Sensitivity analysis for jump type stochastic differential equations with parameters

Atsushi TAKEUCHI

Department of Mathematics, Osaka City University

Sugimoto 3-3-138, Sumiyoshi-ku, Osaka 558-8585, Japan

E-mail: takeuchi@sci.osaka-cu.ac.jp

February 2, 2009

Abstract

Consider jump type stochastic differential equations with parameters. The coefficients of the diffusion and the jump terms satisfy the uniformly non-degenerate condition. The main purpose in this paper is to derive the logarithmic derivatives of the density function with respect to the parameters, which is equivalent to the computations of the Greeks for pay-off functions of asset price dynamics models in mathematical finance. The proof is based on the martingale methods via the Clark-Ocone type formula.

Keywords: Bismut-Elworthy-Li type formulae, heat kernels, jump processes, logarithmic derivatives, Malliavin calculus.

AMS 2000 Mathematics Subject Classifications: 60H30, 60H75, 60H07.

1 Introduction

The Malliavin calculus has played an important role in many fields, as one of powerful tools in infinite dimensional analysis. It has also given us an attractive solution to the hypoelliptic problem for the differential operator associated with a stochastic differential equation, by means of probabilistic methods. It is well known that the Hörmander type conditions on the coefficients of the equation, which is the condition about the Lie algebra generated by the vector fields associated with the coefficients, yield the existence of the smooth density function. See [3, 23] and references therein. Bismut [5] studied the logarithmic derivatives of the density function with respect to the initial point of a stochastic differential equation on Riemannian manifolds. Nowadays, the celebrated formulae in the book are called *the Bismut formulae* after his great contribution. His approach is based upon the Girsanov transform on Brownian motions. The formula has a nice flavour with the precise estimate of heat kernels or large deviation principles. Elworthy and Li [10] also tackled the same problem in more general class of stochastic differential equations on Riemannian manifolds, via the martingale methods based on the Clark-Ocone type formula. The logarithmic derivatives of the density function is equivalent to the Greeks computations for pay-off functions in mathematical finance. Fournié *et al.* [11, 12] applied the Malliavin calculus on the Wiener space to sensitivity analysis for asset price dynamics models. They also applied their results to the numerical simulation of the Greeks.

All works stated above, paid attention to the case of processes without any jumps. There has been a natural and non-trivial question whether a similar approach is applicable to sensitivity analysis in case of jump processes. The interests for jump processes are recently getting more and more in mathematical finance. There are some approaches to tackle the problem on the sensitivities: the Girsanov transformation approach ([19, 20]) for Lévy processes initiated by Bismut [4], the martingale methods based on the Clark-Ocone type formula ([6, 26]) similarly to [10] in case of diffusion processes, and an application of the Malliavin calculus on the Wiener-Poisson space ([1, 7, 9]) as introduced in [2, 8, 24]. In particular, Davis and Johansson [7], and Cass and Friz [6] studied in case of jump diffusion processes, but their approach does not take any effects from the jump term. On the other hand, the author in [26] studied the logarithmic derivatives of the density function with respect to the initial point of stochastic differential equations with jumps, via the martingale methods based on the Clark-Ocone type formula. The formulae obtained there are definitely reflected any effects by jumps.

This paper is a continuation study of [26]. We study the logarithmic derivatives of the density function for jump type stochastic differential equations depending on parameters, in which the effects from the diffusions and the jumps are explicitly reflected. Our approach is based on the martingale methods via the Clark-Ocone type formula, again. In mathematical finance, the sensitivities of pay-off functions with respect to not only the initial point, but also another parameters, have to be studied very carefully. That is our motivation of the present paper. The organization of the present paper is as follows: Section 2 devotes to the introduction of basic facts on stochastic differential equations, and the existence of smooth densities for the solution process. In Section 3, the main result in the paper is introduced, which is proved in the final section. Some typical examples of jump processes are given.

2 Preliminaries

Let us introduce some notations which will be used throughout the paper. Write $\mathbb{R}_0^m = \mathbb{R}^m \setminus \{\mathbf{0}\}$, and denote by $\nu(dz)$ the Lévy measure over $\mathbb{R}_0^m$ such that $\int_{\mathbb{R}_0^m} (|z|^2 \wedge 1)\,\nu(dz) < +\infty$.

Assumption 1. *The Lévy measure $\nu(dz)$ satisfies that*

(i) for any $p > 1$,

$$\int_{\mathbb{R}_0^m} \left\{ |z|\, I_{(|z|\le 1)} + |z|^p\, I_{(|z|>1)} \right\} \nu(dz) < +\infty,$$

(ii) there exist constants $c_1 > 0$ and $\sigma > 0$ such that

$$\inf_{|\theta|=1} \int_{\mathbb{R}_0^m} \left(\left| \frac{z\cdot\theta}{\rho} \right|^2 \wedge 1 \right) \nu(dz) \ge c_1 \rho^{-\sigma}$$

for sufficiently small $0 < \rho < 1$,

(iii) there exists a C^1-density $g(z)$ with respect to the Lebesgue measure on $\mathbb{R}_0^m$ such that

$$\lim_{|z|\to+\infty} |g(z)| = 0.$$

In what follows, we shall impose Assumption 1 on the Lévy measure $\nu(dz)$ without any comments.

Remark 2.1. In [17, 24], the following conditions on the Lévy measure $\nu(dz)$ are assumed.

(iv) there exists $0 < \alpha < 2$ such that

$$\liminf_{\rho\searrow 0} \rho^{-\alpha} \int_{|z|\le\rho} |z|^2\, \nu(dz) > 0,$$

(v) there exists a positive definite matrix $B \in \mathbb{R}^m \otimes \mathbb{R}^m$ such that

$$\liminf_{\rho\searrow 0} \left(\int_{|z|\le\rho} |z|^2\, \nu(dz) \right)^{-1} \int_{|z|\le\rho} z z^*\, \nu(dz) \ge B.$$

The condition (iv) is called *the order condition* on the measure $\nu(dz)$, and the driving Lévy process of the equation (2.1) is called *non-degenerate* under the condition (v). It can be easily checked that the condition (ii) in Assumption 1 is satisfied under (iv) and (v). □

Remark 2.2. Let a, b, $c > 0$, and $0 \leq \beta < 1$. Write

$$\nu(dz) = a\left\{(-z)^{-1-\beta}\, e^{cz} I_{(z<0)} + z^{-1-\beta}\, e^{-bz} I_{(z>0)}\right\} dz.$$

Gamma processes ($c = +\infty,\ \beta = 0$), variance gamma processes ($\beta = 0$), tempered stable processes ($c = +\infty,\ 0 < \beta < 1$), inverse Gaussian processes ($c = +\infty,\ \beta = 1/2$), and CGMY processes are in our position, whose Lévy measure satisfy Assumption 1. Those are often appeared in asset price dynamics models with jumps in mathematical finance. $\square$

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be our underlying probability space, and $W = \{W_t\,;\, t \geq 0\}$ an m-dimensional Brownian motion with $W_0 = \mathbf{0} \in \mathbb{R}^m$. Denote a Poisson random measure over $[0, +\infty) \times \mathbb{R}^m_0$ by $J(dt, dz)$ with the intensity measure $\hat{J}(dt, dz) := dt\, \nu(dz)$, and the natural filtration of W and $J(dt, dz)$ by $\{\mathscr{F}_t\}_{t\geq 0}$. For the simplicity of notations, write

$$\begin{aligned} \tilde{J}(dt, dz) &= J(dt, dz) - \hat{J}(dt, dz), \\ \overline{J}(dt, dz) &= I_{(|z|\leq 1)}\, \tilde{J}(dt, dz) + I_{(|z|>1)}\, J(dt, dz). \end{aligned}$$

Let $a_i(\varepsilon, y) \in C^{1,2}_b\left(\mathbb{R}^l \times \mathbb{R}^d; \mathbb{R}^d\right)$ $(0 \leq i \leq m)$, and $b(\varepsilon, y, z) \in C^{1,2,2}_b\left(\mathbb{R}^l \times \mathbb{R}^d \times \mathbb{R}^m_0; \mathbb{R}^d\right)$ with

$$\inf_{y\in\mathbb{R}^d}\ \inf_{z\in\mathbb{R}^m_0} |\det[I + \partial b(\varepsilon, y, z)]| > 0, \quad \lim_{|z|\searrow 0} b(\varepsilon, y, z) = 0,$$

where $I \in \mathbb{R}^d \otimes \mathbb{R}^d$ is the identity. The symbol ∂ indicates the derivative with respect to the parameter in $\mathbb{R}^d$, while the symbols ∂_ε and ∂_z indicate the derivatives in $\varepsilon \in \mathbb{R}^l$ and $z \in \mathbb{R}^m_0$, respectively. $C^N_b\left(\mathbb{R}^i; \mathbb{R}^j\right)$ denotes the set of $\psi \in C^N\left(\mathbb{R}^i; \mathbb{R}^j\right)$ such that all derivatives of $\partial^\beta \psi$ $(1 \leq |\beta| \leq N)$ are bounded, where $\beta = (\beta_1, \ldots, \beta_i) \in \mathbb{Z}^i_+$, $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_i^{\beta_i}$, and $|\beta| = \sum_{k=1}^i \beta_i$.

