1	1	II′	8_{11}	1	2	1	1	
7_1	1	6	0	3		8_{12}	2	0	1	2	II
7_2	1	2	1	1		8_{13}	2	0	1	1	II′
7_3	1	4	1	2		8_{14}	1	2	0	1	
7_4	1	2	0	2		8_{15}	2	4	0	2	III′
7_5	2	4	0	2	III′	8_{16}	1	2	1	2	
7_6	1	2	1	1		8_{17}	2	0	1	1	II′
7_7	2	0	1	1	II′	8_{18}	2 or 3	0	1	2	I_2 or II
8_1	2	0	1	1	II′	8_{19}	1	6	1	3	
8_2	2	4	0	2	III	8_{20}	1	0	0	1	
8_3	1	0	0	2		8_{21}	2	2	0	1	IV′ or V'_2
8_4	1	2	1	2							

7_7 are of $H(2)$-unknottiing number greater than one and others are of $H(2)$-unknottiing number one. However, soon after the publication of the article [16], he informed of a corrected table for the $H(2)$-unknottiing numbers of knots with up to 7 crossings, that is, he shows $u_2(7_4) = 1$, together with $u_2(3_1\#4_1) = 1$ as Scharlemann's example. Moreover, as is shown in Fig. 11, $u_2(7_2) = 1$.

9.2. **Prime knots with 9 crossings.** Table 5 lists the $H(2)$-unknotting numbers of prime knots with 9 crossings in a similar way to Table 4; the unknotting numbers of 9_{10}, 9_{13} 9_{35} 9_{38} have been determined in [28].

For the $H(2)$-unknotting number one knots except the $(2,9)$-torus knot 9_1, we show diagrams with twisted bands that change to the unknot in Fig. 13; 9_4, 9_5, 9_{15}, 9_{19}, 9_{22}, 9_{43}, 9_{44} are due to Nakajima [26]. For knots with $H(2)$-unknotting number ≤ 2, we show diagrams with twisted bands that change to $H(2)$-unknotting number one knots listed there in Fig. 14; for example, the knot 9_{10} is transformed into 5_2, whose $H(2)$-unknotting

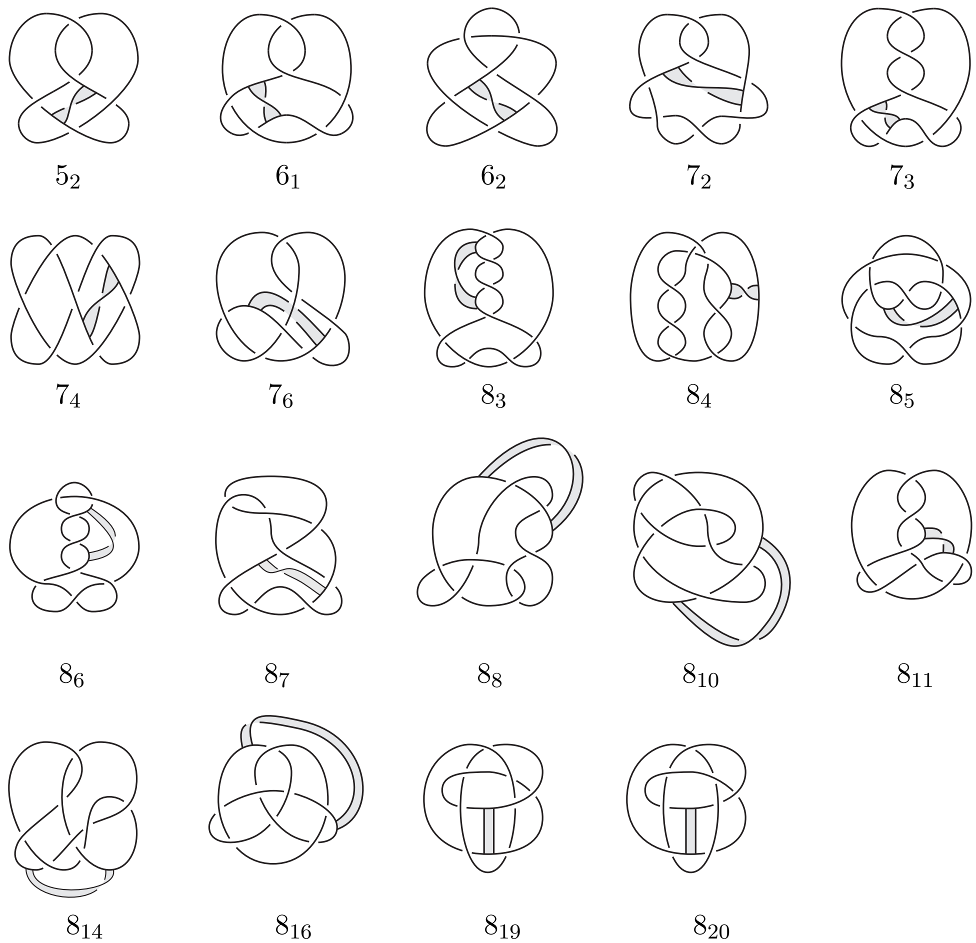
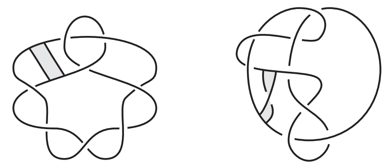


