Signed Gordian distance and Rasmussen invariants

Dedicated to Professor Akio Kawauchi for his 60th birthday

TOSHIFUMI TANAKA

ABSTRACT. We give some criterions for signed Gordian distance by using the Jones polynomial, the Q-polynomial and the Rasmussen invariant of a knot. As a result, we give new calculations of the Gordian distance for knots with low crossing number.

1. Introduction

A *link* is smoothly embedded circles in the 3-sphere $\mathbb{S}^3$. A *knot* is a link with one connected component. We assume that every link is oriented. A diagram of a link is a generic projection of a link to the 2-sphere in $\mathbb{S}^3$ with signed double points, called *positive* (or *negative*) *crossings* as in Figure 1. Let K and K' be knots in $\mathbb{S}^3$. The Gordian distance from K to K', denoted by $d_G(K, K')$, is the minimum number of crossing changes needed to transform a diagram of K into that of K', where the minimum is taken over all diagrams of K and K'. A $+-$ change (or $-+$ change) of a crossing is changing a positive (or a negative) crossing of a diagram. We

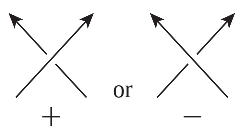


Figure 1

define $d_{+-}(K, K')$ (or $d_{-+}(K, K')$) as the minimum number of $+-$ (or $-+$) changes of crossings needed to transform a diagram of K into that of K' by $d_G(K, K')$ crossing changes, where the minimum is taken over all diagrams of K and K'. (See [2] in the case where K' is the unknot.) The Jones polynomial V is a Laurent polynomial in one variable t of an oriented link can be defined by the following relation.
(1) $V(\bigcirc; t) = 1$;
(2) $t^{-1}V(L_+; t) - tV(L_-; t) = -(t^{-1/2} - t^{1/2})V(L_0; t)$.
Here L_+, L_- and L_0 are three links with diagrams differing only near a crossing as in Figure 2.

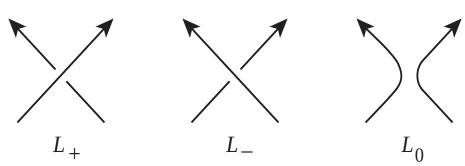
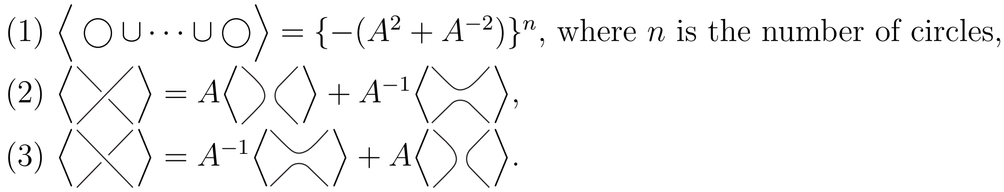


Figure 2

The Jones polynomial can be calculated from the *Kauffman bracket* <> [8]. Let D be an unoriented diagram of a link. Then the Laurent polynomial in A is defined by the following:



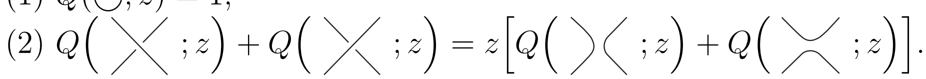
Then the Jones polynomial can be obtained as follows:
$V(L;t) = (-t^{-3/4})^{-w(D)} < D > |_{A=t^{-1/4}}$, where $w(D)$ is the writhe of D.

Set $\omega = e^{\pi\sqrt{-1}/3}$ and $\delta = (\sqrt{5}-1)/2$. In this paper, we show the following.

Theorem 1.1. *Let K and K' be knots in $\mathbb{S}^3$. Suppose $d_{+-}(K,K') = d_G(K,K') = 1$. Set $\overline{V}(t) = \dfrac{tV(K';t) - V(K;t)}{t-1}$. Then $\overline{V}(\omega) = \pm\omega^{\overline{V}'(1)}(\sqrt{-3})^d$ for some non-negative integer d.*

Theorem 1.2. *Let K and K' be knots in $\mathbb{S}^3$.*
Set $V(K;\omega) = (-1)^{s_1}(\sqrt{-3})^{d_1}$; $V(K';\omega) = (-1)^{s_2}(\sqrt{-3})^{d_2}$.
If $d_G(K,K') = d_1 - d_2$, then $d_{-+}(K,K') \equiv s_1 - s_2 \mod 2$.