Let $\varepsilon \in \mathbb{R}^l$ and $x \in \mathbb{R}^d$ be deterministic. Consider a process $\{x_t = x^\varepsilon_t\,;\, t \geq 0\}$ determined by the stochastic differential equation of the form:

$$dx_t = a_0(\varepsilon, x_t)\, dt + a(\varepsilon, x_t) \circ dW_t + \int_{\mathbb{R}^m_0} b(\varepsilon, x_{t-}, z)\, d\overline{J}, \quad x_0 = x, \tag{2.1}$$

where $a(\varepsilon, y) = (a_1(\varepsilon, y), \ldots, a_m(\varepsilon, y))$. The conditions on the coefficients guarantee the existence of the unique solution $\{x_t\,;\, t \geq 0\}$ to (2.1). The infinitesimal generator $\mathscr{L}^\varepsilon$ associated with the Markov process $\{x_t\,;\, t \geq 0\}$ is given by

$$\mathscr{L}^\varepsilon = A^\varepsilon_0 + \frac{1}{2}\sum_{i=1}^m A^\varepsilon_i A^\varepsilon_i + \int_{\mathbb{R}^m_0} \left\{\mathfrak{B}^\varepsilon_z - I_{(|z|\leq 1)} B^\varepsilon_z\right\} \nu(dz),$$

where $A_i^\varepsilon = a_i(\varepsilon, y)\cdot\partial \ (0 \le i \le m)$ and $B_z^\varepsilon = b(\varepsilon, y, z)\cdot\partial \ \left(z \in \mathbb{R}_0^m\right)$ are vector fields over $\mathbb{R}^d$, and the operator $\mathfrak{B}_z^\varepsilon$ is defined by $\mathfrak{B}_z^\varepsilon f(y) = f(y + b(\varepsilon, y, z)) - f(y)$. Write $\mathbb{E}_y[\ \cdot\ ] = \mathbb{E}\left[\ \cdot\ \middle| x_0 = y\right]$. Let $\{Z_t\,;\, t \ge 0\}$ and $\{U_t\,;\, t \ge 0\}$ be $\mathbb{R}^d \otimes \mathbb{R}^d$-valued processes determined by the linear stochastic differential equations of the form:

$$
\begin{aligned}
dZ_t &= \partial a_0(\varepsilon, x_t)\, Z_t\, dt + \partial a(\varepsilon, x_t)\, Z_t \circ dW_t + \int_{\mathbb{R}_0^m} \partial b(\varepsilon, x_{t-}, z)\, Z_{t-}\, d\bar{J},\\
dU_t &= -U_t \left\{ \partial a_0(\varepsilon, x_t) - \int_{|z|\le 1} \left[(I + \partial b)^{-1} - I + \partial b\right](\varepsilon, x_t, z)\, \nu(dz) \right\} dt\\
&\qquad - U_t\, \partial a(\varepsilon, x_t) \circ dW_t + \int_{\mathbb{R}_0^m} U_{t-} \left[(I + \partial b)^{-1} - I\right](\varepsilon, x_{t-}, z)\, d\bar{J},\\
Z_0 &= U_0 = I.
\end{aligned}
$$

It can be easily checked by the Itô formula that $Z_t\, U_t = U_t\, Z_t = I$. The conditions on the coefficients, and the Kolmogorov criterion for random fields implies that

Proposition 2.1 (cf. [13])**.** *For $t \ge 0$, the mapping $\mathbb{R}^d \ni x \longmapsto x_t \in \mathbb{R}^d$ has a C^1-modification, and $Z_t = \partial_x x_t$. Moreover, for any $p > 1$, $T > 0$, and any compact subset $K \subset \mathbb{R}^l$, it holds that*

$$
\sup_{\varepsilon \in K} \mathbb{E}_x \left[\sup_{t \le T} \left(|x_t|^p + \|Z_t\|^p + \|U_t\|^p \right) \right] < +\infty.
$$

Similarly, we can get the following lemma on the differentiability of the process $\{x_t\,;\, t \ge 0\}$ with respect to the parameter $\varepsilon \in \mathbb{R}^l$.

Lemma 2.1. *For $t \ge 0$, the mapping $\mathbb{R}^l \ni \varepsilon \longmapsto x_t \in \mathbb{R}^d$ has a C^1-modification, and the derivative $\partial_\varepsilon x_t^\varepsilon$ satisfies the equation of the form:*

$$
\begin{aligned}
d\partial_\varepsilon x_t &= \partial a_0(\varepsilon, x_t)\, \partial_\varepsilon x_t\, dt + \partial a(\varepsilon, x_t)\, \partial_\varepsilon x_t \circ dW_t + \int_{\mathbb{R}_0^m} \partial b(\varepsilon, x_{t-}, z)\, \partial_\varepsilon x_{t-}\, d\bar{J}\\
&\qquad + \partial_\varepsilon a_0(\varepsilon, x_t)\, dt + \partial_\varepsilon a(\varepsilon, x_t) \circ dW_t + \int_{\mathbb{R}_0^m} \partial_\varepsilon b(\varepsilon, x_{t-}, z)\, d\bar{J}, \qquad (2.2)\\
\partial_\varepsilon x_0^\varepsilon &= \mathbf{0} \in \mathbb{R}^l \otimes \mathbb{R}^d.
\end{aligned}
$$

Moreover, for any $p > 1$, $T > 0$, and any compact subset $K \subset \mathbb{R}^l$, it holds that

$$
\sup_{\varepsilon \in K} \mathbb{E}_x \left[\sup_{t \le T} \|\partial_\varepsilon x_t\|^p \right] < +\infty.
$$

Proof. We shall write $x_t = x_t^\varepsilon$ only in the proof, to emphasize the dependence on $\varepsilon \in \mathbb{R}^l$.

First, we shall study the continuity of $\mathbb{R}^l \ni \varepsilon \longmapsto x_t^\varepsilon \in \mathbb{R}^d$. Let $T > 0$ and $(\varepsilon, \delta) \in \mathbb{R}^l \times \mathbb{R}^l$. Since

$$\begin{aligned} x_t^\varepsilon - x_t^\delta = & \int_0^t \left\{ a_0\left(\varepsilon, x_s^\varepsilon\right) - a_0\left(\delta, x_s^\delta\right) \right\} ds + \int_0^t \left\{ a\left(\varepsilon, x_s^\varepsilon\right) - a\left(\delta, x_s^\delta\right) \right\} \circ dW_s \\ & + \int_0^t \int_{\mathbb{R}_0^m} \left\{ b\left(\varepsilon, x_{s-}^\varepsilon, z\right) - b\left(\delta, x_{s-}^\delta, z\right) \right\} d\bar{J}, \end{aligned}$$

we can get the upper estimate

$$\mathbb{E}_x \left[\sup_{t \le T} \left| x_t^\varepsilon - x_t^\delta \right|^p \right] \le c_{2,p,x,T} \, |\varepsilon - \delta|^p$$

for any $p > 1$, from the conditions on $a_i(\varepsilon, y)$ and $b(\varepsilon, y, z)$. Thus the Kolmogorov criterion tells us that the mapping $\mathbb{R}^l \ni \varepsilon \longmapsto x_t^\varepsilon \in \mathbb{R}^d$ has a continuous modification for each $t \ge 0$.