FIGURE 11. $H(2)$-unknotting number one prime knots with up to 8 crossings.



$8_2 \to 3_1$ $8_{12} \to 7_6$

FIGURE 12. $u_2(8_2)$, $u_2(8_{12}) \leq 2$.

number is one, by adding the twisted band shown in the diagram. The twisted bands for the knots 9_{10}, 9_{46} are due to Nakajima [26], where he also shows $u_2(9_{37}) \leq 2$ using Eq. (20) in Theorem 3.1. For $u_2(9_{49}) \leq 3$, Fig. 15 gives the twisted band that changes to 4_1, whose $H(2)$-unknotting number is two.

TABLE 5. $H(2)$-unknotting numbers of prime knots with up to 9 crossings

K	$u_2(K)$	$\|\sigma\|$	Arf	$u(K)$	Method	K	$u_2(K)$	$\|\sigma\|$	Arf	$u(K)$	Method
9_1	1	8	0	4		9_{26}	1 or 2	2	0	1	
9_2	2	2	0	1	V'$_2$	9_{27}	1	2	0	1	
9_3	1	6	1	3		9_{28}	1 or 2	2	1	1	
9_4	1	4	1	2		9_{29}	1	2	1	2	
9_5	1	2	0	2		9_{30}	2	0	1	1	II_2
9_6	1	6	1	3		9_{31}	1 or 2	2	0	2	
9_7	1	4	1	2		9_{32}	1 or 2	2	1	2	
9_8	1	2	0	2		9_{33}	2	0	1	1	II'
9_9	1	6	0	3		9_{34}	2	0	1	1	II'
9_{10}	2	4	0	3	III	9_{35}	2	2	1	3	I_2
9_{11}	2	4	0	2	III'	9_{36}	1	4	1	2	
9_{12}	2	2	1	1	V'_2	9_{37}	2	0	1	2	II
9_{13}	1	4	1	3		9_{38}	2	4	0	3	III
9_{14}	2	0	1	1	II'	9_{39}	2	2	0	1	V'_2
9_{15}	1	2	0	2		9_{40}	2	2	1	2	
9_{16}	2	6	0	3	IV	9_{41}	2	0	0	2	
9_{17}	1	2	0	2		9_{42}	1	2	0	1	
9_{18}	2	4	0	2	III'	9_{43}	1	4	1	2	
9_{19}	1	0	0	1		9_{44}	1	0	0	1	
9_{20}	2	4	0	2	III'	9_{45}	1 or 2	2	0	1	
9_{21}	1 or 2	2	1	1		9_{46}	2	0	0	2	I_2
9_{22}	1	2	1	1		9_{47}	2	2	1	2	I_2
9_{23}	1 or 2	4	1	2		9_{48}	2	2	1	2	I_2
9_{24}	2	0	1	1	II'	9_{49}	3	4	0	3	V_3
9_{25}	1	2	0	2							