The Q polynomial $Q(K;z)$ is a Laurent polynomial in one variable z of an oriented link can be defined by the following.
(1) $Q(\bigcirc; z) = 1$;



Theorem 1.3. *Let K and K' be knots in $\mathbb{S}^3$. If $Q(K;\delta)/Q(K';\delta) = -(-\sqrt{5})^k$, then $d_G(K,K') > |k|$.*

Two links are *concordant* if there is a smooth embedding

$$(nS^1) \times [0,1] \to S^3 \times [0,1]$$

which restricts to the given links

$$(nS^1) \times \{i\} \to S^3 \times \{i\}$$

where $i = 0, 1$. The set of concordance classes of knots forms an abelian group under connected sum. The group is called the *knot concordance group.*

Recently, Rasmussen has defined an effective concordance invariant $s(K)$ of a knot K from Lee's cohomology [4]. (We call the invariant the *Rasmussen invariant.*) Main properties of Rasmussen invariant are summarized as follows.

Theorem 1.4. *Let K, K_1 and K_2 be knots in S^3. Then we have the following.*
(1) *s induces a homomorphism from the knot concordance group to $\mathbb{Z}$;*
(2) *$|s(K)| \leq 2g_4(K)$, where $g_4(K)$ is the slice genus of K;*
(3) *If K is alternating, then $s(K) = \sigma(K)$, where $\sigma(K)$ is the classical knot signature of K;*
(4) *If K_1 is obtained from K_2 by performing a single positive crossing change, then $s(K_1) - s(K_2) \in \{0, 2\}$.*

We have the following.

Theorem 1.5. *let K and K' be two knots in S^3. Let $s(K, K') = \dfrac{s(K) - s(K')}{2}$. If $s(K, K') \geq 0$, then $d_{+-}(K, K') \geq s(K, K')$ and if $s(K, K') \leq 0$, then $d_{-+}(K, K') \geq -s(K, K')$. In particular, $d_G(K, K') \geq |s(K, K')|$.*

2. Proofs

Proof of Theorem 1.1. By assumption, we assume that a diagram D' of K' is obtained from a diagram D of K by a single $+-$ change of a crossing. We may assume D to have zero writhe adding kinks if necessarily. Note that
$V(K; A^{-4}) = \left\langle \times \right\rangle$ and $A^{-6}V(K; A^{-4}) = \left\langle \times \right\rangle$ since $w(D) = w(D') + 2$. By using the Kauffman bracket relation, we have
(1) $A\left\langle \rangle\langle \right\rangle + A^{-1}\left\langle \asymp \right\rangle = V(K; A^{-4})$,
(2) $A^{-1}\left\langle \rangle\langle \right\rangle + A\left\langle \asymp \right\rangle = A^{-6}V(K; A^{-4})$.
Thus $(A^2 - A^{-2})\left\langle \asymp \right\rangle = A^{-5}V(K'; A^{-4}) - A^{-1}V(K; A^{-4})$.
Then we obtain $\left\langle \asymp \right\rangle = \dfrac{A^{-3}[A^{-2}V(K'; A^{-4}) - A^2V(K; A^{-4})]}{A^2 - A^{-2}} =$
$-A^{-3}\Big[\dfrac{A^{-4}V(K'; A^{-4}) - V(K; A^{-4})}{A^{-4} - 1}\Big] = -A^{-3}\Big[\dfrac{tV(K'; t) - V(K; t)}{t - 1}\Big]$.

Let $\tilde{V}(t) = \dfrac{tV(K'; t) - V(K; t)}{t - 1}$. Then we know that there exists a knot $\tilde{K}(= \asymp)$ such that $V(\tilde{K}; t) = t^n\tilde{V}(t)$ for some integer n. Now $V'(\tilde{K}; t) = nt^{n-1}\tilde{V}(t) + t^n\tilde{V}'(t)$. By substituting 1, we have $V'(\tilde{K}; 1) = n\tilde{V}(1) + \tilde{V}'(1)$. Note that $\tilde{V}(t) = \dfrac{t(V(K'; t) - 1)}{t - 1} - \dfrac{V(K; t) - 1}{t - 1} + 1$. By a result in ([3], §12), we know that $V(K; t) - 1$ and $V(K'; t) - 1$ have $(t - 1)(t^3 - 1)$ as factors. Thus $\tilde{V}'(\tilde{K}; 1) = 0$ and $\tilde{V}(1) = 1$. Therefore we have $n = -\tilde{V}'(1)$, and hence we have $V(\tilde{K}, t) = t^{-\tilde{V}'(1)}\tilde{V}(t)$. By results in [3][5], we know that $V(\tilde{K}; \omega) = \omega^{-\tilde{V}'(1)}\tilde{V}(\omega) = \pm(\sqrt{-3})^d$ for some non-negative integer d.