Next, we shall study the differentiability of x_t^ε in $\varepsilon \in \mathbb{R}^l$. Let $0 \ne \xi, \zeta \in \mathbb{R}$, and $\boldsymbol{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0)^* \in \mathbb{R}^l$ the k-th unit vector. Since

$$\begin{aligned} \frac{x_t^{\varepsilon + \xi \boldsymbol{e}_k} - x_t^\varepsilon}{\xi} = & \int_0^t \frac{a_0\left(\varepsilon + \xi \boldsymbol{e}_k, x_s^{\varepsilon + \xi \boldsymbol{e}_k}\right) - a_0\left(\varepsilon, x_s^\varepsilon\right)}{\xi} ds \\ & + \int_0^t \frac{a\left(\varepsilon + \xi \boldsymbol{e}_k, x_s^{\varepsilon + \xi \boldsymbol{e}_k}\right) - a\left(\varepsilon, x_s^\varepsilon\right)}{\xi} \circ dW_s \\ & + \int_0^t \int_{\mathbb{R}_0^m} \frac{b\left(\varepsilon + \xi \boldsymbol{e}_k, x_{s-}^{\varepsilon + \xi \boldsymbol{e}_k}, z\right) - b\left(\varepsilon, x_{s-}^\varepsilon, z\right)}{\xi} d\bar{J}, \end{aligned}$$

we can get the upper estimate

$$\mathbb{E}_x \left[\sup_{t \le T} \left| \frac{x_t^{\varepsilon + \xi \boldsymbol{e}_k} - x_t^\varepsilon}{\xi} - \frac{x_t^{\varepsilon + \zeta \boldsymbol{e}_k} - x_t^\varepsilon}{\zeta} \right|^p \right] \le c_{3,p,x,T,\varepsilon,k} \, |\xi - \zeta|^p$$

for any $p > 1$. Hence the mapping $\mathbb{R}^l \ni \varepsilon \longmapsto x_t^\varepsilon \in \mathbb{R}^d$ has a C^1-modification with respect to the parameter $\varepsilon \in \mathbb{R}^l$ for each $t \ge 0$, via the Kolmogorov criterion, again.

Furthermore, the conditions on the coefficients enables us to justify that the derivative $\partial_\varepsilon x_t^\varepsilon$ satisfies (2.2). It is an easy work to check the upper estimate of $\partial_\varepsilon x_t$ in the assertion. The proof is complete. $\square$

Corollary 2.1. *The derivative $\partial_\varepsilon x_t$ can be computed as follows:*

$$\begin{aligned}\partial_\varepsilon x_t = Z_t \int_0^t U_s \left(\partial_\varepsilon a_0 (\varepsilon, x_s) + \int_{|z|\leq 1} \left[\left\{ (I+\partial b)^{-1} - I \right\} \partial_\varepsilon b \right] (\varepsilon, x_s, z)\, \nu(dz) \right) ds \\ + Z_t \int_0^t U_s\, \partial_\varepsilon a (\varepsilon, x_s) \circ dW_s + Z_t \int_0^t \int_{\mathbb{R}^m_0} U_{s-} \left[(I+\partial b)^{-1} \partial_\varepsilon b \right] (\varepsilon, x_{s-}, z)\, d\bar{J}.\end{aligned} \tag{2.3}$$

Proof. Obvious by applying the Itô product formula to $U_t\, \partial_\varepsilon x_t$. □

3 Main result

We shall devote to state the main result in the present paper. Throughout this paper, suppose that the coefficients $a_i(\varepsilon, y)$ $(1 \leq i \leq m)$ and $b(\varepsilon, y, z)$ of the equation (2.1) satisfy

Assumption 2. *For each $\varepsilon \in \mathbb{R}^l$, the $\mathbb{R}^d \otimes \mathbb{R}^d$-valued functions*

$$[a\, a^*](\varepsilon, y), \quad [\partial_z b\, \{\partial_z b\}^*](\varepsilon, y, z)$$

are uniformly elliptic on $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^m_0$.

Let $T > 0$, and define $\tilde{b}(\varepsilon, y, z) = \left[(I+\partial b)^{-1} \partial_z b \right] (\varepsilon, y, z)\, z$. Then, the following fact on the existence of the smooth density is well known.

Proposition 3.1 (cf. [21, 25])**.** *Fix $\varepsilon \in \mathbb{R}^l$. If there exist constants $c_4 > 0$ and $\iota > 0$ such that*

$$\inf_{y\in\mathbb{R}^d} \inf_{|\lambda|=1} \left\{ \sum_{i=1}^m \left| a_i(\varepsilon, y) \cdot \frac{\lambda}{\rho} \right|^2 + \int_{\mathbb{R}^m_0} \left(\left| \tilde{b}(\varepsilon, y, z) \cdot \frac{\lambda}{\rho} \right|^2 \wedge 1 \right) \nu(dz) \right\} \geq c_4\, \rho^{-\iota} \tag{3.1}$$

for sufficiently small $0 < \rho < 1$, then the law of the random variable x_T determined by (2.1) *admits a density $p_T(\varepsilon, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ such that $p_t(\varepsilon, x, y)$ is smooth in $y \in \mathbb{R}^d$.*

Remark 3.1. It can be easily checked that Assumption 1 and Assumption 2 imply the condition (3.1) in Proposition 3.1. In fact, since the boundedness of the function $b(\varepsilon, y, z)$ yields that

$$\begin{aligned}1 &\leq \left| \left[(I+\partial b)^{-1} \right]^* (\varepsilon, y, z)\, \lambda \right| \left| [I+\partial b] (\varepsilon, y, z)\, \lambda \right| \\ &\leq c_5 \left| \left[(I+\partial b)^{-1} \right]^* (\varepsilon, y, z)\, \lambda \right|\end{aligned}$$

for $\lambda \in S^{d-1}$, we see that

$$\left| \left[(I+\partial b)^{-1} \partial_z b \right]^* (\varepsilon, y, z) \frac{\lambda}{\rho} \right|^2 \geq c_{6,\varepsilon} \left| \left[(I+\partial b)^{-1} \right]^* (\varepsilon, y, z) \frac{\lambda}{\rho} \right|^2$$
$$\geq c_{7,\varepsilon} \rho^{-2}$$

under Assumption 2. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^m_0} \left(\left| \tilde{b} (\varepsilon, y, z) \cdot \frac{\lambda}{\rho} \right|^2 \wedge 1 \right) \nu (dz) \\
&= \int_{\mathbb{R}^m_0} \left(\left| \left(\left[(I+\partial b)^{-1} \partial_z b \right]^* (\varepsilon, y, z) \frac{\lambda}{\rho} \right) \cdot z \right|^2 \wedge 1 \right) \nu (dz) \\
&\geq c_{8,\varepsilon} \inf_{|\theta|=1} \int_{\mathbb{R}^m_0} \left(\left| \frac{z \cdot \theta}{\rho} \right|^2 \wedge 1 \right) \nu (dz) \\
&\geq c_{9,\varepsilon} \rho^{-\sigma}
\end{aligned}$$

from Assumption 1 (ii), for sufficiently small $0 < \rho < 1$. □

Our goal in the present paper is to study the logarithmic derivatives of $p_T(\varepsilon, x, y)$ with respect to the parameter $\varepsilon \in \mathbb{R}^l$. This can be also regarded as the continuous work of [26], in which the Bismut-Elworthy-Li type formulae (or, the logarithmic derivarives of $p_T(\varepsilon, x, y)$ with respect to the initial point $x \in \mathbb{R}^d$ of the equation (2.1)) are studied.

We shall introduce the main result. Let $D = \{D_s\,;\, s \in [0,T]\}$ be the Malliavin derivative operator. Define

$$\begin{aligned}
F_0(s) &= U_s \left(\partial_\varepsilon a_0 (\varepsilon, x_s) + \int_{|z|\leq 1} \left[\left\{ (I+\partial b)^{-1} - I \right\} \partial_\varepsilon b \right] (\varepsilon, x_s, z)\, \nu (dz) \right), \\
F(s) &= U_s\, \partial_\varepsilon a (\varepsilon, x_s), \\
\kappa (s,z) &= U_{s-} \left[(I+\partial b)^{-1} \partial_\varepsilon b \right] (\varepsilon, x_{s-}, z), \\
N^\varepsilon_t &= \int_0^t (dW_s)^* a(\varepsilon, x_s)^{-1} Z_s F_0(s), \\
L^\varepsilon_t &= \frac{1}{t} \int_0^t (dW_s)^* a(\varepsilon, x_s)^{-1} Z_s \int_0^t F(s) \circ dW_s, \\
H^\varepsilon_t &= \frac{1}{t} \int_0^t \mathrm{Tr} \left[a(\varepsilon, x_s)^{-1} Z_s\, D_s \left(\int_0^t F(s) \circ dW_s \right) \right] ds, \\
V^\varepsilon_t &= \int_0^t \int_{\mathbb{R}^m_0} \frac{1}{g(z)} \mathrm{div}_z \left[g(z) \left[(\partial_z b)^{-1} (I+\partial b) \right] (\varepsilon, x_s, z)\, Z_s\, \kappa (s,z) \right] d\tilde{J},
\end{aligned}$$

where

$$\operatorname{div}_z[\Phi(z)] = \left(\{\operatorname{div}_z[\Phi(z)]\}_k\,;\,1\leq k\leq l\right), \quad \{\operatorname{div}_z[\Phi(z)]\}_k = \sum_{i=1}^{m}\frac{\Phi_{ik}}{\partial z_i}(z)$$

$$\operatorname{Tr}[M] = \left(\{\operatorname{Tr}[M]\}_k\,;\,1\leq k\leq l\right), \quad \{\operatorname{Tr}[M]\}_k = \sum_{i=1}^{m} M_{i,k,i}$$

for an $\mathbb{R}^l\otimes\mathbb{R}^m$-valued function $\Phi = (\Phi_{ik}\,;\,1\leq i\leq m,\,1\leq k\leq l)$, and $M\in\mathbb{R}^m\otimes\mathbb{R}^l\otimes\mathbb{R}^m$. Then we have