9.3. **Composite knots with up to 9 crossings.** Table 6 lists the $H(2)$-unknotting numbers of composite knots with up to 9 crossings in a similar way to Tabtle 4; the unknotting numbers are taken from the table in [33]; $u(3_1!\#5_1) \neq 2$ has not yet been proved.

To evaluate the upper bound we use the following trivail formula:

$$u_2(K\#K') \leq u_2(K) + u_2(K'), \tag{76}$$

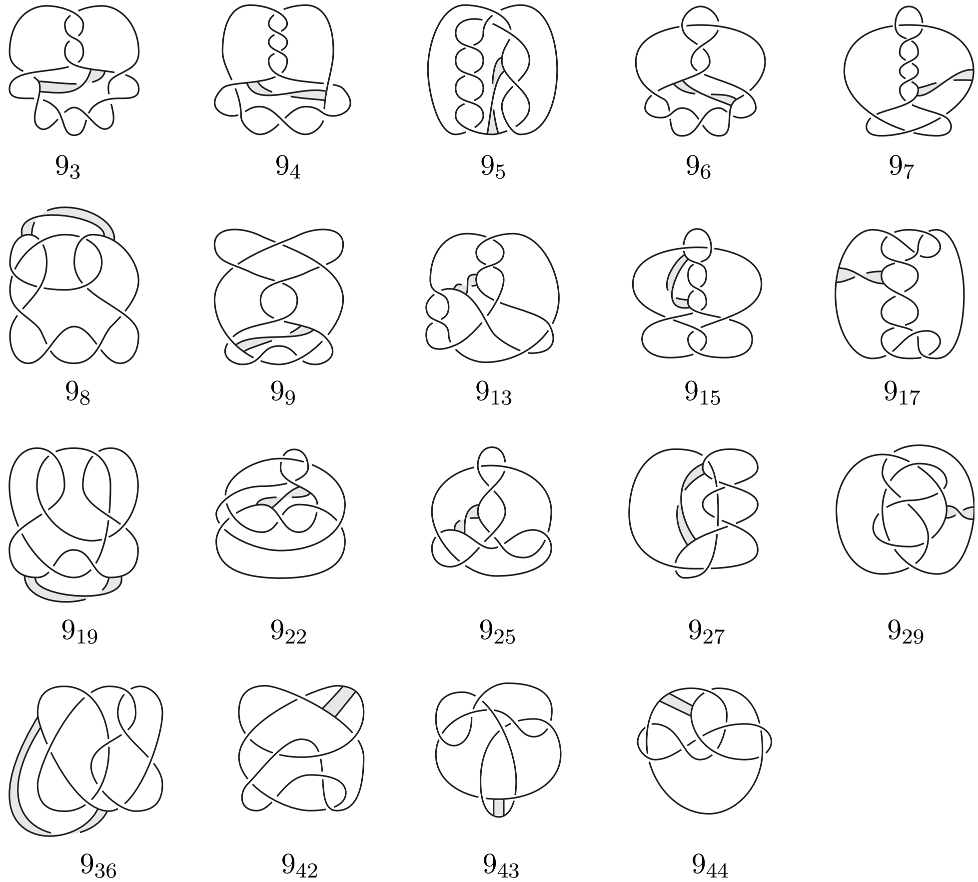


FIGURE 13. $H(2)$-unknotting number one prime knots with 9 crossings.