Proof of Theorem 1.2. We use a method of Traczyk in [2]. By results in [3][5], we know that $V(K; \omega)$ must have the form $\pm(\sqrt{-3})^d$ for some non-negative integer d. Let $K = K_{d_1-d_2}, K_{d_1-d_2-1}, \ldots, K_0 = K'$ be a sequence of crossing changes. Then the exponents of K_i and K_{i-1} in the expression $\pm(\sqrt{-3})^d$ differ by 1 or -1. In fact, let $V(K_j; \omega) = (-1)^{s_j}(\sqrt{-3})^{d_j}$ and $V(K_{j-1}; \omega) = (-1)^{s_{j-1}}(\sqrt{-3})^{d_{j-1}}$. Suppose that $d_j = d_{j-1} + n$ for some integer n. Then, by substituting ω for t in the second relation of the definition of the Jones polynomial, we know that $\omega^{-1}V(K_j; \omega) - \omega V(K_{j-1}; \omega)$ does not have the form $\pm(\sqrt{-3})^d$ if $|n| \geq 2$. We also know that $\omega^{-1}V(K_{j-1}; \omega) - \omega V(K_j; \omega)$ does not have the form $\pm(\sqrt{-3})^d$ if $|n| \geq 2$. Thus we know that $|n| \leq 1$ and by assumption we must have $n = 1$. Moreover, by using the same argument, we know that $\omega^{-1}V(K_j; \omega) - \omega V(K_{j-1}; \omega)$ (or $\omega^{-1}V(K_{j-1}; \omega) - \omega V(K_j; \omega)$) does not have the form $\pm(\sqrt{-3})^d$ if $s_j - s_{j-1} \equiv 1 \mod 2$ (or $s_j - s_{j-1} \equiv 0 \mod 2$.) when $n = 1$. Therefore, we know that if K_{i-1} is obtained from K_i by a $+-$ change of a crossing then the sign is not changed and if K_{i-1} is obtained from K_i by a $-+$ change of a crossing, then the sign is changed.

Thus the $\mod 2$ number of $-+$ changes determines the parity of $s_1 - s_2$.

Proof of Theorem 1.3. We show the theorem by an induction with respect to the Gordian distance. By an argument in the proof of Prososition 4.1 [6], we know that
(1) $Q(K,\delta)/Q(K',\delta) \in \{\pm 1, -(\sqrt{5})^{\pm 1}\}$
if K' is obtained from K by a single crossing change. If $d_G(K,K') = 1$, then we know that $d = 0$ by (1). We assume that the result holds in the case when $d_G(K,K') = m-1$. Suppose that $d_G(K,K') = m$. Then there exists a knot $\overline{K}$ such that $d_G(K,\overline{K}) = m-1$. If $Q(K,\delta)/Q(\overline{K},\delta) = -(-\sqrt{5})^{\overline{d}}$, then $|\overline{d}| < m-1$ and $|d-\overline{d}| \leq 1$ by (1). Thus we have $|d| \leq |d-\overline{d}| + |\overline{d}| < m$. This completes the proof.

Proof of Theorem 1.5. If $s(K,K') \geq 0$, then we need to perform at least $s(K,K')$ positive crossing changes to obtain K' from K by Theorem 1.4(4). Thus we have $d_{+-}(K,K') \geq s(K,K')$. If $s(K,K') \geq 0$, then, by the same idea, we have $d_{+-}(K,K') = d_{-+}(K',K) \geq s(K',K) = -s(K,K')$.

3. Examples

Let $\sigma(K)$ be the signature of a knot K and let K^* be the mirror image of K. We need the following theorem due to K. Murasugi [1].