Theorem 1 (Sensitivity with respect to $\varepsilon\in\mathbb{R}^l$)**.** *Suppose Assumption 1 and Assumption 2. Let* $\varphi:\mathbb{R}^d\to\mathbb{R}$ *be measurable such that* $\mathbb{E}_x\left[|\varphi(x_T)|^2\right] < +\infty$*. Then it holds that*

$$\partial_\varepsilon\mathbb{E}_x[\varphi(x_T)] = \mathbb{E}_x[\varphi(x_T)\,\Gamma_T^\varepsilon], \quad \Gamma_T^\varepsilon := N_T^\varepsilon + L_T^\varepsilon - H_T^\varepsilon - V_T^\varepsilon. \tag{3.2}$$

Remark 3.2. Instead of Assumption 2, suppose that the function $[a\,a^*](\varepsilon,y)$ is uniformly elliptic in $y\in\mathbb{R}^d$, and that the function $[\partial_z b\,\{\partial_z b\}^*](\varepsilon,y,z)$ is not always uniformly elliptic. Then, the weight is computed as $\Gamma_T^\varepsilon = N_T^\varepsilon + L_T^\varepsilon - H_T^\varepsilon$. On the other hand, if the function $[a\,a^*](\varepsilon,y)$ is not always uniformly elliptic, and the function $[\partial_z b\,\{\partial_z b\}^*](\varepsilon,y,z)$ is uniformly elliptic, instead of Assumption 2, then the weight is given by $\Gamma_T^\varepsilon = -V_T^\varepsilon$. $\square$

Example 1 (Lévy processes)**.** Let $m=d=1$, $x\in\mathbb{R}$, and $(\gamma,\sigma_1,\sigma_2)\in\mathbb{R}\times(0,+\infty)\times(0,+\infty)$. Suppose that the measure $\nu(dz)$ satisfies Assumption 1. Consider the $\mathbb{R}$-valued Lévy process $\{x_t\,;\,t\geq 0\}$ defined by

$$x_t = x + \gamma t + \sigma_1 W_t + \sigma_2\int_0^t\int_{\mathbb{R}_0} z\,d\bar{J}.$$

Since we are in position that

$$a_0((\gamma,\sigma_1,\sigma_2),y) = \gamma, \quad a_1((\gamma,\sigma_1,\sigma_2),y) = \sigma_1, \quad b((\gamma,\sigma_1,\sigma_2),y,z) = \sigma_2 z,$$

we have

$$\partial_\gamma x_t = t, \quad \partial_{\sigma_1}x_t = W_t, \quad \partial_{\sigma_2}x_t = \int_0^t\int_{\mathbb{R}_0} z\,d\bar{J}, \quad Z_t = U_t = 1.$$

Then it holds that

$$\begin{aligned}
\left(N_t^\gamma, L_t^\gamma, H_t^\gamma, V_t^\gamma\right) &= \left(\frac{W_t}{\sigma_1}, 0, 0, 0\right), \\
\left(N_t^{\sigma_1}, L_t^{\sigma_1}, H_t^{\sigma_1}, V_t^{\sigma_1}\right) &= \left(0, \frac{W_t^2}{\sigma_1 t}, \frac{1}{\sigma_1}, 0\right),
\end{aligned}$$

$$\left(N_t^{\sigma_2}, L_t^{\sigma_2}, H_t^{\sigma_2}, V_t^{\sigma_2}\right) = \left(-\frac{W_t}{\sigma_1}\int_{|z|\leq 1} z\,\nu\,(dz)\,, 0, 0, \int_0^t\int_{\mathbb{R}_0} \frac{\partial_z\{g(z)\,z\}}{\sigma_2\,g(z)}\,d\tilde{J}\right).$$

Therefore, the corresponding weights Γ_T^γ, $\Gamma_T^{\sigma_1}$, and $\Gamma_T^{\sigma_2}$ on the sensitivities with respect to γ, σ_1, and σ_2 can be computed as follows:

$$\Gamma_T^\gamma = \frac{W_T}{\sigma_1}, \quad \Gamma_T^{\sigma_1} = \frac{W_T^2 - T}{\sigma_1 T}, \quad \Gamma_T^{\sigma_2} = -\frac{W_T}{\sigma_1}\int_{|z|\leq 1} z\,\nu\,(dz) + \int_0^T\int_{\mathbb{R}_0} \frac{\partial_z\{g(z)\,z\}}{\sigma_2\,g(z)}\,d\tilde{J},$$

while the weights on the sensitivity with respect to the initial point $x \in \mathbb{R}$ are given as follows:

$$\Gamma_T^x = \frac{W_T}{\sigma_1 A_T} - \frac{1}{\sigma_2 A_T}\int_0^T\int_{\mathbb{R}_0} \frac{\partial_z\{g(z)\,|z|\}}{g(z)}\,d\tilde{J} + \frac{2}{\sigma_2 A_T^2}\int_0^T\int_{\mathbb{R}_0} z\,dJ,$$

as stated in [26], where $A_T = T + \displaystyle\int_0^T\int_{\mathbb{R}_0} |z|\,dJ$. □

Example 2 (Geometric Lévy processes)**.** Let $m = d = 1$, and suppose that the measure $\nu\,(dz)$ satisfies Assumption 1. Let $\{X_t\,;\,t \geq 0\}$ the $\mathbb{R}$-valued Lévy process given by

$$X_t = \gamma t + \sigma_1 W_t + \sigma_2\int_0^t\int_{\mathbb{R}_0} z\,d\bar{J},$$

where $(\gamma, \sigma_1, \sigma_2) \in \mathbb{R} \times (0, +\infty) \times (0, +\infty)$. For $x > 0$, define $x_t = x\exp[X_t]$, which is called *the geometric Lévy process*. The Itô formula enables us to see that the process $\{x_t\,;\,t \geq 0\}$ satisfies the linear stochastic differential equation of the form

$$dx_t = \left\{\gamma + \int_{|z|\leq 1}\left(e^{\sigma_2 z} - 1 - \sigma_2 z\right)\,\nu\,(dz)\right\} x_t\,dt + \sigma_1 x_t \circ dW_t + \int_{\mathbb{R}_0}\left(e^{\sigma_2 z} - 1\right) x_{t-}\,d\bar{J},$$

which can be regarded as the special case of canonical stochastic differential equations with jumps (cf. [17]). Since we are in position that

$$a_0\left((\gamma, \sigma_1, \sigma_2)\,, y\right) = \left\{\gamma + \int_{|z|\leq 1}\left(e^{\sigma_2 z} - 1 - \sigma_2 z\right)\,\nu\,(dz)\right\} y,$$
$$a_1\left((\gamma, \sigma_1, \sigma_2)\,, y\right) = \sigma_1 y, \quad b\left((\gamma, \sigma_1, \sigma_2)\,, y, z\right) = \left(e^{\sigma_2 z} - 1\right) y,$$

Assumption 2 is not satisfied.