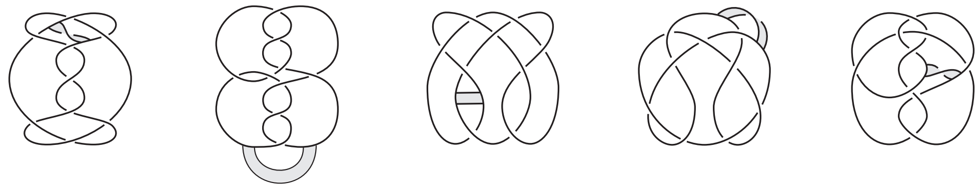
where K, K' are knots. For the $H(2)$-unknotting number one composite knots, we show diagrams with twisted bands that change to the unknot in Fig. 16; $3_1\#4_1$ was given by Lickorish's handwritten table as Scharlemann's example. This example is generalized in [9, Fig. 5]; the compositions of two twisted knots with $H(2)$-unknotting number one are constructed, which was given by Hosokawa. The composite knot $4_1\#5_2$ is one of such examples.

The factor knots of composite knots with up to 9 crossings are of $H(2)$-unknotting number one except $\mathrm{u}_2(4_1) = \mathrm{u}_2(6_3) = 2$. For $\mathrm{u}_2(3_1\#6_3)$, $\mathrm{u}_2(4_1\#4_1) \leq 2$, we show in Fig. 17; adding twisted bands shown there, they become $6_2!$, 5_1, respectively, which are of $H(2)$-unknotting number one.

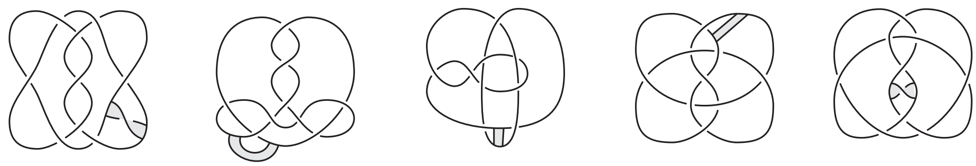
These examples are generalized as follows:

Theorem 9.1. *For any 2-bridge knot K, there exists a 2-bridge knot K' such that* $\mathrm{u}_2(K\#K') = 1$.

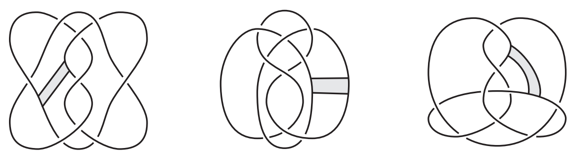
Proof. We can present a 2-bridge knot K as a 4-plat as shown in Fig. 18(a), where β is a pure 3-braid; cf. [15, Exercise 2.1.14], [3, Sec. 12.B]. Construct



$9_{10} \to 5_2$ $9_{16} \to 6_2!$ $9_{23} \to 7_6!$ $9_{31} \to 8_{14}$ $9_{32} \to 3_1!$

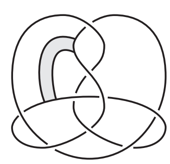


$9_{35} \to 7_4!$ $9_{37} \to 8_{10}$ $9_{38} \to 7_7!$ $9_{40} \to 8_{16}$ $9_{41} \to 8_{16}!$



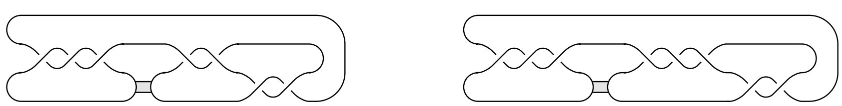
$9_{46} \to 3_1!$ $9_{47} \to 3_1!$ $9_{48} \to 6_1$

FIGURE 14. 9 crossing prime knots with $H(2)$-unknotting number ≤ 2.

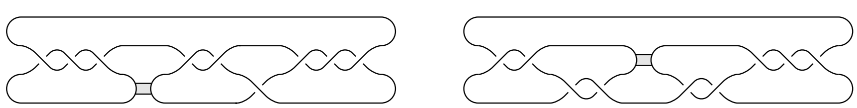


$9_{49} \to 4_1$

FIGURE 15. $u_2(9_{49}) \leq 3$.