Theorem 3.1. *If a diagram of a knot K' is obtained from a diagram of a knot K by a single crossing change, then $\sigma(K) - \sigma(K') \in \{0, 2\}$.*

Example 3.2. We have $d_G(3_1\sharp 4_1, 5_1) = 2$. This is an unknown value in a table in [7]. We can prove it by using Theorem 1.3 and Theorem 3.1 as follows. We know that $d_G(3_1\sharp 4_1, 5_1) \leq 2$ since $d_G(0_1, 4_1) = 1$ and $d_G(3_1, 5_1) = 1$. Suppose that $d_G(3_1\sharp 4_1, 5_1) = 1$. Then by Theorem 3.1, we know that $d_{-+}(3_1\sharp 4_1, 5_1) = 0$ since $\sigma(3_1\sharp 4_1) = -2$ and $\sigma(5_1) = -4$. On the other hand, by Theorem 1.3, we know that $d_{-+}(3_1\sharp 4_1, 5_1) \equiv 1 \mod 2$ since $V(3_1\sharp 4_1;\omega) = \sqrt{-3}$ and $V(5_1;\omega) = -1$. Thus $d_{-+}(3_1\sharp 4_1, 5_1) = 1$. This is a contradiction. We can also prove this by using Theorem 1.2. (For example, we also have $d_G(5_2, 6_1) = 2$ by using the same argument.)

Example 3.3. We have the following.
(1) $d_G(4_1\sharp 4_1, 3_1) = d_G(4_1\sharp 4_1, 3_1^*) = 3$;
(2) $d_G(4_1\sharp 4_1, 5_2) = d_G(4_1\sharp 4_1, 5_2^*) = 3$;
(2) $d_G(4_1\sharp 4_1, 6_3) = 3$.
We know that $d_G(4_1\sharp 4_1, 3_1)$, $d_G(4_1\sharp 4_1, 3_1^*)$, $d_G(4_1\sharp 4_1, 5_2)$, $d_G(4_1\sharp 4_1, 5_2^*) and d_G(4_1\sharp 4_1, 6_3)$ are less than or equal to 3 since 3_1, 4_1, 5_2 and 6_3 have unknotting number one. Hence these equations (1) and (2) are obtained from Theorem 1.4 immediately since $Q(3_1;\delta) = Q(5_2;\delta) = Q(6_3;\delta) = -1$ and $Q(4_1;\delta) = -\sqrt{5}$. These numbers are undecided in the table of I. Darcy [9].

Example 3.4. We obtain the following values by Theorem 1.5.
$d_G(X^*, 10_{145}) = 3$, $d_G(X, 10_{154}) = 4$ and $d_G(X, 10_{161}) = 2$, where $X = 3_1$, 5_2, 6_2, 7_2, 7_6, 8_1, 8_7^*, 8_{14} or 8_{21}.
We cannot use Theorems 1.1, 1.2 and 1.3 to detect them. We have $s(10_{161}) = -6$, $s(10_{145}) = -4$, $s(10_{154}) = 6$. On the other hand, $\sigma(10_{161}) = -4$, $\sigma(10_{145}) = -2$, $\sigma(10_{154}) = 4$. Thus we also cannot use Theorem 3.1.

We list signatures, special values of the Jones polynomial and the Q polynomial for knots with up to 8 crossings (Figure 3.) (Here we set $a = \sqrt{3}$ and $b = \sqrt{5}$.)

K	σ	$V(K;\omega)$	$Q(K;\delta)$	K	σ	$V(K;\omega)$	$Q(K;\delta)$
3_1	-2	$-a$	-1	8_5	4	a	1
4_1	0	-1	$-b$	8_6	-2	1	-1
5_1	-4	-1	b	8_7	2	1	-1
5_2	-2	-1	-1	8_8	2	1	b
6_1	0	a	1	8_9	0	1	$-b$
6_2	-2	1	1	8_{10}	2	a	-1
6_3	0	1	-1	8_{11}	-2	$-a$	-1
7_1	-6	-1	-1	8_{12}	0	-1	1
7_2	-2	1	1	8_{13}	0	-1	1
7_3	4	1	-1	8_{14}	-2	-1	1
7_4	2	$-a$	b	8_{15}	-4	a	-1
7_5	-4	-1	-1	8_{16}	-2	1	b
7_6	-2	-1	1	8_{17}	0	1	-1
7_7	0	$-a$	1	8_{18}	0	3	b
8_1	0	1	-1	8_{19}	6	$-a$	-1
8_2	-4	-1	-1	8_{20}	0	$-a$	1
8_3	0	-1	-1	8_{21}	-2	$-a$	$-b$
8_4	-2	-1	1				

FIGURE 3

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Osaka City University Advanced Mathematical Institute Sugimoto 3-3-138, Sumiyoshi-ku 558-8585 Osaka, Japan.
tanakat@sci.osaka-cu.ac.jp