But this example is also definitely in our position. Write $\psi\,(y) = \exp[y]$. Since

$$\partial_x \mathbb{E}_x\left[\varphi\,(x_T)\right] = \mathbb{E}_x\left[\varphi'\,(x_T)\,\frac{x_T}{x}\right] = \frac{1}{x}\partial_X \mathbb{E}_x\left[(\varphi\circ\psi)\,(X + X_T)\right]\big|_{X=\log x},$$

$$\partial_\gamma \mathbb{E}_x[\varphi(x_T)] = \mathbb{E}_x\left[\varphi'(x_T)\, x_T\, T\right] = \partial_\gamma \mathbb{E}_x[(\varphi\circ\psi)(X+X_T)]\big|_{X=\log x},$$
$$\partial_{\sigma_1} \mathbb{E}_x[\varphi(x_T)] = \mathbb{E}_x\left[\varphi'(x_T)\, x_T\, W_T\right] = \partial_{\sigma_1} \mathbb{E}_x[(\varphi\circ\psi)(X+X_T)]\big|_{X=\log x},$$
$$\partial_{\sigma_2} \mathbb{E}_x[\varphi(x_T)] = \mathbb{E}_x\left[\varphi'(x_T)\, x_T \int_0^T\int_{\mathbb{R}_0} z\bar{J}(ds,dz)\right] = \partial_{\sigma_2} \mathbb{E}_x[(\varphi\circ\psi)(X+X_T)]\big|_{X=\log x}$$

for $\varphi \in C_b^2(\mathbb{R};\mathbb{R})$, the corresponding weights Γ_T^γ, $\Gamma_T^{\sigma_1}$, $\Gamma_T^{\sigma_2}$ for geometric Lévy processes can be computed by using the results done in Example 1 as follows:

$$\Gamma_T^\gamma = \frac{W_T}{\sigma_1}, \quad \Gamma_T^{\sigma_1} = \frac{W_T^2 - T}{\sigma_1 T}, \quad \Gamma_T^{\sigma_2} = -\frac{W_T}{\sigma_1}\int_{|z|\le 1} z\,\nu(dz) + \int_0^T\int_{\mathbb{R}_0} \frac{\partial_z\{g(z)\,z\}}{\sigma_2\, g(z)}\, d\tilde{J},$$
$$\Gamma_T^x = \frac{W_T}{\sigma_1 x A_T} - \frac{1}{\sigma_2 x A_T}\int_0^T\int_{\mathbb{R}_0} \frac{\partial_z\{g(z)\,|z|\}}{g(z)}\, d\tilde{J} + \frac{2}{\sigma_2 x A_T^2}\int_0^T\int_{\mathbb{R}_0} z\, dJ,$$

where $A_T = T + \int_0^T\int_{\mathbb{R}_0} |z|\, dJ$. □

4 Proof of Theorem 1

In this section, we shall give the proof of Theorem 1. Let $T > 0$, $\varepsilon \in \mathbb{R}^l$, and $x \in \mathbb{R}^d$. First, we shall start with $\varphi \in C_b^2(\mathbb{R}^d;\mathbb{R})$, which can be extended to the measurable function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\mathbb{E}_x\left[|\varphi(x_T)|^2\right] < +\infty$, as stated below. Write

$$u^\varepsilon(t,y) = \mathbb{E}_y[\varphi(x_{T-t})]$$

for $t \in [0,T]$ and $y \in \mathbb{R}^d$. The following lemma plays a key role in our argument.

Lemma 4.1. *The following equality holds.*

$$\varphi(x_T) = \mathbb{E}_x[\varphi(x_T)] + \int_0^T [\partial u^\varepsilon a](\varepsilon,x_t)\, dW_t + \int_0^T\int_{\mathbb{R}_0^m} \mathfrak{B}_z^\varepsilon u^\varepsilon(t,x_{t-})\, d\tilde{J}. \tag{4.1}$$

Proof. As stated in [14], the function u^ε is in $C_b^{1,2}\left([0,T)\times\mathbb{R}^d\right)$, and satisfies

$$\partial_t u^\varepsilon + \mathscr{L}^\varepsilon u^\varepsilon = 0, \quad \lim_{t\nearrow T} u^\varepsilon(t,y) = \varphi(y).$$

Let $t \in [0,T)$. Applying the Itô formula to the function u^ε implies that

$$u^\varepsilon(t,x_t) = \mathbb{E}_x[\varphi(x_T)] + \int_0^t [\partial u^\varepsilon a](\varepsilon,x_s)\, dW_s + \int_0^t\int_{\mathbb{R}_0^m} \mathfrak{B}_z^\varepsilon u^\varepsilon(s,x_{s-})\, d\tilde{J}. \tag{4.2}$$

Each term in both hand sides of (4.2) converges to each term in (4.1) as $t \nearrow T$, respectively, because $u^{\varepsilon} \in C_b^{1,2}\left([0,T) \times \mathbb{R}^d\right)$, and

$$u^{\varepsilon}(t, x_t) = \mathbb{E}_x\left[\varphi(x_T) \,\middle|\, \mathscr{F}_t\right] \to \mathbb{E}_x\left[\varphi(x_T) \,\middle|\, \mathscr{F}_T\right] = \varphi(x_T)$$

as $t \nearrow T$ (cf. Theorem I-6.6 in [16]). The proof is complete. $\square$

Lemma 4.2. *It holds that*

$$\mathbb{E}_x\left[\partial\varphi(x_T)\, Z_T \int_0^T F_0(t)\, dt\right] = \mathbb{E}_x\left[\varphi(x_T)\, N_T^{\varepsilon}\right]. \tag{4.3}$$

Proof. Since Lemma 4.1 tells us that the process $\left\{u^{\varepsilon}(t, x_t) = \mathbb{E}_x\left[\varphi(x_T) \,\middle|\, \mathscr{F}_t\right];\, t \in [0,T)\right\}$ is $(\mathscr{F}_t)$-martingale, so is the process $\{\partial_x u^{\varepsilon}(t, x_t);\, t \in [0,T)\}$. In fact, this can be checked by differentiating the both hand sides of the equality (4.2) with respect to $x \in \mathbb{R}^d$, by using $u^{\varepsilon} \in C_b^{1,2}\left([0,T) \times \mathbb{R}^d\right)$. Then, for $t < \tau < T$, we have

$$\mathbb{E}_x\left[\partial_x u^{\varepsilon}(t, x_t)\, F_0(t)\right] = \mathbb{E}_x\left[\partial_x u^{\varepsilon}(\tau, x_\tau)\, F_0(t)\right] = \mathbb{E}_x\left[\partial u^{\varepsilon}(\tau, x_\tau)\, Z_\tau F_0(t)\right],$$

so taking the limit as $\tau \nearrow T$ yields that $\mathbb{E}_x\left[\partial_x u^{\varepsilon}(t, x_t)\, F_0(t)\right] = \mathbb{E}_x\left[\partial\varphi(x_T)\, Z_T F_0(t)\right]$. Therefore, the Fubini theorem and Lemma 4.1 yield that

$$\begin{aligned}
\mathbb{E}_x\left[\partial\varphi(x_T)\, Z_T \int_0^T F_0(t)\, dt\right] &= \int_0^T \mathbb{E}_x\left[\partial_x u^{\varepsilon}(t, x_t)\, F_0(t)\right] dt \\
&= \mathbb{E}_x\left[\int_0^T \left[\partial u^{\varepsilon} a\right](\varepsilon, x_t)\, dW_t\, N_T^{\varepsilon}\right] \\
&= \mathbb{E}_x\left[\varphi(x_T)\, N_T^{\varepsilon}\right],
\end{aligned}$$

which completes the proof. $\square$

Lemma 4.3. *It holds that*

$$\mathbb{E}_x\left[\partial\varphi(x_T)\, Z_T \int_0^T F(s) \circ dW_s\right] = \mathbb{E}_x\left[\varphi(x_T)\, (L_T^{\varepsilon} - H_T^{\varepsilon})\right]. \tag{4.4}$$