$3_1\#4_1$ $3_1\#5_2$



$3_1\#6_2$ $4_1\#5_2$

FIGURE 16. $H(2)$-unknotting number one composite knots.

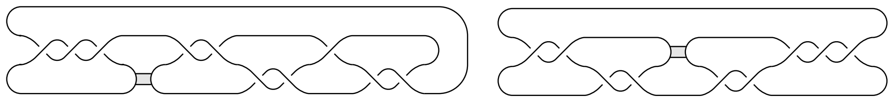
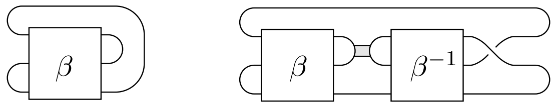


FIGURE 17. $H(2)$-unknotting number two composite knots.

TABLE 6. $H(2)$-unknotting numbers of composite knots with up to 9 crossings.

K	$u_2(K)$	$u(K)$	Method
$3_1\#3_1$	2	2	(I_2) $H_1(\Sigma(K);\boldsymbol{Z}) = \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3$.
$3_1!\#3_1$	2	2	(I_2) $H_1(\Sigma(K);\boldsymbol{Z}) = \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3$.
$3_1\#4_1$	1	2	
$3_1\#5_1$	2	2	(IV) $\sigma(K) = 6$, $\mathrm{Arf}(K) = 0$, $V(K;-1) = -15$, $V'(K;-1) \equiv 8 \pmod{24}$, $V(K;e^{i\pi/3}) = i\sqrt{3}$; or ($V_2$) $\rho(K) = -\sqrt{5}$.
$3_1!\#5_1$	2	3?	(IV) $\sigma(K) = 2$, $\mathrm{Arf}(K) = 0$, $V(K;-1) = -15$, $V'(K;-1) \equiv -8 \pmod{24}$, $V(K;e^{i\pi/3}) = -i\sqrt{3}$; or ($V_2$) $\rho(K) = -\sqrt{5}$.
$3_1\#5_2$	1	2	
$3_1!\#5_2$	2	2	(II) $\sigma(K) = 0$, $\mathrm{Arf}(K) = 1$
$4_1\#4_1$	2	2	(I_2) $H_1(\Sigma(K);\boldsymbol{Z}) = \boldsymbol{Z}_5 \oplus \boldsymbol{Z}_5$.
$3_1\#3_1\#3_1$	3	3	(I_3) $H_1(\Sigma(K);\boldsymbol{Z}) = \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3$.
$3_1!\#3_1\#3_1$	3	3	(I_3) $H_1(\Sigma(K);\boldsymbol{Z}) = \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3$.
$3_1\#6_1$	2	2	(I_2) $H_1(\Sigma(K);\boldsymbol{Z}) = \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_9$.
$3_1!\#6_1$	2	2	(I_2) $H_1(\Sigma(K);\boldsymbol{Z}) = \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_9$.
$3_1\#6_2$	2	2	(III) $\sigma(K) = 4$, $\mathrm{Arf}(K) = 0$
$3_1!\#6_2$	1	2	
$3_1\#6_3$	2	2	(IV) $\sigma(K) = 2$, $\mathrm{Arf}(K) = 0$, $V(K;-1) = -39$, $V'(K;-1) \equiv -8 \pmod{24}$, $V(K;e^{i\pi/3}) = -i\sqrt{3}$.
$4_1\#5_1$	3	3	(V_3) $\rho(K) = -5$
$4_1\#5_2$	1	2	

the connected sum of two 2-bridge knot $K\#K'$ as shown in Fig. 18(b), where β^{-1} is the inverse of β as an element of the 3-braid group. Then it can be changed to the unknot by adding the twisted band shown there. □



(a) (b)

FIGURE 18. (a) A 2-bridge knot K. (b) $K\#K'$.

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