Proof. Write

$$G_t = \int_0^t F(s) \circ dW_s.$$

Since $D_s\varphi(x_T) = \partial\varphi(x_T)\, Z_T\, U_s\, a(\varepsilon, x_s)$ for $s \in [0,T]$ from the chain rule on the operator D, the integration by parts formula implies that

$$
\begin{aligned}
\mathbb{E}_x\left[\partial\varphi\left(x_T\right) Z_T\, G_T\right] &= \mathbb{E}_x\left[\frac{1}{T}\int_0^T D_s\varphi\left(x_T\right) a\left(\varepsilon, x_s\right)^{-1} Z_s\, G_T\, ds\right] \\
&= \mathbb{E}_x\left[\varphi\left(x_T\right)\ \frac{1}{T} D^*\left(a\left(\varepsilon, x_\cdot\right)^{-1} Z_\cdot\, G_T\right)\right],
\end{aligned}
$$

where D^* is the Skorokhod integral operator. Remark that $G_T \in \mathbb{D}_\infty\left(\mathbb{R}^l \otimes \mathbb{R}^d\right)$ from the conditions on $a_i\left(\varepsilon, y\right)\ (1 \le i \le m)$ (cf. [23]). Then, Proposition I-1.3.3 in [23] yields that

$$
\begin{aligned}
& D^*\left(a\left(\varepsilon, x_\cdot\right)^{-1} Z_\cdot\, G_T\right) \\
&= D^*\left(a\left(\varepsilon, x_\cdot\right)^{-1} Z_\cdot\right) G_T - \int_0^T \mathrm{Tr}\left[a\left(\varepsilon, x_s\right)^{-1} Z_s\, D_s G_T\right] ds \\
&= \left\{\int_0^T \left(dW_s\right)^* a\left(\varepsilon, x_s\right)^{-1} Z_s\right\} G_T - \int_0^T \mathrm{Tr}\left[a\left(\varepsilon, x_s\right)^{-1} Z_s\, D_s G_T\right] ds \\
&= T\, L_T^\varepsilon - T\, H_T^\varepsilon,
\end{aligned}
$$

which completes the proof. ☐

Lemma 4.4. *It holds that*

$$
\mathbb{E}_x\left[\partial\varphi\left(x_T\right) Z_T \int_0^T\int_{\mathbb{R}_0^m} \kappa\left(t, z\right) dJ\right] = -\mathbb{E}_x\left[\varphi\left(x_T\right) V_T^\varepsilon\right]. \tag{4.5}
$$

Proof. Write

$$
M_T = \int_0^T\int_{\mathbb{R}_0^m} \kappa\left(s, z\right) dJ, \quad \hat{M}_T = \int_0^T\int_{\mathbb{R}_0^m} \kappa\left(s, z\right) d\hat{J}.
$$

Multiplying M_T by both sides of the equality (4.1), we see

$$
\begin{aligned}
\mathbb{E}_x\left[\varphi\left(x_T\right) M_T\right] &= \mathbb{E}_x\left[\varphi\left(x_T\right)\right] \mathbb{E}_x\left[M_T\right] + \mathbb{E}_x\left[\left\{\int_0^T \left[\partial u^\varepsilon a\right]\left(\varepsilon, x_t\right) dW_t\right\} M_T\right] \\
&\quad + \mathbb{E}_x\left[\left\{\int_0^T\int_{\mathbb{R}_0^m} \mathfrak{B}_z^\varepsilon u^\varepsilon\left(t, x_{t-}\right) d\tilde{J}\right\} M_T\right] \\
&=: \mathbb{E}_x\left[\varphi\left(x_T\right)\right] \mathbb{E}_x\left[\hat{M}_T\right] + I_1 + I_2.
\end{aligned}
$$

As for I_1, we have

$$
I_1 = \mathbb{E}_x\left[\int_0^T \left[\partial u^\varepsilon a_i\right]\left(\varepsilon, x_t\right) M_t\, dW_t\right] + \mathbb{E}_x\left[\int_0^T \left(\int_0^t \left[\partial u^\varepsilon a\right]\left(\varepsilon, x_s\right) dW_s\right) dM_t\right]
$$

$$= \mathbb{E}_x \left[\int_0^T \left(\int_0^t \left[\partial u^\varepsilon a \right] (\varepsilon, x_s)\, dW_s \right) d\hat{M}_t \right].$$

As for I_2, we can get

$$\begin{aligned}
I_2 &= \mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}_0^m} \mathfrak{B}_z^\varepsilon u^\varepsilon (t, x_{t-})\, \kappa(t,z)\, dJ \right] \\
&\quad + \mathbb{E}_x \left[\int_0^T \left(\int_0^t \int_{\mathbb{R}_0^m} \mathfrak{B}_\theta^\varepsilon u^\varepsilon (s, x_{s-})\, d\tilde{J} \right) dM_t \right] + \mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}_0^m} \mathfrak{B}_z^\varepsilon u^\varepsilon (t, x_{t-})\, M_{t-}^\varepsilon\, d\tilde{J} \right] \\
&= \mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}_0^m} \mathfrak{B}_z^\varepsilon u^\varepsilon (t, x_t)\, \kappa(t,z)\, d\hat{J} \right] + \mathbb{E}_x \left[\int_0^T \left(\int_0^t \int_{\mathbb{R}_0^m} \mathfrak{B}_\theta^\varepsilon u^\varepsilon (s, x_{s-})\, d\tilde{J} \right) d\hat{M}_t \right].
\end{aligned}$$

Since

$$\begin{aligned}
&\mathbb{E}_x \left[\left\{ \int_t^T \left[\partial u^\varepsilon a \right] (\varepsilon, x_s)\, dW_s + \int_t^T \int_{\mathbb{R}_0^m} \mathfrak{B}_\theta^\varepsilon u^\varepsilon (s, x_{s-})\, d\tilde{J} \right\} \kappa(t,z) \right] \\
&= \mathbb{E}_x \left[\mathbb{E}_x \left[\int_t^T \left[\partial u^\varepsilon a \right] (\varepsilon, x_s)\, dW_s + \int_t^T \int_{\mathbb{R}_0^m} \mathfrak{B}_\theta^\varepsilon u^\varepsilon (s, x_{s-})\, d\tilde{J} \,\middle|\, \mathscr{F}_t \right] \kappa(t,z) \right] \\
&= \mathbf{0} \in \mathbb{R}^l \otimes \mathbb{R}^d,
\end{aligned}$$

the Fubini theorem and the equality (4.1) in Lemma 4.1 enable us to see that

$$\begin{aligned}
\mathbb{E}_x \left[\varphi(x_T)\, M_T \right] &= \mathbb{E}_x \left[\varphi(x_T)\, \hat{M}_T \right] + \mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}_0^m} \mathfrak{B}_z^\varepsilon u^\varepsilon (t, x_t)\, \kappa(t,z)\, d\hat{J} \right] \\
&\qquad - \mathbb{E}_x \left[\int_0^T \left(\int_t^T \left[\partial u^\varepsilon a \right] (\varepsilon, x_s)\, dW_s \right) d\hat{M}_t \right] \\
&\qquad - \mathbb{E}_x \left[\int_0^T \left(\int_t^T \int_{\mathbb{R}_0^m} \mathfrak{B}_\theta^\varepsilon u^\varepsilon (s, x_{s-})\, d\tilde{J} \right) d\hat{M}_t \right] \\
&= \int_0^T \int_{\mathbb{R}_0^m} \mathbb{E}_x \left[\varphi(x_T)\, \kappa(t,z) \right] d\hat{J} + \int_0^T \int_{\mathbb{R}_0^m} \mathbb{E}_x \left[\mathfrak{B}_z^\varepsilon u^\varepsilon (t, x_t)\, \kappa(t,z) \right] d\hat{J} \\
&= \int_0^T \int_{\mathbb{R}_0^m} \mathbb{E}_x \left[u^\varepsilon (t, x_t + b(\varepsilon, x_t, z))\, \kappa(t,z) \right] d\hat{J}
\end{aligned} \tag{4.6}$$

We shall differentiate both hand sides of (4.6) in $x \in \mathbb{R}^d$. From $\varphi \in C_b^2\left(\mathbb{R}^d; \mathbb{R}\right)$ and the condition on $b(\varepsilon, y, z)$, we see that

$$\partial_x \mathbb{E}_x \left[\varphi(x_T)\, M_T \right] = \mathbb{E}_x \left[\partial \varphi(x_T)\, Z_T\, M_T \right] + \mathbb{E}_x \left[\varphi(x_T)\, \partial_x M_T \right]$$

as for the left hand side of (4.6), while the right hand side of (4.6) is

$$
\begin{aligned}
&\partial_x \int_0^T \int_{\mathbb{R}_0^m} \mathbb{E}_x \left[u^\varepsilon \left(t, x_t + b\left(\varepsilon, x_t, z\right)\right) \kappa\left(t,z\right)\right] d\hat{J} \\
&= \int_0^T \int_{\mathbb{R}_0^m} \mathbb{E}_x \left[\partial_x u^\varepsilon \left(t, x_t + b\left(\varepsilon, x_t, z\right)\right) \kappa\left(t,z\right)\right] d\hat{J} \\
&\quad + \int_0^T \int_{\mathbb{R}_0^m} \mathbb{E}_x \left[\mathfrak{B}_z^\varepsilon u^\varepsilon \left(t, x_t\right) \partial_x \kappa\left(t,z\right)\right] d\hat{J} + \int_0^T \int_{\mathbb{R}_0^m} \mathbb{E}_x \left[u^\varepsilon \left(t, x_t\right) \partial_x \kappa\left(t,z\right)\right] d\hat{J} \\
&=: I_3 + I_4 + I_5.
\end{aligned}
$$

We shall compute I_3, I_4 and I_5. Lemma 4.1 enables us to see that

$$
\begin{aligned}
I_4 &= \mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}_0^m} \mathfrak{B}_z^\varepsilon u^\varepsilon \left(t, x_{t-}\right) d\tilde{J} \int_0^T \int_{\mathbb{R}_0^m} \partial_x \kappa\left(t,z\right) d\tilde{J} \right] \\
&= \mathbb{E}_x \left[\varphi\left(x_T\right) \int_0^T \int_{\mathbb{R}_0^m} \partial_x \kappa\left(t,z\right) d\tilde{J} \right].
\end{aligned}
$$

On the other hand, since $u^\varepsilon\left(t,x_t\right) = \mathbb{E}_x\left[\varphi\left(x_T\right) \mid \mathscr{F}_t\right]$ for $0 \le t < T$, we have

$$
I_5 = \mathbb{E}_x \left[\varphi\left(x_T\right) \int_0^T \int_{\mathbb{R}_0^m} \partial_x \kappa\left(t,z\right) d\hat{J} \right].
$$

Furthermore, multiplying V_T^ε by the equality (4.1) in Lemma 4.1, it holds that

$$
\begin{aligned}
&\mathbb{E}_x \left[\varphi\left(x_T\right) V_T^\varepsilon \right] \\
&= \mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}_0^m} \mathfrak{B}_z^\varepsilon u^\varepsilon \left(s, x_s\right) \operatorname{div}_z \left[g\left(z\right) \left[\left(\partial_z b\right)^{-1} \left(I + \partial b\right) \right] \left(\varepsilon, x_s, z\right) Z_s \, \kappa\left(s,z\right) \right] dz\, ds \right] \\
&= -\mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}_0^m} \partial_z u^\varepsilon \left(s, x_s + b\left(\varepsilon, x_s, z\right)\right) \left[\left(\partial_z b\right)^{-1} \left(I + \partial b\right) \right] \left(\varepsilon, x_s, z\right) Z_s \, \kappa\left(s,z\right) d\hat{J} \right] \\
&= -\mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}_0^m} \partial u^\varepsilon \left(s, x_s + b\left(\varepsilon, x_s, z\right)\right) Z_s \, \kappa\left(s,z\right) d\hat{J} \right] \\
&= -I_3.
\end{aligned}
$$

Here we have used the integration by parts (or, the divergence formula) in the second equality, which can be justified by Assumption 1 (iii). Therefore, we can obtain that

$$
\mathbb{E}_x \left[\partial \varphi\left(x_T\right) Z_T \, M_T \right] + \mathbb{E}_x \left[\varphi\left(x_T\right) \partial_x M_T \right]
$$

$$= -\mathbb{E}_x\left[\varphi\left(x_T\right) V_T^\varepsilon\right] + \mathbb{E}_x\left[\varphi\left(x_T\right) \int_0^T \int_{\mathbb{R}_0^m} \partial_x \kappa\left(t,z\right) d\tilde{J}\right] + \mathbb{E}_x\left[\varphi\left(x_T\right) \int_0^T \int_{\mathbb{R}_0^m} \partial_x \kappa\left(t,z\right) d\hat{J}\right]$$
$$= -\mathbb{E}_x\left[\varphi\left(x_T\right) V_T^\varepsilon\right] + \mathbb{E}_x\left[\varphi\left(x_T\right) \partial_x M_T\right],$$

which completes the proof. □

Proof of Theorem 1. Firstly, consider the case of $\varphi \in C_0^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$, where $C_0^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$ is the set of C^∞-functions on $\mathbb{R}^d$ with the compact support. Remark that

$$\partial_\varepsilon \varphi\left(x_T\right) = \partial\varphi\left(x_T\right) Z_T \left\{\int_0^T F_0\left(s\right) ds + \int_0^T F\left(s\right) \circ dW_s + \int_0^T \int_{\mathbb{R}_0^m} \kappa\left(s,z\right) dJ\right\}$$

from Corollary 2.1. By summing up the equalities (4.3), (4.4) and (4.5), the assertion of Theorem 1 holds for $\varphi \in C_0^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$. The standard density argument (cf. [6, 7]) enables us to extend the assertion of Theorem 1 for $\varphi \in C_0^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$, to the one for square-integrable measurable function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with $\mathbb{E}_x\left[\left|\varphi\left(x_T\right)\right|^2\right] < +\infty$, which we are going to explain below.

Secondly, consider the case of $\varphi \in C_0\left(\mathbb{R}^d;\mathbb{R}\right)$, where $C_0\left(\mathbb{R}^d;\mathbb{R}\right)$ is the set of continuous functions on $\mathbb{R}^d$ with the compact support. Then the function φ can be approximated uniformly and boundedly by the sequence $\{\varphi_n\,;\, n \in \mathbb{N}\}$ in $C_0^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$. Thus we have

$$\left|\mathbb{E}_x\left[\varphi_n\left(x_T\right)\right] - \mathbb{E}_x\left[\varphi\left(x_T\right)\right]\right| \leq \sup_{y\in\mathbb{R}^d} \left|\varphi_n\left(y\right) - \varphi\left(y\right)\right|,$$

which tends to 0 as $n \to +\infty$. On the other hand, the Cauchy-Schwarz inequality and Lemma 2.1 yield that, for any compact set $\Xi \subset \mathbb{R}^l$,

$$\begin{aligned}
&\sup_{\varepsilon\in\Xi} \left|\partial_\varepsilon \mathbb{E}_x\left[\varphi_n\left(x_T\right)\right] - \mathbb{E}_x\left[\varphi\left(x_T\right)\{N_T^\varepsilon + L_T^\varepsilon - H_T^\varepsilon - V_T^\varepsilon\}\right]\right| \\
&\leq \sup_{\varepsilon\in\Xi} \mathbb{E}_x\left[\left|\varphi_n\left(x_T\right) - \varphi\left(x_T\right)\right|^2\right]^{1/2} \sup_{\varepsilon\in\Xi} \mathbb{E}_x\left[\left|N_T^\varepsilon + L_T^\varepsilon - H_T^\varepsilon - V_T^\varepsilon\right|^2\right]^{1/2} \\
&\leq \sup_{y\in\mathbb{R}^d} \left|\varphi_n\left(y\right) - \varphi\left(y\right)\right| \sup_{\varepsilon\in\Xi} \mathbb{E}_x\left[\left|N_T^\varepsilon + L_T^\varepsilon - H_T^\varepsilon - V_T^\varepsilon\right|^2\right]^{1/2}
\end{aligned}$$

which tends to 0 as $n \to +\infty$. Hence, the assertion can be justified for $\varphi \in C_0\left(\mathbb{R}^d;\mathbb{R}\right)$.

Thirdly, study the case of an indicator function $\varphi = I_K$, where K is a compact subset in $\mathbb{R}^d$. Let $N \in \mathbb{N}$, and let $\delta > 0$ be sufficiently small. Define subsets $K_{+\delta}$ and $K_{-\delta}$ in $\mathbb{R}^d$ by

$$K_{+\delta} = \left\{y \in \mathbb{R}^d\,;\, |y - \tilde{y}| \leq \delta \;\; (\tilde{y} \in \partial K)\right\} \cup K,$$

$$K_{-\delta} = \left\{ y \in \mathbb{R}^d ; \, |y - \tilde{y}| > \delta \ \ (\tilde{y} \in \partial K) \right\} \cap K,$$

where ∂K denotes the boundary of K. Then, there exists a pointwise convergent sequence $\{\varphi_n ; n \in \mathbb{N}\}$ in $C_0\left(\mathbb{R}^d ; \mathbb{R}\right)$ to φ such that $0 \le \varphi_n \le 1$, and for every $n \ge N$, $\varphi_n - \varphi = 0$ on $\tilde{K} = K_{+\delta}^c \cup K_{-\delta}$. Since $\{\varphi_n ; n \in \mathbb{N}\}$ is uniformly bounded, the dominated convergence theorem implies that

$$|\mathbb{E}_x [\varphi_n (x_T)] - \mathbb{E}_x [\varphi (x_T)]| \to 0$$

as $n \to +\infty$. It is clear that, for every $n \ge N$ and any compact subset $\Xi \subset \mathbb{R}^l$,

$$\sup_{\varepsilon \in \Xi} \mathbb{E}_x \left[|\varphi_n (x_T) - \varphi (x_T)|^2 I_{(x_T \in \tilde{K})} \right] = 0.$$

Moreover, we see that

$$\mathbb{E}_x \left[|\varphi_k (x_T) - \varphi (x_T)|^2 I_{(x_T \in \tilde{K}^c)} \right] \le 4 \mathbb{P}_x \left[x_T \in \tilde{K}^c \right],$$

because of $0 \le \varphi \le 1$. Since the probability law of the $\mathbb{R}^d$-valued random variable x_T admits a smooth density $p_T(\varepsilon, x, y)$ with respect to the Lebesgue measure over $\mathbb{R}^d$ under Assumption 2, as seen in Proposition 3.1 and Remark 3.1, we have

$$\mathbb{P}_x \left[x_T \in \tilde{K}^c \right] = \int_{\tilde{K}^c} p_T(\varepsilon, x, y) \, dy \le \left(\sup_{y \in \tilde{K}^c} p_T(\varepsilon, x, y) \right) \left| \tilde{K}^c \right|,$$

where $\left|\tilde{K}^c\right|$ denotes the Lebesgue measure of the set $\tilde{K}^c \subset \mathbb{R}^d$. Thus, we can obtain

$$\mathbb{E}_x \left[|\varphi_n (x_T) - \varphi (x_T)|^2 I_{(x_T \in \tilde{K}^c)} \right] \le 4 \left(\sup_{y \in \tilde{K}^c} p_T(\varepsilon, x, y) \right) \left| \tilde{K}^c \right|.$$

Remark that $\left|\tilde{K}^c\right| \to 0$ as $\delta \downarrow 0$. Since $\tilde{K}^c$ is the compact subset in $\mathbb{R}^d$, the density $p_T(\varepsilon, x, y)$ is uniformly bounded in $y \in \tilde{K}^c$. Hence the Cauchy-Schwarz inequality and Lemma 2.1 yield that

$$\begin{aligned}
&\sup_{\varepsilon \in \Xi} |\partial_\varepsilon \mathbb{E}_x [\varphi_n (x_T)] - \mathbb{E}_x [\varphi (x_T) \{N_T^\varepsilon + L_T^\varepsilon - H_T^\varepsilon - V_T^\varepsilon\}]| \\
&\le \sup_{\varepsilon \in \Xi} \mathbb{E}_x \left[|\varphi_n (x_T) - \varphi (x_T)|^2 \right]^{1/2} \sup_{\varepsilon \in \Xi} \mathbb{E}_x \left[|N_T^\varepsilon + L_T^\varepsilon - H_T^\varepsilon - V_T^\varepsilon|^2 \right]^{1/2} \\
&\le 2 \left(\sup_{\varepsilon \in \Xi} \sup_{y \in \tilde{K}^c} p_T(\varepsilon, x, y) \right)^{1/2} \left| \tilde{K}^c \right|^{1/2} \sup_{\varepsilon \in \Xi} \mathbb{E}_x \left[|N_T^\varepsilon + L_T^\varepsilon - H_T^\varepsilon - V_T^\varepsilon|^2 \right]^{1/2},
\end{aligned}$$

which tends to 0 as $n \to +\infty$. Therefore, we can conclude that

$$\partial_\varepsilon \mathbb{E}_x [\varphi(x_T)] = \mathbb{E}_x [\varphi(x_T) \{N_T^\varepsilon + L_T^\varepsilon - H_T^\varepsilon - V_T^\varepsilon\}].$$

Finally, we can immediately extend the assertion of Theorem 1 to the class of finite linear combinations of indicator functions, which can approximate a measurable function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\mathbb{E}_x \left[|\varphi(x_T)|^2 \right] < +\infty$. The proof of Theorem 1 is complete. □

Acknowledgements. The author would like to thank Professor E. Gobet for valuable discussion and encouragement.

References

[1] Bally, V., Bavouzet, M. P. and Messaoud, M. (2007). Integration by parts formula for locally smooth laws and applications to sensitivity computations, *Ann. Appl. Probab.*, **17**, 33–66.

[2] Bichteler, K., Gravereaux, J. -B. and Jacod, J. (1987). *Malliavin Calculus for Processes with Jumps*, Gordon and Breach Science Publishers, New York.

[3] Bismut, J. M. (1981). Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions, *Z. Wahrsc. Verw. Gebiete*, **56**, 469–505.

[4] Bismut, J. M. (1983). Calcul des variations stochastique et processus de saut, *Z. Wahrsc. Verw. Gebiete*, **63**, 147–235.

[5] Bismut, J. M. (1984), *Large Deviations and the Malliavin Calculus*, Birkhäuser, Boston.

[6] Cass, T. R. and Friz, P. K. (2007). The Bismut-Elworthy-Li formula for jump-diffusions and applications to Monte Carlo methods in finance, available at arXiv:math/0604311v3.

[7] Davis, M. H. and Johansson, M. P. (2006). Malliavin Monte Carlo Greeks for jump diffusions, *Stochastic Process. Appl.*, **116**, 101–129.

[8] Di Nunno, G., Oksendal, B. and Proske, F. (2009), *Malliavin Calculus for Lévy Processes with Applications to Finance*, Springer-Verlag, Berlin.

[9] El-Khatib, Y. and Privault, N. (2004). Computations of Greeks in a market with jumps via the Malliavin calculus, *Finance Stoch.*, **8**, 161–179.

[10] Elworthy, K. D. and Li, X. -M. (1994). Formulae for the derivatives of heat semigroups, *J. Funct. Anal.*, **125**, 252–286.

[11] Fournié, E., Lasry, J. M., Lebuchoux, J. and Lions, P. L. (1999). Applications of Malliavin calculus to Monte-Carlo methods in finance, *Finance Stoch.*, **3**, 391–412.

[12] Fournié, E., Lasry, J. M., Lebuchoux, J. and Lions, P. L. (2001). Applications of Malliavin calculus to Monte-Carlo methods in finance II, *Finance Stoch.*, **5**, 201–236.

[13] Fujiwara, T. and Kunita, H. (1985). Stochastic differential equations of jump type and Lévy processes in diffeomorphisms group, *J. Math. Kyoto Univ.*, **25**, 71–106.

[14] Gihman, I. I. and Skorokhod, A. V. (1972). *Stocahstic Differential Equations*, Springer-Verlag, New York.

[15] Gobet, E. (2004). Revisiting the Greeks for Europen and American options, In *Stochastic processes and applications to mathematical finance*, 53–71, World Sci. Publ., River Edge, NJ.

[16] Ikeda, N. and Watanabe, S. (1988). *Stochastic Differential Equations and Diffusion Processes*, 2nd edition, North-Holland / Kodansha.

[17] Ishikawa, Y. and Kunita, H. (2006). Malliavin calcullus on the Wiener-Poisson space and its applications to canonical SDE with jumps, *Stochastic Process. Appl.*, **116**, 1743–1769.

[18] Kawai, R. and Kohatsu-Higa, A. (2008). Computation of Greeks and multi-dimensional density estimation for asset price dynamics with time-changed Brownian motion, submitted for publication.

[19] Kawai, R. and Takeuchi, A. (2008). Greeks formulae for an asset price dynamics model with gamma processes, under revision.

[20] Kawai, R. and Takeuchi, A. (2008). Sensitivity analysis for averaged asset price dynamics with gamma processes, submitted for publication.

[21] Komatsu, T. and Takeuchi, A. (2001). On the smoothness of pdf of solutions to SDE of jump type, *Int. J. Differ. Equ. Appl.*, **2**, 141-197.

[22] Léandre, R. (1985). Régularité de processus desauts dégénérés II, *Ann. Inst. H. Poincaré, Probab. Statist.*, **24**, 209-236.

[23] Nualart, D. (2006). *The Malliavin calculus and related topics*, 2nd edition, Springer-Verlag, Berlin.

[24] Picard, J. (1996). On the existence of smooth densities for jump processes, *Probab. Theory Related Fields*, **105**, 481–511.

[25] Takeuchi, A. (2002). The Malliavin calculus for SDE with jumps and the partially hypoelliptic problem, *Osaka J. Math.*, **39**, 523–559.

[26] Takeuchi, A. (2008). The Bismut-Elworthy-Li type formulae for stochastic differential equations with jumps, submitted for publication.